ABSTRACTS

(Abstracts of paper presented at the Central Regional meeting, Columbus, Ohio, March 23–25, 1967. Additional abstracts appeared in earlier issues.)

34. Generalized multivariate estimators for the mean of a finite population.

PODURI S. R. S. RAO, Information Research Associates, Inc. (Invited)

Just as the ratio estimators are often indicated when the variable y is positively correlated with an auxiliary variable x, so product estimators are indicated when y is negatively correlated with x. For estimating $\bar{Y}$ the following product estimators are considered: $\bar{Y}_1^* = \sum x_i y_i / n\bar{x}$; $\bar{Y}_2^* = \bar{x}\bar{y}/\bar{X}$; $\bar{Y}_3^* = \bar{x}\bar{y}/\bar{X} - (N-n)s_{xy}/Nn\bar{X}$. $\bar{Y}_1^*$ is not a consistent estimator, $\bar{Y}_2^*$ is a biased estimator, and $\bar{Y}_3^*$ is an unbiased estimator. The mean square errors of these estimators are compared when the relation between y and x is of the forms $y = \alpha + \beta x + e$ and $y = \alpha + \beta/x + e$ and when x has a gamma distribution. Similar to the multivariate ratio estimator of Olkin [*Biometrika* **45** (1958), 154–165], multivariate product estimators are formed analogous to the above three estimators and their biases and variances are found. Finally when $x_1, \cdots, x_p$ are positively correlated and $x_{p+1}, \cdots x_q$ are negatively correlated with y, multivariate estimators for $\bar{Y}$ are found and their biases and variances are evaluated. In particular an unbiased multivariate estimator for $\bar{Y}$ is $\tilde{Y} = \tilde{Y}_1 + \tilde{Y}_2$ where

$$\tilde{Y}_1 = \sum W_j r_j \bar{X}_j + [(N-1)n/N(n-1)](\bar{y} - \sum W_j \bar{r}_j \bar{x}_j),$$
$$\tilde{Y}_2 = \sum [W_j/\bar{X}_j][\bar{x}_j\bar{y} - (N-n)(P_j - n\bar{x}_j\bar{y})/Nn(n-1)]$$

with $r_j = y_i/x_{ji}$, $n\bar{r}_j = \sum (y_i/x_{ji})$, $P_\iota = \sum x_{ji}y_\iota$. The optimum weights $\hat{W}_j$ are determined by minimizing $V(\tilde{Y})$. (Received 10 April 1967.)

35. The asymptotic distribution of class of two-sample non-linear rank order statistics in the null case.

SIEGFRIED SCHACH, University of Minnesota. (Invited)

Let $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent samples from the same continuous distribution. Let $h_N(\cdot)$, $N = 1, 2, \cdots$ be a sequence of functions on $[-1, 1]$, symmetric with respect to 0, and periodic with period 1, and let $R_1, R_2, \cdots, R_m$ be the ranks of the X-observations in the combined sample. We find conditions under which a sequence of statistics of the form $T_N = N^{-1} \sum_{i=1}^m \sum_{j=1}^m h_N(R_i - R_j)N^{-1}$ converges in distribution as $N = m + n$ goes to infinity, and we derive the characteristic function of the limiting distribution. If h_N converges to some h with a finite Fourier expansion, the limiting distribution is obtained by writing T_N as a quadratic form in variables $z_i^{(N)}$, where $z_i^{(N)} = 1$ if the ith observation in the ordered combined sample is an X, 0 otherwise, and by diagonalizing the matrix of the quadratic form. Hájek's theorem is used to obtain the limiting distribution of T_N. It is a finite weighted sum of χ^2-variables. If h has an infinite Fourier expansion with an absolutely converging Fourier series, T_N can be represented as a function on the Hilbert H space of real sequences with finite sums of squares. Using Prokhorov's techniques, it is shown that the induced measures on H converge weakly to some Gaussian measure and that T_N has the distribution of an infinite weighted sum of χ^2-variables, where the Fourier coefficients determine the weights. (Received 10 April 1967.)

