# CONFIDENCE INTERVALS FOR THE MEAN OF A FINITE POPULATION[1]

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**1. Introduction.** The admissibility of estimates of the population total with squared error as the loss function were considered by the author, (1965), II, and a certain estimate was shown to be always admissible whatever be the sampling design in the entire class of all estimates. This estimate is equivalent to using the sample mean as estimate of the population mean. Here we consider the allied question of admissibility of the confidence intervals for the population mean, based on the sample mean and the sample standard deviation, which are also commonly used in practice. These confidence intervals are here shown to be also always admissible whatever be the sampling design.

Then, by generalizing, the result is also shown to hold for confidence intervals based on a ratio estimate and a generalized version of the sample standard deviation. It is interesting that as shown in a previous paper (1966), IV, Section 5, with squared error as loss function, this ratio estimate is also admissible as an estimate of the population mean, whatever be the sampling design.

**2. Notation and definitions.** The population $U$ consists of $N$ units $u_1, u_2, \cdots, u_N$; with the unit $u_i$ is associated the variate value $x_i$, $i = 1, 2, \cdots, N$; $x = (x_1, x_2, \cdots, x_N)$ denotes a point in the Euclidean $N$-space $R_N$; a sample $s$ means any subset of $U$; $S$ denotes the set of all possible samples $s$; a probability function $p$ is defined on $S$ such that $p(s) \geqq 0$ for all $s$, and $\sum_{s \varepsilon S} p(s) = 1$. Following Godambe and Joshi (1965), I, the pair $(S, p)$ is called the sampling design. A sample $s$ is drawn from $S$ according to $p$. Then we have

Definition 2.1. An estimate $e(s, x)$ is a real function $e$ defined on $S \times R_N$ which depends on $x$ through only those $x_i$ for which $u_i \varepsilon s$.

The above definitions of sampling design and estimate are wide enough to cover all sampling procedures and classes of estimates; for a brief account we refer to Godambe and Joshi (1965), I, Section 5.

We next define admissibility of a set of confidence intervals for the population mean,

$$\bar{X}_N = N^{-1} \sum_{i=1}^{N} x_i . \tag{1}$$

For a given sampling design $d$, we denote by $\bar{S}$, the subset of $S$, consisting of all those samples $s$ for which $p(s) > 0$. Now let $e_1(s, x)$, $e_2(s, x)$ be two estimates (Definition 2.1) such that $e_1(s, x) \leqq e_2(s, x)$ for all $x \varepsilon R_N$ and all $s \varepsilon \bar{S}$; then $[e_1(s, x), e_2(s, x)]$ denotes the set of confidence intervals

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$$e_1(s, x) \leqq \bar{X}_N \leqq e_2(s, x). \tag{2}$$

For every $x \varepsilon R_N$, let $\bar{S}_{e_1,e_2,x}$ denote the subset of $\bar{S}$ consisting of all those samples $s$ for which (2) holds, and for any alternative set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$ let $\bar{S}_{e_1',e_2',x}$, denote the corresponding subset of $\bar{S}$ consisting of all those samples $s$ for which

$$e_1'(s, x) \leqq \bar{X}_N \leqq e_2'(s, x). \tag{3}$$

We now define,

Definition 2.2. The set of confidence intervals $[e_1(s, x), e_2(s, x)]$ for the population mean is admissible, if there exists no other set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$ such that,

(i) $e_2'(s, x) - e_1'(s, x) \leqq e_2(s, x) - e_1(s, x)$ for all $x \varepsilon R_N$ and for all $s \varepsilon \bar{S}$;

(ii) $\sum_{s\varepsilon\bar{S}_{e_1',e_2',x}} p(s) \geqq \sum_{s\varepsilon\bar{S}_{e_1,e_2,x}} p(s)$ for all $x \varepsilon R_N$;

the strict inequality in (ii) holding for at least one $x \varepsilon R_N$. The sums in (ii) are obviously the inclusion probabilities for the confidence intervals.

We also define a weaker version of admissibility by

Definition 2.3. The set of confidence intervals $[e_1(s, x), e_2(s, x)]$ for the population mean is weakly admissible, if there exists no other set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$, such that,

(i) $e_2'(s, x) - e_1'(s, x) \leqq e_2(s, x) - e_1(s, x)$ for all $x \varepsilon R_N$ and all $s \varepsilon \bar{S}$;

(ii) $\sum_{s\varepsilon\bar{S}_{e_1',e_2',x}} p(s) \geqq \sum_{s\varepsilon\bar{S}_{e_1,e_2,x}} p(s)$,

for almost all (Lebesgue measure) $x \varepsilon R_N$, the strict inequality in (ii) holding on a non-null subset of $R_N$. To distinguish the admissibility, as defined in Definition 2.2, from weak admissibility, we shall refer to the former as strict admissibility.

Note. Throughout the rest of this paper the measure considered will be the Lebesgue measure on $R_N$. So also for any $k$ dimensional subspace of $R_N$, the measure considered will be the Lebesgue measure for the $k$ dimensional subspace. When the measure considered is for a $k$-dimensional subspace, it will be indicated by $(\mu_k)$.

The above definitions of admissibility of confidence intervals are based on the definition for infinite frequency functions with an unknown parameter formulated by Godambe (1961), and subsequently slightly modified by the author (1966).

**3 (I). A Bayes solution.** For a sample $s$, the sample mean $\bar{x}_s$ is given by

$$\bar{x}_s = [n(s)]^{-1} \sum_{i\varepsilon s} x_i \tag{4}$$

where $i \varepsilon s$ is written shortly for $u_i \varepsilon s$ and is so written hereafter, and $n(s)$ denotes the sample size, i.e. the number of units in the sample $s$.

The sample standard deviation is

$$v'(s, x) = [(n(s))^{-1} \sum_{i\varepsilon s} (x_i - \bar{x}_s)^2]^{\frac{1}{2}}. \tag{5}$$

The usual confidence intervals based on the sample standard deviation,

$[e_1(s, x), e_2(s, x)]$ are given by

(6) $$e_1(s, x) = \bar{x}_s - v(s, x),$$
$$e_2(s, x) = \bar{x}_s + v(s, x),$$

where $v(s, x) = k_s \cdot v'(s, x)$, $k_s$ being positive constants. It is usual to take $k_s = 2/[n(s)]^{\frac{1}{2}}$ or $3/[n(s)]^{\frac{1}{2}}$, but our proof will hold for any arbitrary $k_s > 0$. The constants $k_s$ may vary with the sample $s$. As confidence intervals based on the sample standard deviation are used, we assume that for all $s \varepsilon \bar{S}$, $n(s) \geqq 2$.

As a first step towards proving strict admissibility, we now prove

THEOREM 3.1. *The set of confidence intervals* $[e_1(s, x), e_2(s, x)]$ *in* (6) *is weakly admissible.*

OUTLINE OF THE PROOF. As the proof of the theorem is rather long, we shall first give its broad outline and fill in the details later. Suppose the theorem is false; then there exists a set of confidence intervals $[e_1'(s, x)e_2'(s, x)]$ for which (i) and (ii) of Definition 2.3 hold. Let $\bar{e}(s, x) = \frac{1}{2}[e_1'(s, x) + e_2'(s, x)]$, $e_1''(s, x) = \bar{e}(s, x) - v(s, x)$ and $e_2''(s, x) = \bar{e}(s, x) + v(s, x)$. Then by (i) of Definition 2.3, for each $x$, and $s \varepsilon \bar{S}$, $[e_1''(s, x), e_2''(s, x)] \supset [e_1'(s, x), e_2'(s, x)]$. Hence (i) and (ii) of Definition 2.3 hold also for the new set of confidence intervals $[e_1''(s, x), e_2''(s, x)]$. We next consider a prior probability distribution on $R_N$, which is such that all the $x_i$, $i = 1, 2, \cdots, N$, are distributed independently, identically, and normally, with mean $\theta$ and unit variance. Further we assume for $\theta$ itself a prior distribution, with mean zero and variance $\tau^2$. We then work out the set of confidence intervals $[e_\tau(s, x) - v(s, x), e_\tau(s, x) + v(s, x)]$, whose length for each $x$ and $s \varepsilon \bar{S}$ is equal to $2v(s, x)$ and which are a Bayes solution with respect to the above prior distribution. It is shown that the improvement in the unconditional expected inclusion probability of the Bayes set of confidence intervals over the given set $[e_1(s, x), e_2(s, x)]$ is bounded by $2/\tau^2$. Next let $h(s, x) = \bar{e}(s, x) - \bar{x}_s$, and let $E$ be the subset of $R_N$ consisting of all those points $x$ for which, for at least one $s \varepsilon \bar{S}$, $h(s, x) \neq 0$ and $v(s, x) > 0$. It is shown by using the maximizing property of the Bayes solution, that if $E$ is not a null set then, by making $\tau$ sufficiently larger, the unconditional expected inclusion probability of the set of confidence intervals $[e_1''(s, x), e_2''(s, x)]$ can be made less than that of the given set $[e_1(s, x), e_2(s, x)]$. But this contradicts (ii) of Definition 2.3. Hence $E$ must be a null subset, and from this the weak admissibility of the given set of intervals $[e_1(s, x), e_2(s, x)]$, follows. We shall now give the detailed proof.

