TESTS FOR THE EQUALITY OF COVARIANCE MATRICES UNDER THE INTRACLASS CORRELATION MODEL

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1. Introduction and summary. In certain multivariate problems involving several populations, the covariance structure of the populations is such that all covariance matrices can be diagonalized simultaneously by a fixed orthogonal transformation. In the transformed problem one has a number of independent univariate populations. Consequently certain hypotheses in the original problem become equivalent to simultaneous hypotheses on these univariate populations in the transformed model. Using this approach we propose a test procedure for testing the hypothesis of equality of covariance matrices against a certain alternative under the intraclass correlation model. The relative advantages of our procedure over that of Srivastava's procedure [6] are also discussed. Finally we indicate how the problem of testing for the equality of covariance matrices under a more general set up can be reduced to a univariate problem.

2. Tests for the homogeneity of covariance matrices. For $i = 1, 2, \cdots, k$, let the columns of X_i form $(n_i + 1)$ independent and identically distributed random vectors, each of them being distributed as a p-variate normal with unknown mean vector $\boldsymbol{\mu}_i$ and covariance matrix $\Sigma_i = \sigma_i^2[(1 - \rho_i)I + \rho_i \mathbf{e}\mathbf{e}']$, where I is the identity matrix and $\mathbf{e}' = (1, \cdots, 1)$. Also let Γ be a $p \times p$ orthogonal matrix with the first row $\mathbf{e}'/p^{\frac{1}{2}}$. Then it is known [3] that $\Gamma\Sigma_i\Gamma' = \text{diag}(\alpha_i, \beta_i, \cdots, \beta_i)$ where $\alpha_i = \sigma_i^2[1 + (p - 1)\rho_i]$ and $\beta_i = \sigma_i^2(1 - \rho_i)$. Further let $W_i = \Gamma S_i \Gamma'$ where $n_i S_i/(n_i + 1)$ is the maximum likelihood estimate of Σ_i. Now if we set $W_i = (w_{ijk})$, $u_i = w_{i11}$ and $v_i = \sum_{j=2}^{p} w_{ijj}$, then it is evident that $n_i u_i/\alpha_i$ and $n_i v_i/\beta_i$ are independently distributed as χ^2 variates with n_i and $m_i = (p - 1)n_i$ degrees of freedom respectively. In the sequel, we let $F_{1ij} = u_i/u_j$ and $F_{2ij} = v_i/v_j$.

Under the orthogonal transformation Γ, the hypotheses regarding the Σ_i translate into hypotheses about the α_i and β_i. For example, the problem of testing the hypothesis $H:\Sigma_1 = \cdots = \Sigma_k$ is equivalent to testing $\alpha_1 = \cdots = \alpha_k$ and $\beta_1 = \cdots = \beta_k$. Motivated by this equivalence we propose, using Roy's union-intersection principle [5], the following procedure for testing H against $A = \bigcup_{i \neq j} [\Sigma_i \neq \Sigma_j]$ when the sample sizes are equal to N.[1]

Accept H if and only if

$$F_{1ij} \leqq a \quad \text{and} \quad F_{2ij} \leqq b \quad \text{for} \quad i \neq j$$

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[1] The authors have considered also procedures for testing H against certain alternatives when the sample sizes are unequal. (See abstract, *Ann. Math. Statist.* **37** (1966) 1428.)

where a and b are chosen such that

$$P[\max_{i \neq j} F_{1ij} \leqq a \mid H] P[\max_{i \neq j} F_{2ij} \leqq b \mid H] = (1 - \alpha).$$

The critical values a and b can be computed using the method given in [2]. For some special values of α, $N - 1$ and $(p - 1)(N - 1)$, the values of a and b can be obtained from [1]. The above test procedure leads to obvious simultaneous confidence intervals.

Ramachandran [4] showed that Hartley's $F_{\max}$ test is unbiased. Using the same procedure it can be seen that the test proposed in this paper for testing H against A is unbiased. It is also of interest to note that in the univariate case, the problem of testing the hypothesis $\alpha_1 = \cdots = \alpha_k$ against the alternative $\bigcup_{i \neq j=1}^{k} [\alpha_i \neq \alpha_j]$ will reduce to Hartley's $F_{\max}$ test for the equality of variances.

