ASYMPTOTIC EFFICIENCY OF A CLASS OF NON-PARAMETRIC TESTS FOR REGRESSION PARAMETERS[1]

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0. Introduction and summary. For testing hypotheses about α and β in the linear regression model $Y_j = \alpha + \beta x_j + Z_j$, Brown and Mood [18] have proposed distribution-free tests, based on their median estimates. Daniels [6] has also given a distribution-free test for the hypothesis that the regression parameters have specified values. This latter test is an improvement on the Brown and Mood median procedure, although both are based on the signs of the observations. Recently Hájek [10] constructed rank tests, which are asymptotically most powerful, for testing the hypothesis that $\beta = 0$, while α is regarded as a nuisance parameter.

In this paper, a class of rank score tests for the hypothesis $H: \alpha = \beta = 0$, is proposed in Section 2. This class includes as special cases, the Wilcoxon and the normal scores type of tests. In Sections 3 and 4 the limiting distribution of the test statistics is shown to be central χ^2, under H, and non-central χ^2, under a sequence of alternatives tending to the hypothesis at a suitable rate. In Section 5, the Pitman efficiency of the proposed tests relative to the classical F-test, is proved to be the same as the efficiency of the corresponding rank score tests relative to the t-test in the two sample problem.

1. Assumptions and notations. Let $(Y_{n1}, \cdots, Y_{nn})$ be a sequence of random vectors, where Y_{nj}, $j = 1, \cdots, n$, are independent with distributions

$$P_{\alpha\beta}(Y_{nj} \leqq y) = F(y - \alpha - \beta x_{nj}) \tag{1.1}$$

where $P_{\alpha\beta}$ denotes that the probability is being computed for the parameter values α and β.

The x_{nj} are known constants depending on n; and we shall suppress this dependence in our notation, whenever this causes no confusion. The problem here is to construct rank score tests for the hypothesis $H: \alpha = \beta = 0$. The form of F is not known but we shall assume only that $F \varepsilon \mathfrak{F}$, where

$$\begin{aligned} \mathfrak{F} = \{\text{absolutely continuous } F:& \\ &\text{(i) } F'(y) = f(y) \text{ is absolutely continuous,} \\ &\text{(ii) } \textstyle\int_{-\infty}^{\infty} (f'(y)/f(y))^2 f(y)\, dy \text{ is finite,} \\ &\text{(iii) } f(-y) = f(y)\}. \end{aligned} \tag{1.2}$$

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We shall also assume that the constants x_j, $j = 1, \cdots, n$, satisfy the following conditions:

$$\lim [\{\max_j (x_j - \bar{x}_n)^2\}/\{\textstyle\sum_j (x_j - \bar{x}_n)^2\}] = 0, \tag{1.3}$$

$$\lim [n^{-1} \textstyle\sum_j (x_j - \bar{x}_n)^2] < \infty, \qquad |\lim \bar{x}_n| < \infty, \tag{1.4}$$

$$\lim n^{-1} \textstyle\sum_j (x_j - \bar{x}_n)^2 > 0, \tag{1.5}$$

where $\bar{x}_n = n^{-1} \sum_j x_j$, and the summation goes from 1 to n.

All limits in this paper unless otherwise stated are taken as $n \to \infty$. We shall write $\mathcal{L}(X_n \mid P) \to N(a, b^2)$ to denote that the distribution law of $(X_n - a)/b$ tends to the standard normal distribution under P. The following class of functions shall be used in the sequel.

$$\psi(u) = -[g'(G^{-1}(\tfrac{1}{2}u + \tfrac{1}{2}))/g(G^{-1}(\tfrac{1}{2}u + \tfrac{1}{2}))], \qquad 0 < u < 1, \tag{1.6}$$

where G^{-1} is the inverse of G, and G is any known distribution function belonging to the class $\mathfrak{F}$. The (1.6)-function that corresponds to F is

$$\varphi(u) = -[f'(F^{-1}(\tfrac{1}{2}u + \tfrac{1}{2}))/f(F^{-1}(\tfrac{1}{2}u + \tfrac{1}{2}))], \qquad 0 < u < 1. \tag{1.7}$$

Observe that unlike $\psi(u)$ of (1.6), $\varphi(u)$ is not known since it is defined through the unknown F.