(*Abstracts of papers presented at the Western Regional meeting, Missoula, Montana, June 15–17, 1967. Additional abstracts appeared in earlier issues.*)

5. Estimation of a probability density function and its derivatives. P. K. BHATTACHARYA, University of Arizona.

$X_1, \cdots, X_n, \cdots$ are independent continuous random variables with pdf g. f is the pdf of $N(0, 1)$, $\{a_n\}$ and $\{k_n\}$ are sequences of positive numbers which tend to 0 and ∞ respectively as $n \to \infty$, I_n is the indicator function of the interval $[-k_n, k_n]$. Define $g_n(y) = \sum_{i=1}^n f((y - X_i)/a_n)/na_n$ and $g_n^*(y) = \sum_{i=1}^n I_n(X_i)f((y - X_i)/a_n)/a_n \sum_{i=1}^n I_n(X_i)$ when $\sum_{i=1}^n I_n(X_i) > 0$ and $g_n^*(y) = f(y)$ when $\sum_{i=1}^n I_n(X_i) = 0$. The following properties of g_n and g_n^* are proved: (1) If g and its first $r + 1$ derivatives are bounded, then with $k_n = n^u$, $u > 0$, $a_n = n^{-1/(2r+4)}$ and $0 < c < 1/(2r + 4)$, $\sup_{|y| \leq k_n} n^c|g_n^{(r)}(y) - g^{(r)}(y)|$ converges to 0 a.s. 2) If g and g' are bounded, then with $k_n = n^u$, $u > 0$ and $a_n = n^{-c}$, $0 < c < \frac{1}{2}$, g_n^* converges uniformly to g a.s. (Received 24 April 1967.)

6. On a multivariate extension of the Bradley-Terry model for paired comparisons. ROGER R. DAVIDSON, University of Victoria. (Introduced by Robert E. Odeh.)

The Bradley-Terry model for paired comparisons is extended to a multivariate model which yields, for each pair of treatments, the probability associated with each of the 2^p preference patterns that may arise when responses are obtained on p attributes. Denote by $p(\mathbf{s} \mid i, j)$ the probability of obtaining the preference vector $\mathbf{s} = ({}_1s, \cdots, {}_ps)$ whose components ${}_\alpha s$ indicate which treatment in the pair (i, j) is preferred on attribute α, that is, ${}_\alpha s = i$ or j, $\alpha = 1, \cdots, p$. This set of $\binom{t}{2} \cdot 2^p$ probabilities are formulated in terms of a smaller set of parameters for treatment attributes and for associations between attributes. The model adopted, like the Bradley-Terry model, is characterized in a number of different ways: (i) a joint distribution for response to dichotomous variables; (ii) a form of a multivariate logistic distribution; (iii) a multivariate distribution based on modėl suggested by Lehmann. (Received 25 April 1967.)

7. Markovianization of a single state of stationary process (preliminary report). S. W. DHARMADHIKARI, University of Arizona.

Let $\{Y_n\}$ be a stationary process with a finite state-space J. Let δ be a fixed state of J and let $J' = J - \{\delta\}$. We use the numbers $n(\epsilon)$ defined by Gilbert [*Ann. Math. Statist.* **30** (1959) 688–697]. Assume that $n(\delta) < \infty$. Under some further conditions which are natural generalizations of the conditions given previously by this author [*Ann. Math. Statist.* **36** (1965) 524–528] it is shown that $\{Y_n\}$ can be expressed as a function f of another stationary process with state-space $\{\delta_1, \cdots, \delta_n\} \cup J'$ in such a way that, for $i = 1, 2, \cdots, N$, $f(\delta_i) = \delta$ and δ_i is a Markovian state of $\{X_n\}$. Further questions connected with this functional representation are under investigation. (Received 21 April 1967.)

8. The folded cubic association scheme (preliminary report). PETER W. M. JOHN, University of California, Davis.

In the cubic association scheme for $v = 64$ each variety is denoted by a different triple (z_1, z_2, z_3), $z_i = 1, 2, 3, 4$. Two varieties are ith associates, $i = 1, 2, 3$, if their representations have $(3-i)$ coordinates identical. In the folded scheme z_1, z_2, z_3 are elements of $GF(2^2)$ and a fourth coordinate $z_4 = z_1 + z_2 + z_3$ is added. This gives a partially balanced scheme with four associate classes; (z_1, z_2, z_3, z_4) and (z_1', z_2', z_3', z_4') are fourth associates if $z_1 - z_1' = z_2 - z_2' = z_3 - z_3'$; otherwise they are ith associates if $(3-i)$ of the four coordinates are identical. Then $n_1 = 18$, $n_2 = 24$, $n_3 = 18$, $n_4 = 3$. If the first, second and fourth associates are pooled the scheme reduces to a PBIB(2) scheme with $v = 64$, $n_1 = 45$, $n_2 = 18$, $p_{12}^1 = 6$, $p_{12}^2 = 15$. (Received 28 May 1967.)

