AN INTRINSICALLY DETERMINED MARKOV CHAIN[1]

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Consider a Markov chain $X_0, X_1, X_2, \cdots$, on the non-negative integers with $P[X_{t+1} = x - 1 \mid X_t = x] = \gamma(x)$, $P[X_{t+1} = x + 1 \mid X_t = x] = 1 - \gamma(x)$, for $x = 1, 2, \cdots$, and with 0 an absorbing state; that is, $P[X_{t+1} = 0 \mid X_t = 0] = 1$. Thus the process can either go up or down one step and for $x \geqq 1$, $\gamma(x)$ is the probability of going down. For $x = 0, 1, \cdots$, let $q_\gamma(x) = P[X_t = 0$ *for some* $t \geqq 0$ *given* $X_0 = x]$. Hence $q_\gamma(0) = 1$ by definition.

Now let φ be a given function on the interval $(0, 1)$ satisfying $0 \leqq \varphi \leqq 1$. We impose on the function γ of the preceding paragraph the condition

$$\gamma(x) = \varphi(q_\gamma(x)), \qquad x = 1, 2, \cdots. \tag{1}$$

Thus the transition law of the process depends on the probability that the process is absorbed at zero; but the probability that this happens depends on the transition law. The process, if it is determined at all, is determined by its own behavior, i.e., it is determined 'intrinsically'.

We show the possibility of determining a process in this fashion, by virtue of the following result:

THEOREM 1. *If φ is uniformly continuous on $(0, 1)$ with $0 < a = \inf \varphi \leqq \sup \varphi = b < \frac{1}{2}$, then there exists a function γ such that γ and q_γ jointly satisfy (1); these functions, γ and q_γ, are unique if φ is non-increasing.*

For the sake of a phenomenological interpretation, we can imagine that X_t is the fortune of a man who works at gambling. The harder he works each day, the greater the probability of his winning one unit, and the less the probability of losing one unit, these being the only outcomes. Being concerned with the probability of becoming destitute, but at the same time not particularly liking to work, he assesses the latter probability each day, and thereby decides how hard to work, generally working less the lower this probability is. Theorem 1 says that choice of such a procedure, as expressed by φ, is theoretically possible and leads to a unique mode of behavior.

PROOF OF THEOREM 1. The proof will make use of the following elementary lemmas.

LEMMA 1. *If $\gamma = 1 - \bar{\gamma}$ satisfies $0 < \gamma(x) < 1$, $x = 1, 2, \cdots$, then there is at most one function q satisfying*

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$$q(x) = \gamma(x)q(x-1) + \bar{\gamma}(x)q(x+1), \quad x = 1, 2, \cdots, \tag{2}$$

subject to $q(0) = 1$ *and* $q(x) \to 0$ *as* $x \to \infty$.

PROOF. Suppose for two solutions q and q', $\sup_{x \geqq 1} (q(x) - q'(x)) = m > 0$. Since $q(x) - q'(x) \to 0$ as $x \to \infty$, there is a largest integer $x_0 \geqq 1$ such that $q(x_0) - q'(x_0) = m$, and $q(x_0 + 1) - q'(x_0 + 1) < m$. Considering that $\bar{\gamma}(x_0) > 0$,

$$\begin{aligned} m = q(x_0) - q'(x_0) = \gamma(x_0)(q(x_0 - 1) - q'(x_0 - 1)) \\ + \bar{\gamma}(x_0)(q(x_0 + 1) - q'(x_0 + 1)) < m, \end{aligned} \tag{3}$$

and this contradiction completes the proof.

LEMMA 2. *If the function* $\varphi = 1 - \bar{\varphi}$ *on the interval* $(0, 1)$ *satisfies* $0 < \varphi < 1$ *and is non-increasing, then there is at most one function* q *satisfying*

$$q(x) = \varphi(q(x))q(x-1) + \bar{\varphi}(q(x))q(x+1), \quad x = 1, 2, \cdots, \tag{4}$$

subject to $q(0) = 1$, $q(x) \geqq q(x+1)$, $x = 1, 2, \cdots$, *and* $q(x) \to 0$ *as* $x \to \infty$.

PROOF. Proceeding as in the proof of Lemma 1, and employing the fact that if $q(x_0) - q'(x_0) = m > 0$, then $\varphi(q(x_0)) \leqq \varphi(q'(x_0))$, and that $q(x-1) \geqq q(x+1)$, we find

$$\begin{aligned} m = q(x_0) - q'(x_0) \leqq \varphi(q'(x_0))q(x_0 - 1) + \bar{\varphi}(q'(x_0))q(x_0 + 1) \\ - [\varphi(q'(x_0))q'(x-1) - \bar{\varphi}(q'(x_0))q'(x_0 + 1)] < m, \end{aligned} \tag{5}$$

a contradiction as before, and the proof is complete.