2. A class of test statistics. Let R_j be the rank of $|Y_j|$ in the sequence of absolute values $|Y_1|, \cdots, |Y_n|$ of the n observations. Consider a pair of statistics T_1 and T_2 defined by

$$T_1(Y) = n^{-1} \textstyle\sum_j \psi_n(R_j/n + 1) \operatorname{Sign} Y_j, \tag{2.1}$$

$$T_2(Y) = n^{-1} \textstyle\sum_j x_j \psi_n(R_j/n + 1) \operatorname{Sign} Y_j, \tag{2.2}$$

where

$$\psi_n(u) = \psi(j/n + 1), \qquad (j - 1)/n < u \leqq j/n, \tag{2.3}$$

and

$$\lim \textstyle\int_0^1 [\psi_n(u) - \psi(u)]^2 \, du = 0 \tag{2.4}$$

by [9]. Let a symmetric 2×2 matrix $\|\gamma_{kl}\|_n$ be given by

$$\gamma_{11} = \textstyle\int_0^1 \psi^2(u)\, du; \qquad \gamma_{22,n} = n^{-1} \sum_j x_j^2 \int_0^1 \psi^2(u)\, du; \tag{2.5}$$
$$\gamma_{21,n} = n^{-1} \textstyle\sum_j x_j \int_0^1 \psi^2(u)\, du.$$

Define

$$M(\psi) = n(T_1, T_2)\|\gamma_{kl}\|_n^{-1} (T_1, T_2)' \tag{2.6}$$

where $(\mathbf{V})'$ denotes the transpose of $(\mathbf{V})$, and $\|\gamma_{kl}\|_n^{-1}$ is the inverse of $\|\gamma_{kl}\|_n$. We propose $M(\psi)$ as the class of test statistics for the hypothesis $H: \alpha = \beta = 0$.

Observe that M is well defined since both $\|\gamma_{kl}\|_n^{-1}$ and its limit as $n \to \infty$, exist by (1.5).

To every $G \varepsilon \mathfrak{F}$, corresponds one test statistic M. In particular, if G is the logistic distribution function $G(y) = \{1 + \exp(-y)\}^{-1}$, it can easily be checked that $\psi(u) = u$. The corresponding test statistic M, defined through $T_1(Y) = n^{-1} \sum_j (R_j/n + 1)$ Sign Y_j and $T_2(Y) = n^{-1} \sum_j x_j(R_j/n + 1)$ Sign Y_j is said to be of the Wilcoxon type. If G is chosen to be the normal distribution function Φ, then $\psi(u) = \Phi^{-1}(\frac{1}{2}u + \frac{1}{2})$ and the corresponding M-statistic is said to be of the Van der Waerden (normal scores) type. On taking G to be the double exponential distribution function, $\psi(u)$ reduces to unity, and the corresponding M-statistic defined through $T_1(Y) = n^{-1} \sum_j$ Sign Y_j, and $T_2(Y) = n^{-1} \sum_j x_j$ Sign Y_j is said to be of the sign type. We remark that the components T_1 and T_2 of M, are familiar for we recognize T_1 as being equivalent to the usual rank score statistic for the one sample problem [8], while T_2 is similar to Hájek's statistic for the test of symmetry [10].

3. Limiting distribution of M under the hypothesis. We note that under H, the joint distribution of T_1 and T_2 is independent of F, but depends only on the function ψ and hence on G, through which ψ is defined. The following theorem gives the limiting null distribution of M.