9. Estimation of the parameters of the power-function distribution. H. J. MALIK, Canadian Services College, Victoria.

Estimation of the parameter k of the power-function distribution, $k\xi^{-k}x^{k-1}$ (for $0 \leqq x \leqq \xi$, $\xi > 0$, $k \geqq 1$) and zero elsewhere, is considered. Exact distribution of the maximum likelihood estimator $\hat{k}$ is derived. It is shown that the sample geometric mean g is sufficient for k when ξ is known, and $(Y_n, \sum_{i=1}^{n} \log (Y_n/Y_1))$ is joint set of sufficient statistics for (ξ, k), when both are unknown, and where $(Y_1, Y_2, \cdots, Y_n)$ are the order statistics of the random sample $(X_1, X_2, \cdots, X_n)$. It is shown that the statistics $2nk \log (\xi/g)$, $2nk \log (\xi/Y_n)$ and $2nk \log (Y_n/g)$ are chi-square distributed with $2n$, $2n - 2$ and 2 degrees of freedom also following the addition law. Results reveal the unique position of the geometric mean as a measure of location in the power-function distributions. Some tests associated with the parameter k are also suggested. (Received 17 April 1967.)

10. Exact moments of order statistics from a power-function distribution. H. J. MALIK, Canadian Services College, Victoria. (By title)

Let $X_{1,N} < X_{2,N} < \cdots < X_{N,N}$ denote a set of order statistics in a random sample of N independent and identically distributed random variables $X_1, X_2, \cdots, X_N$ from a distribution having a probability density function $k\xi^{-k}x^{k-1}$ (for $0 \leqq x \leqq \xi$, $\xi > 0$, $k \geqq 1$) and zero elsewhere, which is called a power-function distribution. The characteristic function of the kth order statistic is obtained and moments about the origin of the kth order statistic are expressed in terms of gamma function. An exact expression for the covariance of any two order statistics $X_{i,N} < X_{j,N}$ is obtained. Various recurrence relations between the expected values of order statistics are also given. Certain results on order statistics obtained for rectangular and triangular distributions by Sarhan [*Ann. Math. Statist.* **26** 505–511] and Sarhan and Greenberg [*J. Roy. Statist. Soc. Ser. B* **21** 356–363] are special cases of results derived in this paper. (Received 17 April 1967.)

11. Exact distribution of the quotient of independent generalized gamma variables. H. J. MALIK, Canadian Services College, Victoria. (By title)

Let X_1 and X_2 be independently distributed with respective frequency functions $f(x_1; a_1, d_1, p)[x_1 > 0, a_1, d_1, p > 0]$ and $f(x_2; a_2, d_2, p)$ $[x_2 > 0, a_2, d_2, p > 0]$, where $f(x; a, d, p)$ $[x > 0, a, d, p > 0]$ is Stacy's [*Ann. Math. Statist.* **33** 1187–1192] generalization of the gamma distribution. The familiar gamma, chi, chi-squared, exponential and Weibull variates are special cases, as are certain functions of normal variates. This note derives the exact distribution of the quotient $W = X_1/X_2$. (Received 17 April 1967.)

12. A moment-inequality and applications (preliminary report). GOVIND MUDHOLKAR, University of Rochester.

Suppose that the probability density function $f(\mathbf{x}, \theta) = f(x_1 - \theta_1, x_2 - \theta_2, \cdots, x_n - \theta_n)$ of n jointly distributed random variables $X_1, X_2, \cdots, X_n$ is unimodal. Then the purpose of this report is to establish inequalities of the form $E_{\theta}(T^r) \geqq E_{\theta^*}(T^r)$ for certain functions $T = T(X_1, X_2, \cdots, X_n)$ of these random variables and discuss some applications. A particular case, where $X_1, X_2, \cdots, X_n$ are independently distributed with respective densities $f(x_i - \theta_i)$, $i = 1, 2, \cdots, n$, is investigated in greater detail; and conditions under which the joint density is unimodal are obtained. (Received 25 April 1967.)