PROOF. Suppose the theorem is false. Then, by Definition 2.3, there exists a set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$, satisfying

(7) $$e_2'(s, x) - e_1'(s, x) \leqq e_2(s, x) - e_1(s, x) = 2v(s, x) \quad \text{for all} \quad x \varepsilon R_N \quad \text{and all} \quad s \varepsilon \bar{S},$$

and

(8) $$\textstyle\sum_{s\varepsilon\bar{S}_{e_1', e_2', x}} p(s) \geqq \sum_{s\varepsilon\bar{S}_{e_1, e_2, x}} p(s)$$

for almost all $x \varepsilon R_N$, the strict inequality in (8) holding on a non-null subset of $R_N$.

Let,

$$\begin{aligned} \bar{e}(s, x) &= \tfrac{1}{2}[e_1'(s, x) + e_2'(s, x)]; \\ e_1''(s, x) &= \bar{e}(s, x) - v(s, x); \\ e_2''(s, x) &= \bar{e}(s, x) + v(s, x). \end{aligned} \tag{9}$$

Then by (7),

$$e_1''(s, x) \leqq \tfrac{1}{2}[e_1'(s, x) + e_2'(s, x)] - \tfrac{1}{2}[e_2'(s, x) - e_1'(s, x)] = e_1'(s, x), \tag{10}$$

and similarly, $e_2''(s, x) \geqq e_2'(s, x)$. For every $x \varepsilon R_N$ denote by $\bar{S}_{e_1'', e_2'', x}$, the subset of $\bar{S}$, on which

$$e_1''(s, x) \leqq \bar{X}_N \leqq e_2''(s, x). \tag{11}$$

Then by (10)

$$\bar{S}_{e_1'', e_2'', x} \supset \bar{S}_{e_1', e_2', x}. \tag{12}$$

(12) combined with (8) now gives,

$$\textstyle\sum_{s \varepsilon \bar{S}_{e_1,'' e_2'', x}} p(s) \geqq \sum_{s \varepsilon \bar{S}_{e_1, e, 2, x}} p(s) \tag{13}$$

for almost all $x \varepsilon R_N$, the strict inequality in (13) holding on a non-null subset of $R_N$.

We now consider a prior distribution on $R_N$, which is such that all the $x_i$, $i = 1, 2, \cdots, N$, are distributed independently, identically, and normally, with mean $\theta$ and unit variance. We assume further that $\theta$ itself is distributed normally with mean zero and variance $\tau^2$, i.e. $\theta$ is $N(0, \tau^2)$. We now determine the set of confidence intervals $[b_1(s, x), b_2(s, x)]$, which are such that,

$$b_2(s, x) - b_1(s, x) = 2v(s, x) \tag{14}$$

for all $x \varepsilon R_N$ and all $s \varepsilon \bar{S}$, and which subject to (14), are a Bayes solution with respect to the assumed prior distribution, i.e. which maximize the unconditional expected inclusion probability of the confidence intervals. Let $E_\theta$ denote the conditional expectation for given $\theta$, and $E_\tau$ the unconditional expectation. Further for every $x \varepsilon R_N$ let $\bar{S}_{b_1, b_2, x}$ be the subset of $\bar{S}$ on which

$$b_1(s, x) \leqq \bar{X}_N \leqq b_2(s, x). \tag{15}$$

Then, for any $x \varepsilon R_N$, the inclusion probability of these confidence intervals equals $\sum_{s \varepsilon \bar{S}_{b_1, b_2, x}} p(s)$ and its expected value for given $\theta$ is equal to

$$E_\theta\{\textstyle\sum_{s \varepsilon \bar{S}_{b_1, b_2, x}} p(s)\} = \sum_{s \varepsilon \bar{S}} p(s) P_\theta[b_1(s, x) \leqq \bar{X}_N \leqq b_2(s, x) \mid s] \tag{16}$$

where $P_\theta(D \mid s)$ denotes the conditional probability, for given $\theta$ and $s$, of the Borel set $D$ of $R_N$.

Now let,

(17) $$\bar{X}_{N-n(s)} = [N - n(s)]^{-1} \sum_{i \notin s} x_i .$$

We want later to integrate out wrt the variables $x_i$, $i \notin s$. Hence we express the density function on $R_N$ for given $\theta$, in the form

(18) $$f(x, \theta) = L[x_i, i \in s \mid \bar{x}_s]\cdot p(\bar{x}_s - \theta)\cdot L[x_i, i \notin s \mid \bar{X}_{N-n(s)}]\cdot q(\bar{X}_{N-n(s)} - \theta),$$

where,

$$p(\bar{x}_s - \theta) = [n(s)/2\pi]^{\frac{1}{2}}\cdot \exp[-\tfrac{1}{2}n(s)(\bar{x}_s - \theta)^2]$$

and

$$q(\bar{X}_{N-n(s)} - \theta) = \{[N - n(s)]/2\pi\}^{\frac{1}{2}}\exp[-\tfrac{1}{2}(N - n(s))(\bar{X}_{N-n(s)} - \theta)^2].$$

For brevity put

(19) $$L_1 = L[x_i, i \in s \mid \bar{x}_s];$$
$$L_2 = L[x_i, i \notin s \mid \bar{X}_{N-n(s)}].$$

Let $C_{b,s}$ be the subset of $R_N$ defined by

(20) $$C_{b,s} = [x \in R, b_1(s, x) \leqq \bar{X}_N \leqq b_2(s, x)].$$

Then using (18), (19) and (20), we have in the right hand side of (16),

(21) $$P_\theta[b_1(s, x) \leqq \bar{X}_N \leqq b_2(s, x) \mid s] = \int_{C_{b,s}} L_1\cdot L_2\cdot p(\bar{x}_s - \theta)\cdot q(\bar{X}_{N-n(s)} - \theta)\, dx$$

where $dx$ is written for short for $\prod_{i=1}^N dx_i$.

Hence taking expectation wrt the assumed prior distribution of $\theta$, the expected inclusion probability $B_{\tau,s}$ say of the confidence interval $[b_1(s, x), b_2(s, x)]$ for a particular sample $s$, is given by

(22) $$B_{\tau,s} = (2\pi)^{-\frac{1}{2}}\cdot\tau^{-1}\int_{-\infty}^{\infty}\exp(-\theta^2/2\tau^2)\, d\theta \cdot\int_{C_{b,s}} L_1\cdot L_2\cdot p(\bar{x}_s - \theta)\cdot q(\bar{X}_{N-n(s)} - \theta)\, dx.$$

In the right hand side of (22) the integrand is non-negative for all $x$ and $\theta$. Hence by Fubini's theorem, we may interchange the order of integration wrt $\theta$ and $x = (x_1, \cdots, x_N)$. We then get

(23) $$B_{\tau,s} = (2\pi)^{-\frac{1}{2}}\tau^{-1}\int_{C_{b,s}} L_1\cdot L_2\, dx\int_{-\infty}^{+\infty} p(\bar{x}_s - \theta) \cdot q(\bar{X}_{N-n(s)} - \theta)\cdot\exp(-\theta^2/2\tau^2)\, d\theta.$$

Now, substituting the values of $p(\bar{x}_s - \theta)$ and $q(\bar{X}_{N-n(s)} - \theta)$ from (18), we have

