THE GENERALIZED VARIANCE: TESTING AND RANKING PROBLEM[1]

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In this note it is shown that, for a sample from a multivariate normal distribution, the density function of the sample generalized variance possesses a monotone likelihood ratio (MLR). This result is used to construct a uniformly most powerful invariant test for a testing problem concerning the population generalized variance. Also, the result is applied to the problem of ranking multivariate normal populations according to the size of their generalized variances.

Let $X_1, \cdots, X_{n+1}$ be a random sample from a $p(p \leqq n)$ variate normal distribution $N_p(\mu, \Sigma)$, with mean μ and nonsingular covariance matrix Σ. Consider the sufficient statistic $(\bar{X}, S)$ where

$$\bar{X} \equiv (1/n + 1) \textstyle\sum_{i=1}^{n+1} X_i \tag{1}$$

and

$$S \equiv \textstyle\sum_{i=1}^{n+1} X_i'X_i - (n + 1)\bar{X}'\bar{X}, \tag{2}$$

so that $\bar{X}$ and S are independent, $\bar{X}$ is $N_p(\mu, 1/(n + 1)\Sigma)$ and S has a Wishart distribution, $W_p(\Sigma, n)$, with expectation $n\Sigma$. If we set $\theta = \det(\Sigma)$ and $V = \det(\mathrm{S})$, then $\theta(V$, resp.) is the population (sample, resp.) generalized variance. It is well known that V has the same distribution as $\theta \prod_{i=1}^{p} \chi^2_{n-i+1}$ where the factors χ^2_{n-i+1} are independent and have a chi-square distribution with $n - i + 1$ degrees of freedom (see Anderson (1958) p. 171). Let $f_p(v, \theta)$ denote density function of V.

Lemma 1. *The density function, $f_p(v, \theta)$, of the generalized variance has a MLR.*

Proof. The proof is by induction on $p(1 \leqq p \leqq n)$. For $p = 1$, V has the density of a scaled chi-square random variable which is known to have a MLR. Now, it is straightforward to show that

$$f_p(v, \theta) = \textstyle\int_0^\infty f_{p-1}(v, x)h(x, \theta)\, dx \tag{3}$$

where h is the density of a scaled χ^2_{n-p+1} random variable. Noting that $h(x, \theta)$ has a MLR, the result now follows by the induction hypothesis and an application of a result due to Karlin (1956, Lemma 5, p. 125). □

As an application of the above lemma, consider the hypothesis $H_0: \theta \leqq c_1$ and

Received 23 January 1967.

[1] A portion of this work was completed while the author was at Stanford University (1965–66) with the partial support of National Science Foundation Grant, GP-3837. The work was completed at the University of Chicago with the partial support of Research Grant No. NSF 3707 from the Division of Mathematical, Physical and Engineering Sciences of the National Science Foundation, and in part by the Army Research Office, Office of Naval Research, and Air Force Office of Scientific Research by Contract No. Nonr-2121(23), NR 342-043.

the alternative $H_1: \theta > c_2$ where $0 < c_1 \leqq c_2$. After a reduction by sufficiency, the observation available is $(\bar{X}, S)$ where $\bar{X}$ is $N_p(\mu, 1/(n+1)\Sigma)$, S is $W_p(\Sigma, n)$ and $\theta = \det(\Sigma)$. Let G be the group described as follows: An element $g \varepsilon G$ has the form $g = (T, b)$ where T is a $p \times p$ lower triangular matrix with positive diagonal elements and determinant equal to one and b is a p-dimensional row vector. The group operation is defined by

$$(T_2, b_2)(T_1, b_1) \equiv (T_1T_2, b_1T_2 + b_2). \tag{4}$$

Then G operates on the left of sample points and parameter points; the operation being given by,

$$(T, b)(x, s) = (xT + b, T'sT). \tag{5}$$

It is clear that the above hypothesis testing problem is invariant under the group of transformations G. Moreover, it is not hard to show that a maximal invariant in the sample space is $V = \det(S)$ and a maximal invariant in the parameter space is $\theta \equiv \det(\Sigma)$. Let φ_0 be the test which rejects H_0 if V is greater than some constant, K_α, chosen to make the test level $\alpha(0 < \alpha < 1)$.

PROPOSITION 1. *For the above hypothesis testing problem, the test φ_0 is uniformly most powerful invariant. Also, φ_0 is a maximum test and is most stringent.*

PROOF. First note that all invariant tests will be functions of V only and the density of V has a MLR. Since the hypothesis under consideration is one-sided, that φ_0 is uniformly most powerful invariant follows from well known results given in Lehmann (1959).

For the second assertion, we first note that the group G satisfies the conditions of the Hunt-Stein Theorem (see Kiefer (1957)). It then follows immediately that φ_0 is a maximum test and is most stringent (Lehmann (1959), Chapter 8). ▯

Consider independent observations $(\bar{X}_i, S_i)$, $i = 1, \cdots, k$ where $\bar{X}_i$ is $N_p(\mu_i, 1/(n+1)\Sigma_i)$ and is independent of S_i which is $W_p(\Sigma_i, n)$. Let $\theta = (\theta_1, \cdots, \theta_k)$ be the vector of population generalized variances ($\theta_i = \det(\Sigma_i)$) and let $V = (V_1, \cdots, V_k)$ be the vector of sample generalized variances ($V_i = \det(S_i)$). Consider the problem of ranking the k underlying populations according to the size of the associated generalized variance and let φ^* be the decision rule which ranks the populations according to the size of the observed sample generalized variance. It is assumed that the loss function for the ranking problem depends only on $(\theta_1, \cdots, \theta_k)$ and satisfies the assumptions given by Eaton (1967).

PROPOSITION 2. *Within the class of decision rules which depend only on $V = (V_1, \cdots, V_k)$, φ^* is* (i) *minimax,* (ii) *admissible, and* (iii) *the uniformly best decision rule within the class of rules which are invariant under permutations of the vector $V = (V_1, \cdots, V_k)$.*

PROOF. Noting that the sample sizes are equal and using Lemma 1, the conclusions follow immediately from Theorems 4.2 and 4.3 given by Eaton (1967). ▯

Now, let G^k denote the direct product of k copies of the group G defined above. Then G^k operates on the left of a sample point $((\bar{X}_1, S_1), \cdots, (\bar{X}_k, S_k))$ coordinatewise as defined by (5). Similarly, G^k operates on the left of a parameter point $((\mu_1, \Sigma_1), \cdots, (\mu_k, \Sigma_k))$. It is clear that the ranking problem is invariant under the group G^k and that the maximal invariant in the sample (parameter, respectively) space is $V = (V_1, \cdots, V_k)(\theta = (\theta_1, \cdots, \theta_k)$, respectively).

PROPOSITION 3. *The decision rule φ^* is minimax within the class of all decision rules.*

PROOF. Since the ranking problem is invariant under G^k, this result follows from a direct application of a general minimax theorem due to Kiefer (1957, p. 587). □

Since φ^* is minimax within the class of all decision rules, it follows that φ^* is a most economical decision rule (see Eaton (1967) and Hall (1958 and 1959)).

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