THEOREM 3.1. *Under the assumptions of Section* 2,

$$\lim P_0(M \leqq y) = P(\chi_2^2 \leqq y)$$

where χ_2^2 denotes the central chi-square random variable, with 2 *degrees of freedom, and P_0 denotes that the probability is computed under $H: \alpha = \beta = 0$.*

PROOF. It suffices to prove that $\mathfrak{L}(n^{\frac{1}{2}}(T_1, T_2) \mid P_0)$ tends to the bivariate normal distribution with covariance matrix $\|\gamma_{kl}\| = \lim \|\gamma_{kl}\|_n$. The idea of the proof is as in [10] to replace T_i, $i = 1, 2$, by sums of independent random variables and apply the central limit theorem.

Now introduce two statistics,

$$S_1^{(0)} = n^{-1} \textstyle\sum_j \psi_n(F^*(|Y_j|)) \text{ Sign } Y_j \tag{3.1}$$

and

$$S_2^{(0)} = n^{-1} \textstyle\sum_j x_j \psi_n(F^*(|Y_j|)) \text{ Sign } Y_j \tag{3.2}$$

where F^* is the distribution function of $|Y_j|$, i.e. $P_0\{|Y_j| \leqq y\} = F^*(y) = 2F(y) - 1$ for $y > 0$. Because the vectors $(R_1, \cdots, R_n)$, $(|Y_1|, \cdots, |Y_n|)$ and (Sign $Y_1, \cdots$, Sign Y_n) are mutually independent, and $E_0(\text{sign } Y_j) = 0$ for all j, we have

$$\begin{aligned} E_0\{n^{\frac{1}{2}}(T_2 - S_2^{(0)})\}^2 &= \text{Var}_0\{n^{\frac{1}{2}}(T_2 - S_2^{(0)})\} \\ &= n^{-1} \textstyle\sum_j x_j^2 E_0[\psi_n(R_j/n + 1) - \psi_n(U_j)]^2 \end{aligned}$$

where $U_j = F^*(|Y_j|)$ are independent random variables uniformly distributed on $(0, 1)$. Clearly

$$E_0[\psi_n(R_j/n + 1) - \psi_n(U_j)]^2 = E_0[\psi_n(R_1/n + 1) - \psi_n(U_1)]^2$$

for every $j = 1, \cdots, n$. Using Lemma 2.1 of [9], and (1.4), it is seen that

$$n^{\frac{1}{2}}(T_i - S_i^{(0)}) \to 0 \quad \text{in} \quad P_0\text{-probability}, \qquad i = 1, 2. \tag{3.3}$$

Let

$$S_1 = n^{-1} \textstyle\sum_j \psi(U_j) \text{ Sign } Y_j \tag{3.4}$$

and

$$S_2 = n^{-1} \textstyle\sum_j x_j \psi(U_j) \text{ Sign } Y_j, \tag{3.5}$$

where $U_j = F^*(|Y_j|)$. Due to the independence of $|Y_j|$ and Sign Y_j, we have

$$\begin{aligned} E_0[n^{\frac{1}{2}}(S_2^{(0)} - S_2)]^2 &= \text{Var}_0\{n^{\frac{1}{2}}(S_2^{(0)} - S_2)\} \\ &= n^{-1} \textstyle\sum_j x_j^2 E_0[\psi_n(F^*(|Y_j|)) - \psi(F^*(|Y_j|))]^2 \\ &= n^{-\frac{1}{2}} \textstyle\sum_j x_j^2 \int_0^1 [\psi_n(u) - \psi(u)]^2 \, du \to 0. \end{aligned}$$

Hence

$$n^{\frac{1}{2}}(S_i^{(0)} - S_i) \to 0 \quad \text{in} \quad P_0\text{-probability}, \qquad i = 1, 2. \tag{3.6}$$

On combining (3.3) and (3.6), we obtain that for $i = 1, 2$,

$$n^{\frac{1}{2}}(T_i - S_i) \to 0 \quad \text{in} \quad P_0\text{-probability}. \tag{3.7}$$