(24) $$(2\pi)^{-\frac{1}{2}}\tau^{-1}p(\bar{x}_s - \theta)\cdot q(\bar{X}_{N-n(s)} - \theta)\cdot\exp(-\theta^2/2\tau^2)$$
$$= (2\pi)^{-\frac{1}{2}}\tau^{-1}\{n(s)/2\pi\}^{\frac{1}{2}}\cdot\{[N - n)s)]/2\pi\}^{\frac{1}{2}}$$
$$\cdot\exp\{-\tfrac{1}{2}[n(s)(\bar{x}_s - \theta)^2 + (N - n(s))[\bar{X}_{N-n(s)} - \theta]^2 + \theta^2/\tau^2]\}.$$

We now put

(25) $$g = 1 + (N\tau^2)^{-1}$$

and

(26) $$g_s = 1 + (n(s) \cdot \tau^2)^{-1}.$$

Then the expression in the square bracket in the right hand side of (24) is equal to

$$\begin{aligned} &Ng\theta^2 - 2\theta\{n(s)\bar{x}_s + [N - n(s)] \cdot \bar{X}_{N-n(s)}\} + n(s)\bar{x}_s^2 + [N - n(s)]\bar{X}^2_{N-n(s)} \\ &\quad = Ng[\theta - [n(s) \cdot \bar{x}_s + [N - n(s)] \cdot \bar{X}_{N-n(s)}]/Ng]^2 \\ &\qquad + \{[N - n(s)]/Ng\} \cdot n(s) \cdot g_s \bar{X}^2_{N-n(s)} + (n(s)/Ng)[N - n(s) + \tau^{-2}]\bar{x}_s^2 \\ &\qquad - 2\{[N - n(s)] \cdot n(s)/Ng\} \cdot \bar{X}_{N-n(s)} \cdot \bar{x}_s \\ &\quad = Ng\{\theta - [n(s) \cdot \bar{x}_s + [N - n(s)] \cdot \bar{X}_{N-n(s)}]/Ng\}^2 \\ &\qquad + \{[N - n(s)]/Ng\} \cdot n(s) \cdot g_s[\bar{X}_{N-n(s)} - \bar{x}_s/g_s]^2 + (\bar{x}_s^2/g_s) \cdot \tau^{-2}. \end{aligned}$$

Using (26) we get,

the right hand side of (24)

(27) $$\begin{aligned} &= (Ng/2\pi)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}Ng\{\theta - [n(s) \cdot \bar{x}_s + [N - n(s)] \cdot X_{N-n(s)}]/Ng\}^2\} \\ &\quad \cdot \{[N - n(s)]/2\pi Ng\}^{\frac{1}{2}} \cdot [n(s)g_s]^{\frac{1}{2}} \cdot \exp\ \{-\tfrac{1}{2}[[N - n(s)]] \cdot n(s)g_s/Ng] \\ &\quad \cdot [\bar{X}_{N-n(s)} - \bar{x}_s/g_s]^2\} \cdot (2\pi g_s)^{-\frac{1}{2}} \cdot \tau^{-1} \exp\ (-\bar{x}_s^2/2g_s\tau^2) \\ &= F_3 \cdot F_2 \cdot F_1 \quad \text{say}, \end{aligned}$$

where $F_3$, $F_2$ and $F_1$ respectively denote the first, second and third factors in the right hand side of (27).

We now substitute the right hand side of (27) in (23) and integrate out wrt $\theta$. $F_3$ is the only factor which involves $\theta$ and its integral is $=1$. Thus (23) reduces to

(28) $$B_{\tau,s} = \int_{C_{b,s}} L_1 \cdot L_2 \cdot F_1 \cdot F_2 \cdot dx.$$

Now by an orthogonal transformation of co-ordinates in the $[N - n(s)]$ dimensional space of the variates $x_i$, $i \not\in s$, we obtain that $\bar{X}_{N-n(s)}$ is independent of the other $N - n(s) - 1$ co-ordinates. Let $x'$ denote the group of the remaining $[N - n(s) - 1]$ transformed variables. Then for each fixed $\bar{X}_{N-n(s)}$,

(29) $$\int L_2\, dx' = \int L[x_i, i \not\in s \mid \bar{X}_{N-n(s)}]\, dx' = 1.$$

Using (29), we now integrate out in (28) with respect to the variables $x'$. Let $R_{n(s)}$ denote the $n(s)$-dimensional space of the variates $x_i$, $i \in s$. Also we have

(30) $$\bar{X}_N = [N - n(s)]N^{-1}\bar{X}_{N-n(s)} + n(s)N^{-1}\bar{x}_s.$$

Hence it is seen from (20) that for given values of $x_i$, $i \in s$, $\bar{X}_{N-n(s)}$ varies over the linear interval $I_{b,s}$ given by

$$(31)\quad I_{b,s} : N[N - n(s)]^{-1}b_1(s, x) - n(s)[N - n(s)]^{-1}\bar{x}_s \leq \bar{X}_{N-n(s)} \leq N[N - n(s)]^{-1}b_2(s, x) - n(s)[N - n(s)]^{-1}\bar{x}_s .$$

Thus by integrating out wrt the variables $x'$, and using (29), we have from (28),

$$(32)\qquad B_{\tau,s} = \int_{R_{n(s)}} L_1 \cdot F_1 \cdot dx_s \int_{I_{b,s}} F_2 \cdot d\bar{X}_{N-n(s)}$$

where $dx_s$ is written for short for $\prod_{i\varepsilon s} dx_i$.

Considering the expression for $F_2$ in (27), it is seen, that at each point of $R_{n(s)}$, the right hand side of (32) is maximized by taking $I_{b,s}$ to be centered at the point $\bar{x}_s/g_s$. Further since by (14), $b_2(s, x) - b_1(s, x) = 2v(s, x)$ the length of the interval $I_{b,s}$ in (31) must be $2N[N - n(s)]^{-1} \cdot v(s, x)$. Hence the Bayes solution is obtained by taking in (31),

$$N[N - n(s)]^{-1}b_1(s, x) - n(s)[N - n(s)]^{-1}\bar{x}_s = \bar{x}_s/g_s - N[N - n(s)]^{-1}v(s, x).$$

This gives,

$$(33)\qquad \begin{aligned} b_1(s, x) &= (\bar{x}_s/g_s)[1 - n(s)N^{-1} + n(s)N^{-1} \cdot g_s] - v(s, x) \\ &= \bar{x}_s \cdot g/g_s - v(s, x). \end{aligned}$$

And similarly, $b_2(s, x) = \bar{x}_s \cdot g/g_s + v(s, x)$. Clearly since the confidence interval $[b_1(s, x), b_2(s, x)]$ maximizes $B_{\tau,s}$ for given $s$, the set of confidence intervals $[b_1(s, x), b_2(s, x)]$ maximizes the total expected inclusion probability viz., $B_\tau = \sum_{s\varepsilon\bar{S}} p(s)B_{\tau,s}$.

Thus $b_1(s, x)$ and $b_2(s, x)$ in (33) give the Bayes set of confidence intervals. Since $g_s = 1 + 1/n(s)\tau^2$ and $g = 1 + 1/N\tau^2$, comparison of (6) and (33) indicates that the improvement in the total expected inclusion probability would be of the order of $1/\tau^2$. We shall now obtain a precise upper bound for this improvement.

For a particular sample $s$, let $A_{\theta,s}$ denote the conditional inclusion probability for given $\theta$, of the confidence interval $[e_1(s, x), e_2(s, x)]$, and let $A_{\tau,s}$ denote the expected inclusion probability. Let the corresponding probabilities summed over all samples be $A_\theta$ and $A_\tau$ respectively; then

$$(34)\qquad \begin{aligned} A_\theta &= \sum_{s\varepsilon\bar{S}} p(s)A_{\theta,s}, \\ A_\tau &= \sum_{s\varepsilon\bar{S}} p(s)A_{\tau,s}. \end{aligned}$$

Consider the probability $A_{\theta,s}$. We note that by a well known property of the normal distribution for given $s$, $v(s, x)$ is distributed independently of both $\theta$ and $\bar{x}_s$. We shall consider the conditional probabilities for given $s$, and given $v(s, x) = v$.