13. A model for controlling a lethal growth process. MARCEL F. NEUTS, Purdue University. (By title)

A host carries a lethal growth, containing N particles initially, which increases according to a Yule process. We may keep the process under control by applying periodic radio-chemotherapy, but the dosage administered is limited by the toxicity of the treatments. We assume that the host has a mortality rate dependent on the size of the growth and on the number and intensity of the treatments already received. This paper discusses the explicit time-dependent behavior of the growth process and shows how the intensity and spacing of treatments may be chosen so as to maximize the survival probability of the host in $[0, T]$ or his expected lifetime. Explicit numerical solutions will also be reported on. (Received 12 May 1967.)

14. Weak convergence of a 2-sample empirical process; and a new approach to Chernoff-Savage theorems. RON PYKE and GALEN SHORACK, University of Washington.

Let $X_1, \cdots, X_n$ and $Y_1, \cdots, Y_m$ be independent random samples from continuous df's F and G respectively. Let $N = m + n, \lambda_N = m/N$ and let F_m, G_n and H_N be the empirical df's of the X_i's, the Y_i's and the combined sample respectively. Let $K_N = F_n(H_N^{-1})$ and $K = F(H^{-1})$ where $H = \lambda_N F + (1 - \lambda_N)G$. Since $K_N(j/N)$, $j = 1, \cdots, N$, equals the proportion of X_i's less than or equal to the jth order statistic of the combined sample, it is natural to study the 2-sample empirical process $\{L_N(t) \mid 0 \leqq t \leqq 1\}$ defined by $L_N(t) = N^{\frac{1}{2}} \cdot [K_N(t) - K(t)]$. A basic identity relating L_N to a (random) linear combination of the two 1-sample empirical processes is given. Statistics of the type studied by Chernoff and Savage [*Ann. Math. Statist.* **29** (1958) 972–994] can then be viewed as functionals on the L_N-process after an appropriate summation by parts. The asymptotic normality of these statistics is then established; the Chernoff-Savage condition on the growth of a function J and its first two derivatives is replaced by the requirement that there exists a non-negative function r on $[0, 1]$ non-decreasing (non-increasing) on $[0, \frac{1}{2}]$ ($[\frac{1}{2}, 1]$) for which $\int_0^1 [t(1 - t)]^{-\frac{1}{2}}[r(t)]^{-1}\, dt < \infty$ and $\int_0^1 r\,d|\nu| < \infty$ where $|\nu|$ is a total variation measure corresponding to J. The weak convergence of the L_N-process is also established under mild restrictions on F and G. Analogous results are obtained for the c-sample problem. (Received 17 April 1967.)

15. A Chernoff-Savage theorem for random sample sizes. RON PYKE and GALEN SHORACK, University of Washington.

A theorem in the spirit of Chernoff and Savage [*Ann. Math. Statist.* **29** (1958) 972–994] on the asymptotic normality of 2-sample linear rank statistic is established for random sample sizes under mild conditions on the underlying df's of the two samples. The proof follows the Pyke and Shorack (see preceding abstract) approach to the fixed sample size problem in which the statistics are viewed as functionals defined on stochastic processes. The processes are also shown to converge weakly to a limiting process. The analogous results are given for the case of c-sample statistics. (Received 25 April 1967.)

16. Moments, cumulants and FORMAC. JOHN S. WHITE, General Motors Corporation.

This paper demonstrates the use of the IBM FORMAC (FOrmula MAnipulation Compiler) programming system as a tool for symbolic manipulation of complicated mathematical expressions. One of the problems considered is that of expressing moments in terms of cumulants and vice versa. If $M(t) = E(\exp(Xt)) = 1 + m_1 + m_2t^2/2 + m_3t^3/3! + \cdots$

is the moment generating function of a random variable X then $K(t) = k_1 + k_2t^2/2 + k_3t^3/3! + \cdots = \log (M(t))$ is its cumulant generating function. A FORMAC program whose output gives symbolic expressions for m_j in terms of k_i and k_j in terms of m_i is exhibited. Tables of m_j and k_j for $j \leqq 15$ are included. The bivariate problem with $M(s, t) = \sum m_{ij} s^it^j/(i!j!)$, $K(s, t) = \log M(s, t) = \sum k_{ij}s^it^j/(i!j!)$ is also considered. (Received 25 April 1967.)

