NOTES

HOW TO SURVIVE A FIXED NUMBER OF FAIR BETS[1]

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Suppose a gambler with initial capital b_0 wants to maximize his probability of still having a positive capital after n_0 successive independent bets, under two conditions: (a) the minimal stake is one dollar; (b) bets are fair and their probability of success is at most $\frac{1}{2}$.

A bet is determined by the stake c and the odds k: the gambler wins $kc - c$ with probability $1/k$ and loses c otherwise. If b_{m-1} denotes the gambler's capital after $m - 1$ bets, he must choose for the mth bet c_m $(1 \leqq c_m \leqq b_{m-1})$ and k_m $(k_m \geqq 2)$. For simplicity of presentation we make the inessential restriction that all b_m, c_m and k_m are integers. In a fair roulette (without zero) k can only be a divisor of 36. A bet $c = 1$, $k = 2$ is called conservative.

A situation is a pair (n, b) where b is the capital and n the number of bets to go. A strategy for (n_0, b_0) is a rule prescribing which bet should be made in the initial situation (n_0, b_0) and in each situation which may evolve from it. Under the stated conditions there exists for each (n_0, b_0) a (possibly non-unique) optimal strategy which leads to a (unique) maximal probability of survival (pos) denoted by $p(n_0, b_0)$. The independence of bets implies that for $n > 1$ and $b \geqq 1$

$$p(n, b) = \max_{c,k} \{(1/k)p(n-1, b + kc - c) + (1 - 1/k)p(n-1, b-c)\}.$$

Theorem 1. *The* pos $q(n, b)$ *for the conservative strategy* (i.e. $c = 1$ *and* $k = 2$ *in each situation*) *is for every* $n \geqq 1$ *a concave function of* b.

Proof. The theorem holds for $n = 1$ as $q(1, 0) = 0$, $q(1, 1) = \frac{1}{2}$ and $q(1, b) = 1$ for $b \geqq 2$. We proceed by induction. The definition of q implies that

$$q(n-1, \beta) \geqq q(n, \beta) \tag{1}$$

and

$$q(n, b) = \tfrac{1}{2}q(n-1, b+1) + \tfrac{1}{2}q(n-1, b-1). \tag{2}$$

Substituting (1) with $\beta = b \pm 1$ into (2) we obtain $q(n, \lambda\beta_1 + (1-\lambda)\beta_2) \geqq \lambda q(n, \beta_1) + (1-\lambda)q(n, \beta_2)$, first for $\lambda = \frac{1}{2}$ and then by well known arguments for all $\lambda \in (0, 1)$ and all β_1, β_2 such that both sides of the inequality are defined.

Theorem 2. *The conservative strategy is optimal for all* n_0 *and* b_0.

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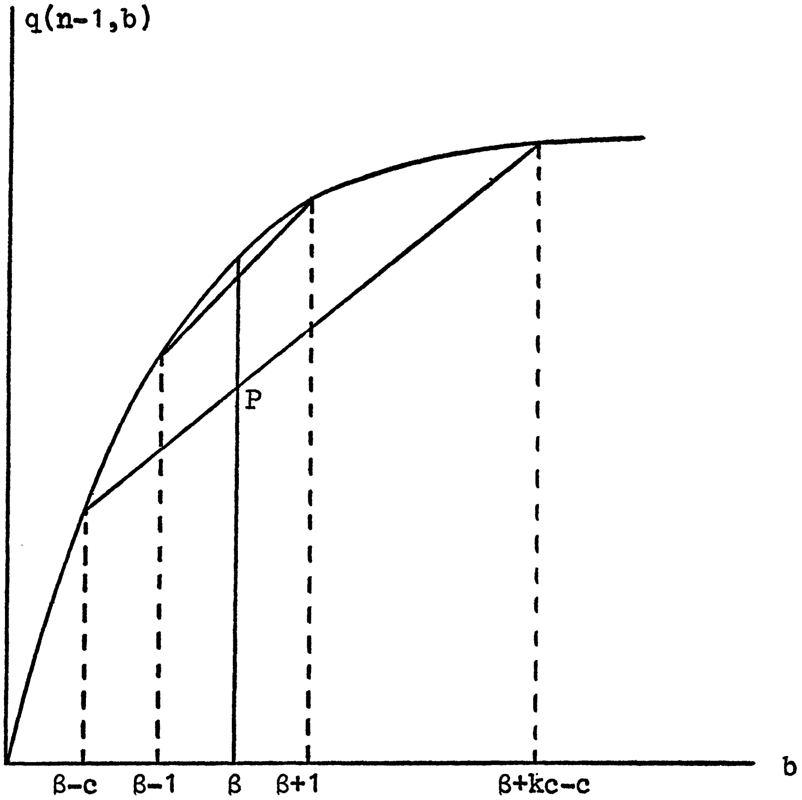

FIG. 1

PROOF. This is trivial for $n_0 = 1$. Suppose it holds for $n_0 = n - 1$. The pos from (n, β) for the bet (c, k) followed by $(n - 1)$ conservative bets is represented by the ordinate of the point of intersection P of the vertical in β and the chord connecting the points on the graph of $q(n - 1, \cdot)$ with abscissae $\beta - c$ and $\beta + kc - c$ (see Figure 1). As the function is concave, the choice $c = 1$, $k = 2$ is seen to be optimal under our conditions $c \geqq 1$, $k \geqq 2$.

REMARK 1. Very similar and somewhat more general results were obtained independently by Freedman [2].

REMARK 2. $q(n, b)$ is determined recursively from (2) and the boundary conditions $q(n, 0) = 0$ for all n, $q(0, b) = 1$ for all $b \geqq 1$. No closed expression for q seems to exist, but we have

$$q(n, b) = \sum_{j=n+1}^{\infty} \lambda_j^{(b)}$$

where $\lambda_j^{(b)}$ are the well-known first passage probabilities for the symmetric random walk given in [1]; p. 254–256.

REMARK 3. Suppose bets are unfair, in the sense that there is a fixed $\alpha < 1$

such that the gambler gains $kc - c$ with probability α/k, and loses c otherwise. It then turns out that bold bets become attractive for small α. For $n_0 = 3$, $b_0 = 1$ the conservative strategy is only optimal for $\alpha > 2 - 2/3^{\frac{1}{2}} \approx .84$. For $n_0 = 13$, $b_0 = 1$ an initial bet $c_1 = 1$, $k_1 = 3$ must be made even for an ordinary roulette with one zero ($\alpha = 36/37$).

REFERENCES

[1] FELLER, W. (1957). *An Introduction to Probability Theory and Its Applications* (2nd edition). **1** Wiley, New York.

[2] FREEDMAN, D. (1967). Timid play is optimal. *Ann. Math. Statist.* **38** 1281–1283.