As $\bar{X}_{N-n(s)}$ and $\bar{x}_s$ are distributed independently of $v(s, x)$, conditionally for given $s$, $\theta$ and $v(s, x) = v$,

$$(35)\qquad (\bar{X}_{N-n(s)} - \bar{x}_s) \text{ is } N(0, N/[N - n(s)] \cdot n(s)).$$

Now by (6), $e_1(s, x) \leqq \bar{X}_N \leqq e_2(s, x)$ holds if, and only if, $|\bar{X}_N - \bar{x}_s| \leqq v(s, x)$. Hence using (30),

$$e_1(s, x) \leqq \bar{X}_N \leqq e_2(s, x), \quad \text{if and only if} \quad |\bar{X}_{N-n(s)} - \bar{x}_s| \leqq N[N - n(s)]^{-1} \cdot v(s, x). \tag{36}$$

Let $\alpha(s, v, \theta)$ denote the conditional probability that (36) holds for given $s$, the given value $v(s, x) = v$ and for given $\theta$. Then using (35),

$$\begin{aligned} \alpha(s, v, \theta) &= P_\theta[|\bar{X}_{N-n(s)} - \bar{x}_s| \leqq N[N - n(s)]^{-1} \cdot v \mid v(s, x) = v] \\ &= \int_{-N[N-n(s)]^{-1} \cdot v}^{N[N-n(s)]^{-1} \cdot v} f(t)\, dt, \end{aligned} \tag{37}$$

where

$$f(t) = (2\pi)^{-\frac{1}{2}} \cdot \{[N - n(s)] \cdot n(s)\}^{\frac{1}{2}} N^{-\frac{1}{2}} \exp[-\tfrac{1}{2}[N - n(s)] \cdot n(s) N^{-1} \cdot t^2].$$

It is seen that $\alpha(s, v, \theta)$ is independent of $\theta$. Hence we may put $\alpha(s, v, \theta) = \alpha(s, v)$. Now transform the variables by putting

$$z = \{[N - n(s)] \cdot n(s)/N\}^{\frac{1}{2}} \cdot t.$$

Let

$$c_s = \{N \cdot n(s)/[N - n(s)]\}^{\frac{1}{2}} \cdot v.$$

We thus get from (37),

$$\alpha(s, v) = (2\pi)^{-\frac{1}{2}} \int_{-c_s}^{c_s} \exp(-z^2/2)\, dz. \tag{38}$$

Let $\phi_s(v)$ be the distribution function of $v(s, x)$ which is independent of $\theta$. Then

$$A_{\theta,s} = \int_{v=0}^{v=\infty} \alpha(s, v)\, d\phi_s(v). \tag{39}$$

Since the right hand side of (39) is independent of $\theta$, we have

$$A_{T,s} = \int_{v=0}^{v=\infty} \alpha(s, v)\, d\phi_s(v). \tag{40}$$

Now consider the expression for $B_{T,s}$ in (32). In the integral wrt $\bar{X}_{N-n(s)}$, transform the variable by putting

$$z = \{[N - n(s)] \cdot n(s) \cdot g_s/Ng\}^{\frac{1}{2}}[\bar{X}_{N-n(s)} - \bar{x}_s/g_s]. \tag{41}$$

As observed below equation (32) in the interval $I_{b,s}$, $(\bar{X}_{N-n(s)} - \bar{x}_s/g_s)$ varies from $-N[N - n(s)]^{-1} \cdot v$ to $N[N - n(s)]^{-1} \cdot v$. Therefore the variable $z$ in (41) varies from $-d_s$ to $+d_s$, where

$$d_s = \{N \cdot n(s) \cdot g_s/[N - n(s)] \cdot g\}^{\frac{1}{2}} \cdot v = c_s(g_s/g)^{\frac{1}{2}} \tag{42}$$

in which $c_s$ is the constant in (38).

Put

$$\beta(s, v) = \int_{I_{b,s}} F_2 \cdot d\bar{X}_{N-n(s)}, \tag{43}$$

where the integral in the right hand side is the same as the inner integral in the right hand side of (32). Noting the value of $F_2$ from (27), it is seen that by making the transformation in (41) we get,

$$\beta(s, v) = (2\pi)^{-\frac{1}{2}} \int_{-d_s}^{d_s} \exp(-z^2/2)\, dz. \tag{44}$$

Since $\exp(-z^2/2)$ strictly decreases as $|z|$ increases, (44), (42) and (38), give,

$$\begin{aligned} \beta(s, v) &< \alpha(s, v)(g_s/g)^{\frac{1}{2}} = \alpha(s, v)\{[1 + 1/n(s)\cdot\tau^2]/[1 + 1/N\tau^2]\}^{\frac{1}{2}} \\ &< \alpha(s, v)\{1 + 1/n(s)\cdot\tau^2\}^{\frac{1}{2}} < \alpha(s, v)\cdot[1 + 1/2n(s)\cdot\tau^2] \\ &\leqq \alpha(s, v)\cdot[1 + 1/2\tau^2] \end{aligned} \tag{45}$$

since $n(s) \geqq 1$ for all $s \varepsilon \bar{S}$. Here we have used the values of $g_s$ and $g$ in (25).

Now for given $s$, $\beta(s, v)$ depends on $v$ only. Hence in the outer integral in the right hand side of (32), we may transform the variables by taking as independent variables, $\bar{x}_s$ and $v$. Let $x'$ denote the group of the remaining $(n(s) - 2)$ transformed variables. Using the value of $F_1$ in (27) it is seen that the integration wrt $\bar{x}_s$ yields a factor equal to unity. Moreover the integral of $L_1 = L[x_i, i \varepsilon s \mid \bar{x}_s]$ wrt $x'$ must be equal to the probability density of $v(s, x)$.

We thus get from (32),

$$B_{\tau,s} = \int_{v=0}^{v=\infty} \beta(s, v)\, d\phi_s(v) < A_{\tau,s}[1 + 1/2\tau^2] \qquad \text{by (45) and (40).} \tag{46}$$

Summing over all samples $s \varepsilon \bar{S}$, we have

$$\begin{aligned} B_\tau = \textstyle\sum_{s\varepsilon\bar{S}} p(s)B_{\tau,s} &< (1 + 1/2\tau^2) \textstyle\sum_{s\varepsilon\bar{S}} p(s)A_{\tau,s} \\ &= (1 + 1/2\tau^2)A_\tau . \end{aligned} \tag{47}$$

As $A_\tau$ is the expected value of a probability $A_\tau \leqq 1$. Thus finally we have from (47),

$$B_\tau - A_\tau < A_\tau/2\tau^2 < 1/2\tau^2. \tag{48}$$

Thus $1/2\tau^2$ is an upper bound for the improvement in the expected inclusion probability. Using this result we shall complete the proof of the theorem in part II of this section.

**3 (II). Weak admissibility.** We have denoted by $A_{\theta,s}$ and $A_{\tau,s}$ the conditional and unconditional expected inclusion probabilities for a given sample $s$ of the confidence interval $[e_1(s, x), e_2(s, x)]$. Let $A''_{\theta,s}$ and $A''_{\tau,s}$ denote the corresponding inclusion probabilities for the confidence interval $[e_1''(s, x), e_2''(s, x)]$ in (9). Next using the frequency on $R_N$ in (18) we obtain the expressions for $A_{\tau,s}$ and $A''_{\tau,s}$ which correspond to the expression for $B_{\tau,s}$ in (32). Then, corresponding to (31), let $I_{e,s}$ denote the linear interval of variation of $\bar{X}_{N-n(s)}$, given by

$$\begin{aligned} I_{e,s} : N[N - n(s)]^{-1}e_1(s, x) - n(s)[N - n(s)]^{-1}\bar{x}_s &\leqq \bar{X}_{N-n(s)} \\ &\leqq N[N - n(s)]^{-1}e_2(s, x) - n(s)[N - n(s)]^{-1}\bar{x}_s . \end{aligned} \tag{49}$$

Similarly let $I''_{e,s}$ denote the interval

$$(50)\quad I''_{e,s} : N[N - n(s)]^{-1}e_1''(s, x) - n(s)[N - n(s)]^{-1}\bar{x}_s \leqq \bar{X}_{N-n(s)} \leqq N[N - n(s)]^{-1}e_2''(s, x) - n(s)[N - n(s)]^{-1}\bar{x}_s .$$

Then proceeding step by step, exactly as from (22) to (32), we obtain in place of (32),

$$(51)\quad A_{\tau,s} = \int_{R_{n(s)}} L_1 \cdot F_1\, dx_s \int_{I_{e,s}} F_2 \cdot d\bar{X}_{N-n(s)} ;$$

$$A''_{\tau,s} = \int_{R_{n(s)}} L_1 \cdot F_1\, dx_s \int_{I''_{e,s}} F_2 \cdot d\bar{X}_{N-n(s)} .$$

Hence, as both integrands in (51) are integrable, we may combine them and have,

$$(52)\quad A_{\tau,s} - A''_{\tau,s} = \int_{R_{n(s)}} L_1 \cdot F_1\, dx_s \{\int_{I_{e,s}} F_2\, d\bar{X}_{N-n(s)} - \int_{I''_{e,s}} F_2\, d\bar{X}_{N-n(s)}\}.$$

Here $F_1$ and $F_2$ are the factors which occur in (27), and $L_1$ is the function defined by (19).