Because the $\psi(U_j), j = 1, \cdots, n$, are independent identically distributed random variables, which are also independent of sign Y_j, the central limit theorem gives immediately

$$\mathcal{L}(n^{\frac{1}{2}}T_1 \mid P_0) \to N(0, \gamma_{11}) = N(0, \gamma_1^2). \tag{3.8}$$

On the other hand, it is clear that the general central limit theorem (Loève [17], Theorem B, p. 280) applies to $n^{\frac{1}{2}}S_2$, and on account of (3.7), we also obtain that,

$$\mathcal{L}(n^{\frac{1}{2}}T_2 \mid P_0) \to N(0, \gamma_{22}) = N(0, \gamma_2^2) \tag{3.9}$$

where $\gamma_{22} = \lim \gamma_{22,n}$ defined in (2.5). To prove the joint asymptotic normality of $n^{\frac{1}{2}}T_1$ and $n^{\frac{1}{2}}T_2$, it is sufficient, because of (3.7) and a well known theorem (Cramér [4], p, 299), to establish the joint asymptotic normality of $n^{\frac{1}{2}}S_1$ and $n^{\frac{1}{2}}S_2$.

But for any arbitrary λ_1 and λ_2, $n^{\frac{1}{2}}(\lambda_1 S_1 + \lambda_2 S_2) = n^{-\frac{1}{2}} \sum_j x_j^* \psi(U_j)$ Sign Y_j where x_j^* satisfy conditions (1.3) and (1.4). It follows that

$$\mathcal{L}(n^{\frac{1}{2}}(\lambda_1 T_1 + \lambda_2 T_2) \mid P_0)$$

tends to a normal distribution, and hence that $\mathcal{L}(n^{\frac{1}{2}}(T_1, T_2) \mid P_0)$ tends to the bivariate normal distribution with zero means, and covariance matrix $\|\gamma_{kl}\|$. The assertion of the theorem follows by the usual transformation, see for example Sverdrup [21].

As a direct consequence of Theorem 3.1, it follows that the critical function

$$\begin{aligned} \varphi(M) &= 1 \quad \text{if} \quad M > \chi^2_{2,\epsilon} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

where $\chi^2_{2,\epsilon}$ is the $100(1 - \epsilon)\%$ point of the χ^2 distribution with 2 degrees of freedom, provides an asymptotic level ϵ test of H.

4. Limiting distribution of M under near alternatives. In order to determine the efficiency of the class $M(\psi)$ of test statistics, it is necessary to find its distribution under a sequence of alternatives tending to the hypothesis, at a suitable rate. In this section we discuss the distribution of M for alternatives tending to H at the rate of $n^{-\frac{1}{2}}$; and for this, we shall follow the method based on Le Cam's contiguity lemma [10] and [15]. First we give the set-up under which the contiguity principle is applicable.

Let $P_n = \prod_{j=1}^n P_j$ be the distributions of $(Y_1, \cdots, Y_n)$ under a sequence K_n of alternatives defined by

$$K_n : \alpha_n = an^{-\frac{1}{2}}; \qquad \beta_n = bn^{-\frac{1}{2}} \tag{4.1}$$

and let

$$r_j = p_j(Y_j)/p_0(Y_j) \qquad \text{for} \quad p_0(y) > 0 \tag{4.2}$$

where $p_j, j = 1, \cdots, n$, are densities corresponding to P_j, and p_0 corresponds to the distribution P_0, under the hypothesis.

Define

$$W_n = 2 \textstyle\sum_j (r_j^{\frac{1}{2}} - 1). \tag{4.3}$$

With the above notation, we shall prove the following:

LEMMA 4.1. *If* (1.2), (1.3) *and* (1.4) *hold, and P_n are the distributions under K_n given in* (4.1), *then P_n are contiguous to P_0.*

PROOF. The lemma will be proved if we show that

(i) $\lim \max_j P_0(|r_j - 1| > \epsilon) = 0$ for every $\epsilon > 0$, and

(ii) $\mathcal{L}(W_n \mid P_0) \to N(-\frac{1}{4}\sigma^2, \sigma^2)$.