17. On finite mixtures. Sidney J. Yakowitz, University of Arizona.

This paper continues a study initiated by H. Teicher [Identifiability of finite mixtures, *Ann. Math. Statist.* 3 (1963) 1265–1269]. Teicher (1963) reveals a useful sufficiency condition that finite mixtures of a family $\mathfrak{F}$ of cdf's be identifiable. We reveal a simple necessary and sufficient condition that finite mixtures of an arbitrary family $\mathfrak{F}$ of cdf's is identifiable, namely that $\mathfrak{F}$ be linearly independent in $\langle\mathfrak{F}\rangle$, the vector space which is the span of $\mathfrak{F}$ over the real numbers. An easy consequence is that the set of finite mixtures of $\mathfrak{F}$ is identifiable if and only if the image of $\mathfrak{F}$ is linearly independent in any vector-isomorphic image of $\langle\mathfrak{F}\rangle$. Further results on identifiability included in our paper are that sets of finite mixtures of $\mathfrak{F}$ are identifiable if $\mathfrak{F}$ is any of the following: the n-dimensional exponential family, the n-dimensional Gaussian family, the union of the last two families, the Cauchy family, the non-degenerate negative binomial family, and the translation parameter family generated by any distribution function. (Received 31 March 1967.)

(Abstracts of papers to be presented at the Annual meeting, Washington, D. C., December 27–30, 1967. Additional abstracts appeared in the June issue and will appear in future issues.)

3. Equivalent strongly regular graphs and partially balanced association schemes. R. C. Bose, University of North Carolina.

Given a partially balanced two class association scheme or the corresponding strongly regular graph G with parameters $n_1 = r(k-1)$, $p_{11}^1 = (r-1)(t-1) + k - 2$, $p_{11}^2 = rt$ (*Pacific J. Math.* 13 (1963) 319–418; *Ann. Math. Statist.* 30 (1959) 21–38), it is shown that if (i) $2(p_{11}^1 + p_{11}^2) = 3n_1 - n_2 - 1$ or equivalently either $k = 2t$ or $k - 1 - 2t = t/(r-1)$ and (ii) the vertices of G can be partitioned into the sets V_1 and V_2 such that each vertex in V_1 is adjacent to exactly half the vertices in V_2 and vice versa, then another strongly regular graph G^* (or the corresponding partially balanced association scheme) with the same parameters as G can be obtained by the following process: Delete all edges of G connecting a vertex in V_1 to an adjacent vertex in V_2, and connect each vertex in V_1 with all non-adjacent vertices in V_2 by new edges. G^* will be defined to be equivalent to G, but may be non-isomorphic to G. This generalizes a result of Seidel who considered the special cases $n_1 = 12$, $p_{11}^1 = 6$, $p_{11}^2 = 4$ and $n_1 = 6$, $p_{11}^1 = 2$, $p_{11}^2 = 2$, obtaining the exceptional graphs of Hoffman, Chang and Shrikhande (*IBM J.* 4 (1960) 487–496; *Science Record. Math. New Series.* 4 (1960) 12–18; *Ann. Math. Statist.* 30 (1959) 781–798, *Sankhyā* 23 (1961) 115–116). By this method non-isomorphic association schemes or strongly regular graphs with the same parameters can be obtained from a Latin square scheme with characteristics $(r, 2r)$, SLB scheme with characteristics $(r, 2r)$ or scheme $NL_g(2g)$ (*Ann. Math. Statist.* 38 (1967) 571–581). (Received 17 May 1967.)

4. Bayesian estimation of regression parameters for a class of symmetric distributions. George C. Tiao and David R. Lund, University of Wisconsin and Wisconsin State University, Eau Claire.