We next define functions $U(s, x)$ and $U''(s, x)$ on $R_{n(s)}$ as follows: Let,

$$(53)\quad f_s = \{[N - n(s)] \cdot n(s)/2\pi N\}^{\frac{1}{2}} \cdot \exp \cdot \{-\tfrac{1}{2}[N - n(s)] \cdot n(s) N^{-1}[\bar{X}_{N-n(s)} - \bar{x}_s]^2\}.$$

Since as $\tau \to \infty$, $g_s \to 1$, and $g \to 1$, it is seen from the expression for $F_2$ in (27), that $f_s = \lim_{\tau\to\infty} F_2$ . Now put

$$(54)\quad U(s, x) = \int_{I_{e,s}} f_s \cdot d\bar{X}_{N-n(s)} ;$$

$$U''(s, x) = \int_{I''_{e,s}} f_s \cdot d\bar{X}_{N-n(s)} .$$

Note that $U(s, x)$ and $U''(s, x)$ depend on $x$ through only those $x_i$ for which $i \varepsilon s$. Hence they are estimates according to Definition 2.1. We shall now show that for all $x \varepsilon R_N$ , $U(s, x) \geqq U''(s, x)$.

Substituting for $e_1(s, x)$, $e_2(s, x)$ by (6), the interval $I_{e,s}$ can be also expressed as

$$(55)\quad I_{e,s} : \bar{x}_s - N[N - n(s)]^{-1}v(s, x) \leqq \bar{X}_{N-n(s)} \leqq \bar{x}_s + N[N - n(s)]^{-1}v(s, x).$$

Similarly substituting from (9), the interval $I''_{e,s}$ is expressed as

$$(56)\quad N[N - n(s)]^{-1}[\bar{e}(s, x) - \bar{x}_s] + \bar{x}_s - N[N - n(s)]^{-1}v(s, x) \leqq \bar{X}_{N-n(s)} \leqq N[N - n(s)]^{-1}[\bar{e}(s, x) - \bar{x}_s] + \bar{x}_s + N[N - n(s)]^{-1}v(s, x).$$

Considering the expression for $f_s$ in (53), it is seen from (55) and (56) that we have always,

$$(57)\quad U(s, x) - U''(s, x) \geqq 0.$$

Moreover, in (57) this sign of equality must hold at any point $x \varepsilon R_N$ , at which either $v(s, x) = 0$, or $h(s, x) = \bar{e}(s, x) - \bar{x}_s = 0$, and the sign of strict inequality must hold at any point $x \varepsilon R_N$ , at which both $v(s, x) > 0$ and $h(s, x) \neq 0$. [(57) means that the given intervals $[e_1(s, x), e_2(s, x)]$ are a Bayes solution when the prior distribution of $\theta$ is taken to be a uniform (improper) density on $-\infty < \theta < \infty$.]

Now there are two possible alternatives viz.,

(i) for every sample $s \varepsilon \bar{S}$, the subset of $R_N$ on which the strict inequality in (57) holds is a null $(\mu_N)$ subset of $R_N$, or

(ii) there exists at least one sample $s \varepsilon \bar{S}$, for which the strict inequality in (57) holds on a non-null $(\mu_N)$ subset of $R_N$.

Suppose (ii) is true. For any arbitrary positive constant $a$, let $T_a$ denote the subset of $R_{n(s)}$ defined by

(58) $$x = (x_i, i \varepsilon s) \varepsilon T_a, \quad \text{if, and only if,} \quad |x_i| \leqq a \quad \text{for all } i \varepsilon s.$$

Now $U(s, x)$ and $U''(s, x)$ depend on $x$ only through those $x_i$ for which $i \varepsilon s$. Hence since the strict inequality in (57) is assumed to hold on a non-null $(\mu_N)$ subset of $R_N$, it holds on a non-null $(\mu_{n(s)})$ subset of $R_{n(s)}$. This implies that there exists a positive constant $k$, $(k > 0)$, and a number $a$, such that,

(59) $$\int_{T_a} L_1 \, dx_s \cdot [U(s, x) - U''(s, x)] = k.$$

Let $T_a^c$ be the complement of $T_a$, i.e., $T_a^c = R_{n(s)} - T_a$. Then we have from (52),

(60) $$\begin{aligned} A_{\tau,s} - A''_{\tau,s} &= \int_{T_a} L_1 F_1 \, dx_s \{\int_{I_{e,s}} F_2 \, d\bar{X}_{N-n(s)} - \int_{I''_{e,s}} F_2 \, d\bar{X}_{N-n(s)}\} \\ &\quad + \int_{T_a^c} L_1 F_1 \, dx_s \{\int_{I_{e,s}} F_2 \, d\bar{X}_{N-n(s)} - \int_{I''_{e,s}} F_2 \, d\bar{X}_{N-n(s)}\} \\ &= J_1 + J_2, \quad \text{say,} \end{aligned}$$

where $J_1$ and $J_2$ respectively denote the first and second integrals in the right hand side.

Now noting the expression for $F_1$ from (27), $J_1$ can be expressed as,

(61) $$J_1 = [(2\pi)^{\frac{1}{2}}\tau]^{-1} \int_{T_a} G_\tau(s, x) \, dx_s,$$

where

(62) $$\begin{aligned} &G_\tau(s, x) \\ &\quad = g_s^{-\frac{1}{2}} \exp \cdot (-\bar{x}_s^2/2\tau^2) \cdot L_1 \cdot \{ \int_{I_{e,s}} F_2 \, d\bar{X}_{N-n(s)} - \int_{I''_{e,s}} F_2 \, d\bar{X}_{N-n(s)}\}. \end{aligned}$$

As $\tau \to \infty$, $F_2 \to f_s$. In the integrals within the curly bracket in the right hand side of (62), the intervals $I_{e,s}$, $I''_{e,s}$ are each of finite length $= 2v(s, x)$. Also from (27), since $n(s) \geqq 1$, for all $\tau$ say $\geqq 1$, we have

$$F_2 \leqq \{[N - n(s)]/(2\pi)N\}^{\frac{1}{2}}[2n(s)]^{\frac{1}{2}} = M_2, \quad \text{say.}$$

Thus the integrand in these integrals are bounded uniformly in $\tau$ for $\tau \geqq 1$, by $M_2$ which has finite integrals on $I_{e,s}$ and $I''_{e,s}$. Hence by the dominated convergence theorem, we can take limits under the integral sign. We thus have by (54),

(63) $$\begin{aligned} \lim_{\tau\to\infty} \int_{I_{e,s}} F_2 \, d\bar{X}_{N-n(s)} &= \int_{I''_{e,s}} f_s \, d\bar{X}_{N-n(s)} = U(s, x); \\ \lim_{\tau\to\infty} \int_{I''_{e,s}} F_2 \, d\bar{X}_{N-n(s)} &= \int_{I''_{e,s}} f_s \, d\bar{X}_{N-n(s)} = U''(s, x). \end{aligned}$$

NOTE. The results in (63) also remain valid if the length $2v(s, x)$ of the intervals $I_{e,s}$ and $I''_{e,s}$ were infinite, as in that case the integrals in the left hand side and right hand side of (63) all become equal to unity. Since $F_2$ and $f_s$ are probability densities, these results can also be proved alternatively, by a simple application of the Helly-Bray theorem, see for instance page 182 of *Probability Theory* by Loève (1960).