For (i), write $r_j = f(Y_j - h_j)/f(Y_j)$ where $h_j = n^{-\frac{1}{2}}(a + bx_j)$ and we may assume $h_j \neq 0$. Again the dependence on n of h_j is suppressed. Then

$$\begin{aligned}\max_j P_0(|r_j - 1| > \epsilon) &\leqq \max_j \epsilon^{-1} E_0 |r_j - 1| \\ &= \max_j \epsilon^{-1} |h_j| \textstyle\int_{\infty}^{\infty} |h_j^{-1}\{f(y - h_j) - f(y)\}| \, dy.\end{aligned}$$

Now

$$|h_j^{-1}\{f(y - h_j) - f(y)\}| \leqq |h_j^{-1}| \textstyle\int_{y-h_j}^{y} |f'(x)| \, dx,$$

and

$$\textstyle\int_{-\infty}^{\infty} |h_j^{-1}\{f(y - h_j) - f(y)\}| \, dy \leqq \int_{-\infty}^{\infty} |f'(y)| \, dy \quad \text{for all} \quad j = 1, \cdots, n.$$

Hence we have that

$$\max_j P_0\{|r_j - 1| > \epsilon\} \leqq \max_j \epsilon^{-1} |h_j| \textstyle\int_{-\infty}^{\infty} |f'(y)| \, dy \to 0.$$

To prove (ii), define

$$S_n^* = n^{-1} \textstyle\sum_j (a + bx_j) f'(Y_j)/f(Y_j), \tag{4.4}$$

where $f'(x) = d/dxf(x)$, and write W_n in the form

$$W_n = 2 \sum_j [\{s(Y_j - h_j)/s(Y_j)\} - 1], \tag{4.5}$$

where $s(x) = f^{\frac{1}{2}}(x)$. It can be seen that

$$E_0 W_n \sim -\sigma_n^2/4 \tag{4.6}$$

and that

$$\mathrm{Var}_0 \, (n^{\frac{1}{2}} S_n^*) = \sigma_n^2 \tag{4.7}$$

where $\sigma_n^2 = \sum_j h_j^2 \int_{-\infty}^{\infty} (f'(y)/f(y))^2 f(y) \, dy = \sum_j h_j^2 \int_0^1 \varphi^2(u) \, du$. Furthermore,

$$\begin{aligned} (4.8) \quad & E_0(W_n - E_0 W_n - n^{\frac{1}{2}} S_n^*) \\ & \leqq 4 \sum_j h_j^2 \int_{-\infty}^{\infty} [h_j^{-1}\{s(y - h_j) - s(y)\} - s'(y)]^{12} \, dy \end{aligned}$$

and the right hand side of (4.8) tends to zero by (1.4) and Lemma 4.3 of [10]. The limiting distribution of $n^{\frac{1}{2}} S_n^*$ is of course normal with zero mean and variance $\sigma^2 = \lim \sigma_n^2$. It follows from (4.6) and (4.8) that $\mathcal{L}(W_n \mid P_0) \to N(-\frac{1}{4}\sigma^2, \sigma^2)$, and the proof of the lemma is therefore complete.