Srivastava [6] considered a problem similar to the one considered in this paper. His test procedure is based on the statistic $(\max F_{1ij})(\max F_{2ij})$. The exact evaluation of the critical values associated with the above procedure is not known. A second disadvantage of his test procedure is that it can lead to confidence intervals on parametric functions of the form $\alpha_j \beta_j / \alpha_i \beta_i$ and not on α_j / α_i and β_j / β_i. His procedure is thus not quite useful in drawing inference on sub-hypotheses of the form $\alpha_i = \alpha_j$, $\beta_i = \beta_j$ when H is rejected. Finally the following intuitive argument based on test notions demonstrates yet another undesirable feature of his test. For example if $k = 2$ and both α_2/α_1 and β_2/β_1 are much greater than unity whereas $\alpha_2\beta_2/\alpha_1\beta_1$ is approximately unity, then the hypothesis is very false, but Srivastava's test tends to accept the hypothesis. Consequently his test can be expected to have a poor power function.

3. Test procedures under a more general set up. Let Σ_θ $(\theta \varepsilon H)$ be a family of covariance matrices indexed by the parameter θ $(\theta \varepsilon H)$ and satisfying $\Sigma_{\theta_1} = \Sigma_{\theta_2}$ if and only if $\theta_1 = \theta_2$. Then it can be easily proved that there exists an orthogonal matrix Γ (independent of θ) such that $\Gamma \Sigma_\theta \Gamma'$ is a diagonal matrix for each $\theta \varepsilon H$ if and only if $\Sigma_{\theta_1} \Sigma_{\theta_2} = \Sigma_{\theta_2} \Sigma_{\theta_1}$ for all $\theta_1, \theta_2 \varepsilon H$. This result enables us to characterize all the families of covariance matrices for which test procedures for testing the equality of covariance matrices (or mean vectors) can be obtained in a simple manner by diagonalizing Σ_θ, through an orthogonal transformation Γ, and then constructing test procedures using a method similar to that given in the preceding section. The following are a few families of covariance matrices which satisfy the above mentioned necessary and sufficient condition.

(a) *Intraclass correlation model.* This model has already been dealt with in the preceding section.

(b) *Successive correlation model.* Under this model the covariance matrices are of the following form

$$\Sigma_{(\rho, \sigma^2)} = \sigma^2 \{\rho_{ij}\}_{p \times p}$$

where

$$\rho_{ij} = 1 \quad \text{if} \quad i = j$$

$$= \rho \quad \text{if} \quad |i - j| = 1$$
$$= 0 \quad \text{otherwise.}$$

An orthogonal matrix which diagonalizes $\Sigma_{(\rho,\sigma^2)}$ is given by $\Gamma_1 = \{a_{ij}\}_{p\times p}$ where $a_{ij} = (2/p)^{\frac{1}{2}} \sin(ij\pi/(p+1))(i, j = 1, \cdots, p)$.

(c) *Circular serial correlation model.* In this case the covariance matrices are of the following form.

$$\Sigma_{(\rho_1,\cdots,\rho_q,\sigma^2)} = \sigma^2\{\rho_{ij}\}_{p\times p}$$

where $\rho_{ij} = \rho_{|i-j|} = \rho_{p-|i-j|}$ for $|i - j| = 0, 1, \cdots, q$. $(q \leqq p/2)$.

When p is even, an orthogonal matrix which diagonalizes $\Sigma_{(\rho_1,\cdots,\rho_q,\sigma^2)}$ is given by $\Gamma_2 = \{a_{ij}\}$ where, for $i = 1, \cdots, p$,

$$\begin{aligned} a_{ij} &= (2/p)^{\frac{1}{2}} \cos(i(j+1)\pi/p), && \text{if } j \text{ is odd and } \leqq p-3, \\ &= (2/p)^{-\frac{1}{2}} \sin(ij\pi/p), && \text{if } j \text{ is even and } \leqq p-2, \\ &= p^{-\frac{1}{2}}, && \text{if } j = p-1, \\ &= (-1)^i p^{-\frac{1}{2}}, && \text{if } j = \text{p}. \end{aligned}$$

When p is odd, an orthogonal matrix which diagonalizes $\Sigma_{(\rho_1,\cdots,\rho_q,\sigma^2)}$ is given by $\Gamma_2^* = \{a_{ij}\}$ where, for $i = 1, \cdots, p$,

$$\begin{aligned} a_{ij} &= (2/p)^{\frac{1}{2}} \cos(i(j+1)\pi/p), && \text{if } j \text{ is odd and } \leqq p-2, \\ &= (2/p)^{\frac{1}{2}} \sin(ij\pi/p), && \text{if } j \text{ is even and } \leqq p-1, \\ &= p^{-\frac{1}{2}}, && \text{if } j = p. \end{aligned}$$

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