Hence, since $g_s \to 1$ as $\tau \to \infty$, at every point of $R_{n(s)}$,

$$\text{(64)} \qquad \lim_{\tau\to\infty} G_{\tau,s} = L_1[U(s, x) - U''(s, x)] = \text{integrand in the right hand side of (59).}$$

Next we can also take the limit of the integral in the right hand side of (61) under the integral sign. The integral of $F_2$ wrt $\bar{X}_{N-n(s)}$ on any interval $\leqq 1$. Similarly for all $\tau$, $g_s > 1$. Moreover

$$L_1 = L[x_i, i \varepsilon s \mid \bar{x}_s] \leqq (2\pi)^{-[n(s)-1]/2} = M_1, \quad \text{say.}$$

Hence $G_\tau(s, x)$ is bounded in absolute magnitude, uniformly in $\tau$, by $M_1$. Since the set $T_a$ is of finite measure, $M_1$ has a finite integral on $T_a$, $= M_1 \cdot (2a)^{n(s)}$.

Hence, again by the dominated convergence theorem, we can take the limit under the integral sign. We thus get from (64) and (59),

$$\text{(65)} \qquad \lim_{\tau\to\infty} \int_{T_a} G_\tau(s, x)\, dx_s = k.$$

It follows from (65), that we can find a value $\tau_0$ of $\tau$ such that

$$\int G_\tau(s, x)\, dx_s \geqq k/2, \quad \text{for all} \quad \tau \geqq \tau_0.$$

Hence by (61),

$$\text{(66)} \qquad J_1 \geqq (2\pi)^{-\frac{1}{2}} \cdot k/2\tau \quad \text{for all} \quad \tau \geqq \tau_0.$$

We turn now to the second integral in the right hand side of (60). By virtue of the maximizing property of the Bayes solution, the integrand is everywhere reduced, if we replace the interval $I''_{e,s}$ by the Bayes interval $I_{b,s}$. The resulting integrand is non-positive for at all points of $R_{n(s)}$. Hence we can extend the integration from the set $T_a^c$ to the whole space $R_{n(s)}$. We thus get,

$$\begin{aligned} J_2 &\geqq \int_{T_a^c} L_1 \cdot F_1\, dx_s\{ \int_{I_{e,s}} F_2\, d\bar{X}_{N-n(s)} - \int_{I_{b,s}} F_2\, d\bar{X}_{N-n(s)}\} \\ \text{(67)} \qquad &\geqq \int_{R_{n(s)}} L_1 \cdot F_1\, dx_s\{ \int_{I_{e,s}} F_2\, d\bar{X}_{N-n(s)} - \int_{I_{b,s}} F_2\, d\bar{X}_{N-n(s)}\} \\ &= A_{\tau,s} - B_{\tau.s}, \qquad \text{by (51) and (32).} \end{aligned}$$

Adding up (67) and (66), we have from (60),

$$\text{(68)} \qquad A_{\tau,s} - A''_{\tau,s} \geqq (2\pi)^{-\frac{1}{2}} \cdot k/2\tau + A_{\tau,s} - B_{\tau,s}.$$

(68) holds only for the particular sample $s$ for which the alternative (ii) stated below equation (57), is assumed to hold. To distinguish this sample from other samples, we shall denote it by $s'$ and write (68) as

$$\text{(69)} \qquad A_{\tau,s'} - A''_{\tau,s'} \geqq (2\pi)^{-\frac{1}{2}} \cdot k/2\tau + A_{\tau,s'} - B_{\tau,s'}.$$

By the property of the Bayes solution, for every other sample $s \varepsilon \bar{S}$,

$$A''_{\tau,s} \leqq B_{\tau,s},$$

and therefore

$$A_{\tau,s} - A''_{\tau,s} \geqq A_{\tau,s} - B_{\tau,s}. \tag{70}$$

Multiply both sides of (69) by $p(s')$ and both sides of (70) by $p(s)$ and sum up over all samples $s \varepsilon \bar{S}$. Then denoting the total conditional (for given $\theta$) and unconditional expected inclusion probabilities for the confidence intervals $[e_1''(s, x), e_2''(s, x)]$, by $A_\theta''$, $A_\tau''$, respectively, we have as in (34),

$$\begin{aligned} A_\theta'' &= \textstyle\sum_{s\varepsilon\bar{S}} p(s) A''_{\theta,s}, \\ A_\tau'' &= \textstyle\sum_{s\varepsilon\bar{S}} p(s) A''_{\tau,s}. \end{aligned} \tag{71}$$

Then by summing up (69) and (70) over all samples $s$, and then using (48), we have,

$$\begin{aligned} A_\tau - A_\tau'' &\geqq p(s')(2\pi)^{-\frac{1}{2}} \cdot k/2\tau + A_\tau - B_\tau \\ &> p(s')(2\pi)^{-\frac{1}{2}} \cdot k/2\tau - 1/2\tau^2, \quad \text{for all} \quad \tau \geqq \tau_0. \end{aligned} \tag{72}$$

Since $s' \varepsilon \bar{S}$, $p(s') > 0$ by definition of $\bar{S}$. Also $k > 0$ by assumption. Hence the right hand side of (72) can be made $>0$ by making $\tau$ sufficiently large. But this contradicts the inequality (13). (13) implies that $A_\theta'' \geqq A_\theta$ for all $\theta$. Hence, we must have, i.e.,

$$\begin{aligned} A_\tau'' &\geqq A_\tau; \\ A_\tau - A_\tau'' &\leqq 0 \quad \text{for all} \quad \tau. \end{aligned} \tag{73}$$

It follows from the contradiction, that the alternative (ii) stated below equation (57), cannot hold for any sample $s \varepsilon \bar{S}$. Hence, for every sample $s \varepsilon \bar{S}$, the set of points $x \varepsilon R_N$, on which both $v(s, x) > 0$ and also $h(s, x) = \bar{e}(s, x) - \bar{x}_s \neq 0$, must be a null $(\mu_N)$ subset of $R_N$. As the total number of samples $s$ is finite, it follows that the set $E$ of $R_N$ is also a null $(\mu_N)$ set, where $E$ is defined as,

(74) $x \varepsilon E$, if and only if, for at least one $s \varepsilon \bar{S}$, both the following inequalities hold, viz.:

(a) $v(s, x) > 0$,

(b) $h(s, x) = \bar{e}(s, x) - \bar{x}_s \neq 0$.

We now complete the proof of Theorem 3.1 by returning to equation (52). By considering the term in curly brackets, it is seen that the integrand in the right hand side of (52) vanishes at every point $x \varepsilon R_N$, at which $v(s, x) = 0$. It also vanishes at any point at which the intervals $I_{e,s}$ and $I''_{e,s}$ become identical, i.e. by (55) and (56) at any point $x$ at which $h(s, x) = 0$. By (74), the set of all points $x \varepsilon R_N$ at which both inequalities (a) and (b) in (74) hold for any one sample $s \varepsilon \bar{S}$, is a null $(\mu_N)$ subset of $R_N$. Since $v(s, x)$ and $h(s, x)$ depend on $x$ only through those $x_i$ for which $i \varepsilon s$ it follows that the subset $E_s$ of $R_{n(s)}$ on which both the inequalities (a) and (b) in (74) hold for a particular $s$, is a null $(\mu_{n(s)})$ subset of $R_{n(s)}$. It therefore follows from (52), that $A_{\tau,s} - A''_{\tau,s} = 0$ for every $s \varepsilon \bar{S}$. Therefore, summing over all $s \varepsilon \bar{S}$ we get

$$A_\tau - A_\tau'' = 0. \tag{75}$$

But, for every $\theta$, the prior density is positive ($>0$) for all $x \varepsilon R_N$. Hence since the strict inequality in (13) holds on a non-null set of $R_N$, we must have $A_\theta'' > A_\theta$ for every $\theta$, $-\infty < \theta < \infty$, and therefore,

$$A_\tau'' > A_\tau. \tag{76}$$

It thus follows from (75), that the strict inequality in (13) and hence in (8) cannot hold on a non-null subset of $R_N$. Thus no set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$ can exist, which satisfies (i) and (ii) of Definition 2.3. This proves that the given set of confidence intervals $[e_1(s, x), e_2(s, x)]$ is weakly admissible. This completes the proof of Theorem 3.1.