We shall now apply the contiguity principle to obtain the limit distribution of (T_1, T_2) under the sequence K_n of alternatives. The main result of this section is the following:

Theorem 4.1. *Under the assumptions of Lemma* 4.1, $\mathcal{L}(n^{\frac{1}{2}}(T_1, T_2) \mid P_n)$ *tends to the bivariate normal distribution with mean vector* (μ_1, μ_2) *and covariance matrix* $\|\gamma_{kl}\|$, *where the means are defined by:*

$$\begin{aligned} (4.9) \qquad \mu_1 &= \lim n^{-1} \sum_j (a + bx_j) \int_0^1 \psi(u)\varphi(u) \, du, \\ \mu_2 &= \lim n^{-1} \sum_j x_j(a + bx_j) \int_0^1 \psi(u)\varphi(u) \, du \end{aligned}$$

and the functions ψ *and* φ *are defined in* (1.6) and (1.7) *respectively.*

Proof. We shall first prove that $\mathcal{L}(n^{\frac{1}{2}} T_i \mid P_n) \to N(\mu_i, \gamma_i^2)$, $i = 1, 2$, then show that (T_1, T_2) has a joint asymptotic normal distribution under K_n. By the contiguity principle the first one will follow if we prove that

(a) $\mathcal{L}(n^{\frac{1}{2}} T_i, W_n \mid P_0)$, $i = 1, 2$, tend to bivariate normal distributions with certain correlation coefficients ρ_i, and W_n is as defined in (4.3);

(b) $\mu_i = \rho_i \sigma \gamma_i$, where σ^2 is the limit of σ_n^2 defined in (4.7).

By (3.7) and (4.8) it is sufficient to consider $\mathcal{L}(n^{\frac{1}{2}}(S_i, S_n^*) \mid P_0)$. Also due to the symmetry of the density function f, we may write S_n^* of (4.4), as

$$\begin{aligned} S_n^* &= n^{-1} \sum_j (a + bx_j) f^{*\prime}(|Y_j|)/f^*(|Y_j|) \operatorname{Sign} Y_j \\ &= n^{-1} \sum_j (a + bx_j)\varphi(U_j) \operatorname{Sign} Y_j \end{aligned}$$

where f^* is the density function corresponding to the distribution function F^* of $|Y_j|$. Now

$$n(S_2, S_n^*) = [\sum_j x_j \psi(U_j), \sum_j (a + bx_j)\varphi(U_j)] \operatorname{Sign} Y_j,$$

so that

$$\mathrm{Cov}_0\,[n^{\frac{1}{2}}(S_2\,,\,S_n^{\,*})] = n^{-1}\sum_j x_j(a+bx_j)\int_0^1\psi(u)\varphi(u)\,du$$

which, by (1.4) tends to a finite limit. Furthermore it can be seen that under conditions (1.3) and (1.4) the bivariate central limit theorem (Cramér [5], p. 114, Theorem 21a) applies to $n^{\frac{1}{2}}(S_i\,,\,S_n^{\,*})$, $i = 1, 2$, giving that $\mathcal{L}(n^{\frac{1}{2}}(S_i\,,\,S_n^{\,*})\,|\,P_0)$ tend to the bivariate normal distributions with correlation coefficients ρ_i where

$$\rho_1 = \lim\,\{n^{-1}\sum_j(a+bx_j)\int_0^1\psi(u)\varphi(u)\,du \cdot[n^{-1}\sum_j(a+bx_j)^2\int_0^1\psi^2(u)\,du\int_0^1\varphi(u)\,du]^{-\frac{1}{2}}\},$$

$$\rho_2 = \lim\,\{n^{-1}\sum_j x_j(a+bx_j)\int_0^1\psi(u)\varphi(u)\,du \cdot[n^{-1}\sum_j(a+bx_j)^2\int_0^1\psi^2(\mathrm{u})\,du\int_0^1\varphi^2(u)\,du]^{-1}\}.$$

Finally it is immediate that $\mu_i = \rho_i\sigma\gamma_i$, hence we have proved that $\mathcal{L}(n^{\frac{1}{2}}T_i\,|\,P_n) \rightarrow N(\mu_i\,,\,\gamma_i^2)$, $i = 1, 2$.