In this paper we consider the linear model $y = X\theta + \mathbf{e}$ where $\mathbf{y}$ is an $n \times 1$ vector of observations, X is an $n \times k$ matrix of fixed elements with rows x_u', θ is a $k \times 1$ vector of regression parameters, and $\mathbf{e}$ is an $n \times 1$ error vector. We assume that the components of $\mathbf{e}$ are independently drawn from a member of the class of distributions $p(e \mid \sigma, \beta) = K \cdot \exp\{-\frac{1}{2}|e/\sigma|^{2/(1+\beta)}$ for $(-1 < \beta \leqq 1)$. For a given value of β and locally uniform priors for θ and $\ln \sigma$, the posterior distribution of θ is $p(\theta \mid \mathrm{y}, \beta) \propto [M(\theta)]^{-n(1+\beta)/2}$ where $M(\theta) = \sum_{u=1}^{n} |y_u - \mathbf{x}_u'\theta|^{2/(1+\beta)}$. Properties of this distribution are discussed. In particular, methods are given for approximating the distribution. Numerical examples are given for the special case of comparing two means and for the simple regression model $y_u = \theta_1 + x_u\theta_2 + e_u$ where $x_u = \pm 1$. (Received 15 May 1967.)

(Abstracts of papers not connected with any meeting of the Institute.)

1. Existence of optimal stopping rules for rewards of the form $c_n|S_n|^\alpha$ (preliminary report). PAUL FEDER and GORDON SIMONS, Stanford University.

Let $X_1, X_2, \cdots$ be a sequence of independently and identically distributed random variables with mean zero and finite second moment. Let $S_n = \sum_1^n X_i$. Let $c_n \geqq 0$ be a sequence of constants satisfying the sole condition $0 \leqq \limsup_{n\to\infty} n^\alpha c_n < \infty$ for some given α, $0 < \alpha \leqq 2$. Let t denote a stopping variable. A stopping variable exists which maximizes $Ec_t|S_t|^\alpha$. In addition, when $\alpha = 1$, a stopping variable exists which maximizes Ec_tS_t. The so-called "functional equation rule" produces one such (optimal) stopping variable. These results constitute a generalization of work done by H. Teicher and J. Wolfowitz appearing in *Wahrscheinlichkeitstheorie* (1966). (Received 12 April 1967.)

2. On the distribution of the number of conceptions. S. N. SINGH and K. B. PATHAK, Banaras Hindu University.

A probability distribution for the number of conceptions during the period $(0, T)$ has been derived and applied to two examples. If X denotes the number of conceptions during the period $(0, T)$, then its distribution function is given by $G(x, T) = 1 - \alpha + \alpha\beta F(x, T) + [\alpha(1-\beta)/h] \int_0^s F(x, T-l)\, dl + [\alpha(1-\beta)/h] \int_s^h F(x-1, T-l)\, dl$ where

$$F(r, T) = \sum_{x=0}^{r} [\sum_{m=0}^{x} e^{-r(T-xh)}[r(T-xh)]^m/m! - \sum_{m=0}^{x-1} \{[r(T-(x-1)\cdot h)]^m/m!\}\, e^{r(T-(x-1)\cdot h)}]$$

and s, h, α and β, are, respectively, the postpartum amenorrhea period, rest period, the proportion of the fecund couples and the proportion of the fecund couples exposed to the risk of conception at the start of the observational period. (Received 3 April 1967.)

3. An estimator using auxiliary information in sample surveys. SURENDRA K. SRIVASTAVA, Lucknow University.

For estimating the mean $\bar{Y}$ of a character y of a finite population with the help of information on an auxiliary character x whose population mean $\bar{X}$ is known, the estimator $\bar{y} = \bar{y}(\bar{x}/\bar{X})^\alpha$ is suggested where α is set equal to a good guess of $-\rho C_y/C_x$, C_y, C_x being the coefficients of variations of y and x respectively and ρ the correlation coefficient between them. The variance of the estimator $\bar{y}$ decreases as the value of $|\alpha - \text{opt } \alpha|$ decreases and is minimum when $|\alpha - \text{opt } \alpha| = 0$, where opt $\alpha = -\rho C_y/C_x$. The variance of $\bar{y}$ remains smaller than the variance of the ratio estimator as long as $|\alpha - \text{opt } \alpha| < |1 - \rho C_y/C_x|$ and that of the product estimator as long as $|\alpha - \text{opt } \alpha| < |1 + \rho C_y/C_x|$. In most cases the absolute bias of $\bar{y}$ remains smaller than that of the ratio or product estimators, that would have been used. The bias and variance expressions used are up to the order of n^{-1} only. If α is chosen equal to $(1 - 2\rho C_y/C_x)$, the bias of $\bar{y}$ up to the order of n^{-1} becomes zero. (Received 10 April 1967.)