During the course of the proof, we have also proved the following result which will be required for proving the further extentions of Theorem 3.1. If $[e_1''(s, x), e_2''(s, x)]$ is a set of confidence intervals satisfying $e_2''(s, x) - e_1''(s, x) = 2v(s, x)$, and also equation (13) except for the requirement that (13) holds on a non-null subset of $R_N$, and $E$ is the subset of $R_N$ defined by (74), then,

COROLLARY 3.1. *$E$ is a null $(\mu_N)$ subset of $R_N$.*

REMARK 3.1. It will be seen that in the proof of Theorem 3.1 we have used only the following properties of the function $v(s, x)$ defined by (5) and (6), viz.,

(i) $v(s, x)$ is non-negative;

(ii) $v(s, x)$ is an estimate (Definition 2.1) i.e. it depends on $x$ through only those $x_i$ for which $i \varepsilon s$;

(iii) under the assumed prior distribution on $R_N$, for each $s \varepsilon \bar{S}$, $v(s, x)$ is distributed independently of both $\theta$ and $\bar{x}_s$.

Let $v^*(s, x)$ be any other function defined on $S \times R$ which also satisfied the conditions (i), (ii) and (iii) mentioned above. Then it follows that Theorem 3.1 and in particular Corollary 3.1 continue to hold for the set of confidence intervals obtained by substituting $v^*(s, x)$ for $v(s, x)$ in (6). This remark will be relevant for the proof of the theorem contained in the next section.

**4. Weak admissibility under constraints.** The argument in the present section runs closely parallel to that in a previous paper [(1965) III, Sections 4 and 5] in which the admissibility of a certain estimate was proved. We consider the hyperplanes in $R_N$ obtained by assigning fixed values to some $k$ of the variates. Let $Q_{N-k}^\alpha$ be the hyperplane in which, say the last $k$ variates $x_{N-k+t}$, $t = 1, 2, \cdots, k$, have fixed values $\alpha_{N-k+t}$, respectively. Let $\bar{S}_k$ be the subset of $\bar{S}$, consisting of all those samples which contains each of the last $k$ units $u_{N-k+t}$, $t = 1, 2, \cdots, k$, i.e. $s \varepsilon \bar{S}_k$, if, and only if, $u_{N-k+t} \varepsilon s$ for all $t = 1, 2, \cdots, k$ and $s \varepsilon \bar{S}$. We denote by $\bar{S}_k \cdot \bar{S}_{e_1, e_2, x}$ the intersection of the set $\bar{S}_k$ with the set $\bar{S}_{e_1', e_2, x}$ on which (2) holds. The intersections of $\bar{S}_k$ with the sets $\bar{S}_{e_1', e_2', x}$ on which (3) holds, and $S_{e_1'', e_2'', x}$ on which (11) holds, are denoted similarly. Now suppose that for $x \varepsilon Q^{\alpha}_{N-k}$, and $s \varepsilon \bar{S}_k$, estimates $e_1'(s, x)$, $e_2'(s, x)$ exist such that,

$$\sum_{s \varepsilon \bar{S}_k \cdot \bar{S}_{e_1', e_2', x}} p(s) \geqq \sum_{s \varepsilon \bar{S}_k \cdot \bar{S}_{e_1, e_2, x}} p(s) \tag{77}$$

holds for almost all $(\mu_{N-k}) x \varepsilon Q^{\alpha}_{N-k}$, and let $\bar{e}(s, x)$ be as in (9).

Let $E^{\alpha}_{N-k}$ be the subset of $Q^{\alpha}_{N-k}$ defined by

$$x \varepsilon E^{\alpha}_{N-k}, \quad \text{if, and only if,} \quad x \varepsilon Q^{\alpha}_{N-k} \tag{78}$$

and,

(a) $h(s, x) = \bar{e}(s, x) - \bar{x}_s \neq 0$,

(b) $v(s, x) > 0$,

both hold for at least one $s \varepsilon \bar{S}_k$. We now prove,

THEOREM 4.1. *If $e_1'(s, x)$, $e_2'(s, x)$ are estimates satisfying* (77), *and $E^{\alpha}_{N-k}$ is the subset of $Q^{\alpha}_{N-k}$ defined by* (78), *then $E^{\alpha}_{N-k}$ is a null $(\mu_{N-k})$ set.*

OUTLINE OF THE PROOF. As the following proof is rather long, we shall first give a brief outline. Let $Q^{\alpha'}_{N-k}$ be the hyperplane corresponding to another set of values of the last $k$ variates $x_i$, $i = N - k + 1, \cdots, N$. We establish suitably a 1–1 correspondence between the points $x' \varepsilon Q^{\alpha'}_{N-k}$ and the points $x \varepsilon Q^{\alpha}_{N-k}$. Then defining $e_1''(s, x)$, $e_2''(s, x)$ as in (9), for $x \varepsilon Q^{\alpha}_{N-k}$, the definitions of $e_1''(s, x)$, $e_2''(s, x)$ and $v(s, x)$ for $s \varepsilon \bar{S}_k$, are extended to the hyperplanes $Q^{\alpha'}_{N-k}$ by fixing the values of these estimates at the point $x' \varepsilon Q^{\alpha'}_{N-k}$ in terms of their values at the point $x \varepsilon Q^{\alpha}_{N-k}$. The fixation is done in such a way, that corresponding to the non-null $(\mu_{N-k})$ subset of $Q^{\alpha}_{N-k}$, on which (a) and (b) in (78) hold, there exists a non-null $(\mu_{N-k})$ subset in each hyperplane $Q^{\alpha'}_{N-k}$ on which also both (a) and (b) in (78) hold. Thus, by this construction, there exists a non-null $(\mu_N)$ subset $E$ of $R_N$, on which the inequalities (a) and (b) in (78) hold at each point for at least one $s \varepsilon \bar{S}_k$. In this process the function obtained by extending the estimate $v(s, x)$ ceases to be the sample standard deviation. We therefore denote it by $v^*(s, x)$. In place of the set of confidence intervals $[e_1(s, x), e_2(s, x)]$, the extended set of confidence intervals is $[e_1^*(s, x), e_2^*(s, x)]$, where $e_1^*(s, x) = \bar{x}_s - v^*(s, x)$, and $e_2^*(s, x) = \bar{x}_s + v^*(s, x)$. For samples $s \varepsilon (\bar{S} - \bar{S}_k)$, the definitions of $e_1''(s, x)$, $e_2''(s, x)$ and $v^*(s, x)$ are extended simply by putting them respectively equal to $e_1(s, x)$, $e_2(s, x)$ and $v(s, x)$. As a result of this construction the inequality corresponding to (13) holds for the sets of confidence intervals $[e_1''(s, x), e_2''(s, x)]$ and $[e_1^*(s, x), e_2^*(s, x)]$, and the set $E$ of $R_N$ on which both inequalities (a) and (b) in (78) hold at each point for at least one $s \varepsilon \bar{S}$, is non-null $(\mu_N)$. The contradiction is then established by showing that the new function $v^*(s, x)$ satisfies all the conditions (i), (ii) and (iii) in Remark 3.1, and therefore by Corollary 3.1 read with Remark 3.1 the set $E$ cannot be a non-null set. We now give the detailed proof.

PROOF. For $x \varepsilon Q^{\alpha}_{N-k}$, we define $e_1''(s, x)$ and $e_2''(s, x)$ as in (9). Then from (12) and (77) we get,

$$\sum_{s\varepsilon \bar{S}_k \cdot S_{e''_1, e''_2, x}} p(s) \geqq \sum_{s\varepsilon \bar{S}_k \cdot S_{e_1, e_2, x}} p(s) \tag{79}$$

for almost all $(\mu_{N-k})x \;\varepsilon\; Q^{\alpha}_{N-k}$. Now in (77) and (79), the estimates $e_1'(s, x)$, $e_2'(s, x)$, and hence $e_1''(s, x)$, $e_2''(s, x)$ and $\bar{e}(s, x)$ are defined only for samples $s \;\varepsilon\; \bar{S}_k$ and points $x \;\varepsilon\; Q^{\alpha}_{N-k}$. We now extend their definitions to other points $x \;\varepsilon\; R_N$ and to samples $s \not\varepsilon\; \bar{S}_k$ as follows.