For the joint asymptotic distribution of T_1 and T_2 under K_n, note that

$$(4.10)\qquad \lim\mathcal{L}(n^{\frac{1}{2}}(\lambda_1T_1+\lambda_2T_2),\,W_n\,|\,P_0) = \lim\mathcal{L}(n^{\frac{1}{2}}(\lambda_1S_1+\lambda_2S_2),\,S_n^{\,*}\,|\,P_0)$$

which by Theorem 21a of [5], is a bivariate normal. We have also shown that $\mathcal{L}(n^{\frac{1}{2}}(\lambda_1S_1+\lambda_2S_2)\,|\,P_0)$ is asymptotically normal. This fact, together with (4.10) implies that $\mathcal{L}(n^{\frac{1}{2}}(\lambda_1S_1+\lambda_2S_2)\,|\,P_n)$ is also asymptotically normal. Since λ_1 and λ_2 are arbitrary, it follows that $n^{\frac{1}{2}}(T_1\,,\,T_2)$ has a limiting bivariate normal under K_n. This completes the proof of Theorem 4.1.

We are now in the position to give the limit distribution of M, and this is stated in the following:

THEOREM 4.2. *Under the assumptions of Lemma* 4.1, $\mathcal{L}(M\,|\,P_n) \rightarrow \mathcal{L}(\chi_2^{\,2}(\Delta^2))$ *where* $\chi_2^{\,2}(\Delta^2)$ *denotes the non-central chi-square random variable with* 2 *degrees of freedom and non-centrality parameter* Δ^2 *given by*

$$(4.11)\qquad \Delta^2 = \lim\,(a^2+2ab\bar{x}_n+b^2n^{-1}\textstyle\sum_j x_j^2)(\int\psi\varphi)^2/\int\psi^2.$$

PROOF. The proof follows directly from Theorem 4.1, and Δ^2 is obtained by straightforward computation.

5. Asymptotic efficiency of M-tests. We employ a measure of relative efficiency of two test statistics due to Pitman (see Noether [19]). If under the same sequence of alternatives, the two test statistics have non-central chi square limit distributions, with the same degrees of freedom, it has been shown by Andrews [2] and Hannan [11], that their relative asymptotic efficiency is given by the ratio of their non-centrality parameters. To find the asymptotic efficiency of the M-tests relative to the classical F-test, we need therefore, to compute the non-centrality parameter of the latter.

The classical test statistic $\tilde{M}$ for H is based on a quadratic function in the least squares estimates $\tilde{\alpha}$ and $\tilde{\beta}$ of α and β. H is rejected if

$$(5.1)\qquad \tilde{M} = n(\tilde{\alpha},\,\tilde{\beta})\|\tau_{kl}\|_n^{-1}(\tilde{\alpha},\,\tilde{\beta})'$$

is too large, where $\|\tau_{kl}\|_n^{-1}$ is the inverse of $\|\tau_{kl}\|_n$ defined by

$$(5.2)\quad \tau_{11} = \tau_1^2 = (\textstyle\sum_j x_j^2)/\sum_j(x_j - \bar{x}_n)^2; \qquad \tau_2^2 = \{n^{-1}\sum_j(x_j - \bar{x}_n)^2\}^{-1};$$
$$\tau_{21} = \tau_{12} = -\bar{x}_n/n^{-1}\textstyle\sum_j(x_j - \bar{x}_n)^2.$$

If, as is usual in classical tests, the distribution function F is assumed to be Φ, the normal df, then the estimates $\tilde{\alpha}$ and $\tilde{\beta}$ being linear functions of normal random variables, are themselves normal random variables, so that, under H, $\tilde{M}$ has a central χ^2 distribution with 2 degrees of freedom. And under any given alternative of the form $\alpha = a$, $\beta = b$, $\tilde{M}$ has a non-central χ^2 distribution with 2 degrees of freedom, and non-centrality parameter given by (see e.g. Lehmann [16], p. 284)

$$\tilde{\Delta}_n^2 = n(a + b\bar{x}_n)^2 + b\textstyle\sum_j(x_j - \bar{x}_n)^2.$$