Now let C be the class of functions q on the non-negative integers with $q(x) \geqq q(x+1)$, $x = 0, 1, \cdots$, and satisfying

$$[a/(1-a)]^x \leqq q(x) \leqq [b/(1-b)]^x, \quad x = 0, 1, 2, \cdots, \tag{6}$$

where a and b are as in the hypothesis of Theorem 1. If γ satisfies

$$a \leqq \gamma(x) \leqq b, \quad x = 1, 2, \cdots, \tag{7}$$

then q_γ is given as the unique solution of

$$q(x) = \gamma(x)q(x-1) + \bar{\gamma}(x)q(x+1), \quad x = 1, 2, \cdots, \tag{8}$$

subject to $q \varepsilon C$. It is obvious that q_γ satisfies (8) and will be unique by Lemma 1 if $q_\gamma \varepsilon C$. To prove that $q_\gamma \varepsilon C$, let $q_b = q_\gamma$ in the case where $\gamma(x) = b$, $x = 1, 2, \cdots$. Considering that b represents the maximum probability of going down one step, clearly $q_\gamma \leqq q_b$ for γ satisfying (7). Also, in that case, X_t is the sum of independent, identically distributed random variables with mean $1 - 2b > 0$. The strong law of large numbers implies $q_b(x) \to 0$ as $x \to \infty$. It is now easy to verify that $q_b(x) = [b/(1-b)]^x$ since the latter function satisfies (8) when $\gamma(x) = b$, and is the unique solution by Lemma 1. Similarly, $q_a \leqq q_\gamma$ where q_a is defined as q_γ for the case where $\gamma(x) = a$, $x = 1, 2, \cdots$. It remains to remark that $q_\gamma(x) \geqq q_\gamma(x+1)$ since $q_\gamma(x+1) = q_\gamma(x)$ Pr $[X_t = x$ *for at least one* $t \geqq 0$ *given* $X_0 = x + 1]$.

Consider then the mapping $U: C \to C$ defined by $(Uq)(x) = q_\gamma(x)$, where γ is given by $\gamma(x) = \varphi(q(x))$, $x = 1, 2, \cdots$. In view of (1) we have to show that U has a unique fixed point. It is easy to check that C is a compact, convex, subset of the Hilbert cube consisting of all sequences $\{\xi_n\}$ such that $|\xi_n| \leqq k/n$ for a suitable constant k. We will show that U is continuous. Then the Brouwer fixed point theorem implies (see Lemma 3 [1], p. 453) that U has a fixed point.

Since φ is uniformly continuous, $q' \to q$ in sup norm implies $\varphi(q'(x)) \to \varphi(q(x))$ uniformly in $x \geqq 1$. Thus all we require in order to establish the continuity of U, is that for γ_n and γ satisfying (7), $\gamma_n \to \gamma$ uniformly in x implies $q_n = q_{\gamma_n} \to q_\gamma$. Suppose to the contrary, that $\sup_{x \geqq 1} | q_n(x) - q_\gamma(x)|$ fails to converge to zero as $n \to \infty$. Then there is a subsequence $\{q_{n_k}\}$ such that $q_{n_k} \to q_0 \neq q_\gamma$ with $q_0 \varepsilon C$, since C is compact. Each q_{n_k} satisfies an equation like (8) with γ_{n_k} replacing γ, so adding and subtracting q_0 and γ in an obvious way, we write, (omitting the argument $x \geqq 1$)

$$
(9) \quad q_0 - (q_0 - q_{n_k}) = \gamma q_0^- + \bar{\gamma} q_0^+ + (\gamma_{n_k} - \gamma) q_0^- + (\bar{\gamma}_{n_k} - \bar{\gamma}) q_0^+ + \gamma_{n_k}(q_{n_k}^- - q_0^-) + \bar{\gamma}_{n_k}(q_{n_k}^+ - q_0^+),
$$

where q^- and q^+ stand, respectively, for the functions $q(x-1)$ and $q(x+1)$. Considering (9) as $k \to \infty$, it becomes apparent that q_0 satisfies (8), hence $q_0 = q_\gamma$ by Lemma 1. This contradiction establishes the continuity of U. Application of Lemma 2 completes the proof of Theorem 1.

REFERENCE

[1] DUNFORD, NELSON, and SCHWARTZ, JACOB T. (with the assistance of WILLIAM G. BADE and ROBERT G. BARTLE) (1958). *Linear Operators, Part I*, Interscience, New York.