4. A note on Olkin's multivariate ratio estimator. SURENDRA K. SRIVASTAVA, Lucknow University.

In the multivariate ratio estimator for the mean of y based on p auxiliary variables $(x_1, \cdots, x_p)$, $\bar{Y}^*_{MR} = W_1 \bar{Y}^*_{R_1} + \cdots + W_p \bar{Y}^*_{R_p}$, $(W_1 + \cdots + W_p = 1)$, given by Olkin [*Biometrika* **45** (1958) 154–165], where $\bar{Y}^*_{R_i}$ is the simple ratio estimator based on the auxiliary variable x_i only, the optimum weights are determined by minimizing the variance of $\bar{Y}^*_{MR}$, and in practice the estimates of these weights from the sample drawn are used in $\bar{Y}^*_{MR}$ [Cochran (1963) 185, *Sampling Techniques, Second Edition*, Wiley, New York]. With the estimated weights, the variance of $\bar{Y}^*_{MR}$ and its estimate is not known and a survey statistician will hesitate to use such as estimator. In many cases these weights can be determined from past data or experience. It has been shown that moderate departures of used weights from optimum weights does not result in any considerable increase in the optimum variance of $\bar{Y}^*_{MR}$. In case $p = 2$, $V(\bar{Y}^*_{MR}) \leqq V(\bar{Y}^*_{R_1})$ if and only if W_2 lies between zero and twice optimum W_2. A comparison of multivariate ratio, univariate ratio, and mean per unit estimators has also been made considering a simple cost function for the survey. (Received 10 April 1967.)

5. On attachment models for virus and bacteriophage. R. C. SRIVASTAVA, Ohio State University.

Let n_{00} be the number of A-particles and v_{00} be the number of B-particles. Suppose in a suitable medium B-particles attach themselves to A-particles and also let $m = V_{00}/n_{00}$ be the multiplicity of B-particles and r the saturation capacity of an A-particle. Further let $n_i(t)(i = 0, 1, \cdots, n)$ be the number A-particles with exactly i B-particles attached to them and $v_0(t)$ be the number of free B-particles at time t, $0 < t < t_0$. Then under certain conditions, it is shown by Gani [*Biometrics* **21** (1965) 134–139 and Research report, The University of Sheffield] that

$$P(n_0, \cdots, n_r; t) = (n_{00}!/n_0! \cdots n_r!) \prod_{i=1}^{r} (a_{0i}(t))^{n_i}$$

where $P(n_0, \cdots, n_r; t)$ denotes the probability that there are $n_0, \cdots, n_r$ A-particles with $0, \cdots, r$ B-particles attached to them respectively at time $t \geqq 0$ and $a_{0j}(t) = \binom{r}{j} e^{-r\alpha\rho(t)} \cdot (e^{\alpha\rho(t)-1})^j$ where

$$\rho(t) = \int_0^t v_0(t)\, dt = \alpha^{-1} \log((r - m \exp(-b\alpha t))/(r - m)); \; b = n_{00}(r - m).$$

This model has been suggested by Gani to investigate the mechanism of phage attachment to bacteria and also antibody attachment to virus. In this paper we study some of the properties of this model. It is proved that under certain conditions, the joint distributions of $N(t) = (n_0(t), \cdots, n_r(t))'$ tends to a multiple Poisson distribution and certain continuous functionals of $N(t)$ converge to the corresponding functionals of a Gaussian process as n_{00} tends to infinity. (Received 1 May 1967.)

6. A characterization of normality. M. V. TAMHANKAR, University of Poona.

While studying absolutely continuous probability distributions on an n-dimensional Euclidean space it turns out that the concept of independence can be used to characterize a normal distribution. A probability distribution may be specified by giving the joint distribution of the rectangular coordinates, $X_1, \cdots, X_n$ or of the polar coordinates R, $\Theta_1, \cdots, \Theta_{n-1}$. Under some regularity conditions on the n-dimensional distribution the following theorem is proved. THEOREM. $X_1, \cdots, X_n$ *are mutually independent and R is independent of* $(\Theta_1, \cdots, \Theta_{n-1})$ *if and only if* $X_1, \cdots, X_n$ *are independently normally distributed with zero means and the same variances.* It is found, however, that if the conditions are relaxed to some extent then the class of distributions so obtained is wider and the general density function of a distribution in this class is of the form $ax^b \exp(cx^2)$ where a, b, c satisfy suitable conditions. (Received 4 April 1967.)