It follows that under the sequence of alternatives K_n, the non-centrality parameter reduces to

$$(5.3)\qquad \tilde{\Delta}^2 = \lim\,(a^2 + 2ab\bar{x}_n + b^2n^{-1}\textstyle\sum_j x_j^2).$$

For any distribution function $F \varepsilon \mathfrak{F}$, the limit distribution of $\tilde{M}$ will still be χ^2, this is a consequence of theorem of Eicker [7], which gives general conditions for the least squares estimates to be asymptotically normally distributed. It can be verified [1] that Eicker's conditions are implied by our general assumptions of Section 1. We summarize the above facts in

LEMMA 5.1 *Under the assumptions of Lemma* 4.1, $\mathfrak{L}(\tilde{M} \mid P_n) \to \mathfrak{L}(\chi_2^2(\tilde{\Delta}^2))$ *where* $\chi_2^2\ (\tilde{\Delta}^2)$ *denotes the non-central chi-square random variable with* 2 *degrees of freedom and non-centrality parameter* $\tilde{\Delta}^2$ *given by* (5.3).

From (4.11) and (5.3), it follows that the asymptotic efficiency of the M-tests relative to the classical $\tilde{M}$-test is given by

$$(\textstyle\int_0^1 \psi(u)\varphi(u)\,du)^2/(\int_0^1 \psi^2(u)\,du).$$

If the common variance of the Y_j is σ_0^2, instead of unity, as we assumed, the efficiency becomes

$$(5.4)\qquad e_{M,\tilde{M}}(\psi) = \sigma_0^2(\textstyle\int_0^1 \psi(u)\varphi(u)\,du)^2/(\int_0^1 \psi^2(u)\,du).$$

On taking $\psi(u) = u$ (Wilcoxon), the efficiency expression on the right hand side of (5.4) reduces, after integration by parts, to

$$(5.5)\qquad e_{Mw.\tilde{M},}(F) = 12\sigma_0^2(\textstyle\int_{-\infty}^{\infty} f^2(y)\,dy)^2,$$

and on choosing $\psi(u) = \Phi^{-1}(\frac{1}{2}u + \frac{1}{2})$ (Van der Waerden) the efficiency simplifies to

$$(5.6)\qquad e_{Mn\cdot sc.\tilde{M},}(F) = \sigma_0^2 \textstyle\int_{-\infty}^{\infty} f^2(y)\,dy/\Phi'(\Phi^{-1}(F(y)))$$

where Φ' is the density of the standard normal distribution. Another special case of interest is obtained by taking $\psi(u) = \text{sign}\,u = 1$. The efficiency, in this case, becomes

$$(5.7)\qquad e_{Ms,\tilde{M}}(F) = 4\sigma_0^2f^2(0).$$

We recognize (5.5) and (5.6) as the efficiencies of the Wilcoxon and the normal scores tests for shift relative to the t-test. Efficiency (5.5) has been studied in detail by Hodges and Lehmann [13], and (5.6) by Chernoff and Savage [3]. The question of choice between M_w and $M_{n \cdot sc}$ as test statistics has been fully discussed in [14]. In general, provided the functions ψ and φ are non-decreasing on $(0, 1)$, the efficiency expression in (5.4) is the same as that of the corresponding rank score tests relative to the student's t-test or the F-test in the one sample [8] or the c-sample [20] problem. This is an immediate consequence of the similarity between the Chernoff-Savage $J(u)$ function used in [20], and Hájek's $\varphi(u)$ and $\psi(u)$ functions.

In [6] Daniels proposed a distribution-free test for H which is related to the Hodges' bivariate sign test [12] for symmetry. It would be of interest to compare Daniels' test and the M-tests with respect to their efficiency behaviour, but such a comparison does not seem to be readily feasible because the asymptotic distribution of Daniels' test under a sequence of alternatives is not known.

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