Let $Q^{\alpha'}_{N-k}$ be the hyperplane $\subset R_N$ given by $x_{N-k+t} = \alpha'_{N-k+t}$, $t = 1, 2, \cdots, k$. We establish a 1–1 correspondence between the points $x \;\varepsilon\; Q^{\alpha}_{N-k}$ and $x' \;\varepsilon\; Q^{\alpha'}_{N-k}$, by putting

$$x_r' = x_r + a, \qquad r = 1, 2, \cdots, N - k. \tag{80}$$

The constant '$a$' can be fixed (uniquely) so that,

$$\bar{X}_N' - \bar{x}_s' = \bar{X}_N - \bar{x}_s, \tag{81}$$

for all $x \;\varepsilon\; Q^{\alpha}_{N-k}$, $s \;\varepsilon\; \bar{S}_k$, where

$$\bar{x}_s' = [n(s)]^{-1} \sum_{i\varepsilon s} x_i', \qquad \bar{X}_N' = N^{-1} \sum_{i=1}^{N} x_i'. \tag{82}$$

Such a constant always exists. Indeed let

$$\bar{\alpha}' = k^{-1} \sum_{r=N-k+1}^{N} \alpha_r', \qquad \bar{\alpha} = k^{-1} \sum_{r=N-k+1}^{N} \alpha_r . \tag{83}$$

We have, since $n(s) \geqq k$,

$$\bar{x}_s' = [n(s)]^{-1}[k\bar{\alpha}' + n(s)\bar{x}_s - k\bar{\alpha} + (n(s) - k)a],$$

and

$$\bar{X}_N' = N^{-1}[k\bar{\alpha}' + N\bar{X}_N - k\bar{\alpha} + (N - k)a].$$

Hence, $\bar{X}_N' - \bar{x}_s' = [\bar{X}_N - \bar{x}_s - k([n(s)]^{-1} - N^{-1})(\bar{\alpha}' - \bar{\alpha} - a)]$, so that (81) is satisfied if

$$a = \bar{\alpha}' - \bar{\alpha}. \tag{84}$$

We introduce now a new estimate $v^*(s, x)$ defined as follows: for $x \;\varepsilon\; Q^{\alpha}_{N-k}$ and $s \;\varepsilon\; \bar{S}_k$,

$$v^*(s, x) = v(s, x), \tag{85}$$

and for $x' \;\varepsilon\; Q^{\alpha'}_{N-k}$, $s \;\varepsilon\; \bar{S}_k$,

$$v^*(s, x') = v(s, x). \tag{86}$$

Since by (85) and (86) $v^*(s, x)$ is defined for every hyperplane $Q^{\alpha'}_{N-k}$, it is defined for all $x \;\varepsilon\; R_N$, for all $s \;\varepsilon\; \bar{S}_k$. In place of the original set of confidence intervals $[e_1(s, x), e_2(s, x)]$, given by (6), we consider a new set $[e_1^*(s, x), e_2^*(s, x)]$, where

$$\begin{aligned} e_1^*(s, x) &= \bar{x}_s - v^*(s, x), \\ e_2^*(s, x) &= \bar{x}_s + v^*(s, x). \end{aligned} \tag{87}$$

We denote by $\bar{S}_{e_1^*,e_2^*,x}$, the subset of $\bar{S}$, for each $x \varepsilon R_N$, for which

$$e_1^*(s, x) \leqq \bar{X}_N \leqq e_2^*(s, x) \quad \text{holds.} \tag{88}$$

We now extend the definitions of $\bar{e}(s, x)$, $e_1''(s, x)$ and $e_2''(s, x)$ to other hyperplanes $Q_{N-k}^{\alpha'}$ by putting

$$\begin{aligned} \bar{e}(s, x') &= \bar{e}(s, x) + \bar{x}_s' - \bar{x}_s ; \\ e_1''(s, x') &= \bar{e}(s, x') - v^*(s, x'); \\ e_2''(s, x') &= \bar{e}(s, x') + v^*(s, x'). \end{aligned} \tag{89}$$

Since for $x \varepsilon Q_{N-k}^{\alpha}$, $e_1^*(s, x)$ and $e_2^*(s, x)$ coincide with $e_1(s, x)$ and $e_2(s, x)$, $\bar{S}_{e_1^*,e_2^*,x} = \bar{S}_{e_1,e_2,x}$ for every $x \varepsilon Q_{N-k}^{\alpha}$. Hence (79) implies that

$$\sum\nolimits_{\bar{S}\varepsilon_k \cdot \bar{S}_{e''_1,e''_2,x}} p(s) \geqq \sum\nolimits_{s\varepsilon\bar{S}_k \cdot \bar{S}_{e_1^*,e_2^*,x}} p(s) \tag{90}$$

for almost all $(\mu_{N-k})$ $x \varepsilon Q_{N-k}^{\alpha}$.

Moreover there exists a non-null $(\mu_{N-k})$ subset of $Q_{N-k}^{\alpha}$ viz., $E_{N-k}^{\alpha}$, such that at each point of this subset, both the following inequalities hold for at least one $s \varepsilon \bar{S}_k$, viz.:

$$\begin{aligned} &\text{(a)} \quad \bar{e}(s, x) = h(s, x) - x_s \neq 0; \\ &\text{(b)} \quad v^*(s, x) > 0, \quad \text{for} \quad x \varepsilon E_{N-k}^{\alpha}. \end{aligned} \tag{91}$$

Now (90) and (91) are easily seen to hold for every other hyperplane $Q_{N-k}^{\alpha'}$. Indeed, for any point $x' \varepsilon Q_{N-k}^{\alpha'}$, by (86) and (87),

$$\begin{aligned} e_1^*(s, x') &= \bar{x}_s' - v^*(s, x') = \bar{x}_s - v^*(s, x) + \bar{x}_s' - \bar{x}_s); \\ e_2^*(s, x') &= \bar{x}_s' + v^*(s, x') = \bar{x}_s + v^*(s, x) + \bar{x}_s' - \bar{x}_s). \end{aligned}$$

Moreover, by (81), $\bar{x}_N' = \bar{x}_N + (\bar{x}_s' - \bar{x}_s)$, and by (89), (86) and (9), $e_1''(s, x') = e_1''(s, x) + (\bar{x}_s' - \bar{x}_s)$ and $e_2''(s, x') = e_2''(s, x) + (\bar{x}_s' - \bar{x}_s)$. Thus for every sample $s \varepsilon \bar{S}_k$, at every point $x' \varepsilon Q_{-Nk}^{\alpha'}$, $e_1^*(s, x') \leqq \bar{X}_N' \leqq e_2^*(s, x')$ holds, if and only if, at the point $x \varepsilon Q_{N-k}^{\alpha}$ corresponding to $x'$ according to (80), $e_1^*(s, x) \leqq \bar{X}_N \leqq e_2^*(s, x)$. Similarly, $e_1''(s, x') \leqq \bar{X}_N' \leqq e_2''(s, x')$ holds, if and only if, $e_1''(s, x) \leqq \bar{X}_N \leqq e_2''(s, x)$ holds. Hence,

$$\begin{aligned} \bar{S}_k \cdot \bar{S}_{e_1^*,e_2^*,x'} &= \bar{S}_k \cdot \bar{S}_{e_1^*,e_2^*,x}, \\ \bar{S}_k \cdot \bar{S}_{e_1'',e_2'',x'} &= \bar{S}_k \cdot \bar{S}_{e_1'',e_2'',x}. \end{aligned} \tag{92}$$

(92) implies that (90) holds for almost all $(\mu_N)$ $x \varepsilon R_N$. Moreover since every hyperplane $Q_{N-k}^{\alpha'}$ contains a non-null $(\mu_{N-k})$ subset on which (91) holds, the subset of $R_N$ for which (91) holds is non-null $(\mu_N)$.

Consider samples $s \not\varepsilon \bar{S}_k$, i.e. $s \varepsilon (\bar{S} - \bar{S}_k)$. We have not so far defined the values of $\bar{e}(s, x)$, $e_1''(s, x)$, $e_2''(s, x)$, and $v^*(s, x)$ for $s \varepsilon (\bar{S} - \bar{S}_k)$. We now put, for $s \varepsilon (\bar{S} - \bar{S}_k)$ and all $x \varepsilon R_N$,

$$\begin{aligned} \bar{e}(s, x) &= \bar{x}_s ; \\ v^*(s, x) &= v(s, x); \\ e_1''(s, x) &= \bar{e}(s, x) - v^*(s, x); \\ e_2''(s, x) &= \bar{e}(s, x) + v^*(s, x); \end{aligned} \tag{93}$$

and $e_1^*(s, x)$ and $e_2^*(s, x)$ as given by (87). It follows that for $s \varepsilon (\bar{S} - \bar{S}_k)$, and all $x \varepsilon R_N$,

$$e_1''(s, x) = e_1^*(s, x); \qquad e_2''(s, x) = e_2^*(s, x).$$

Hence, denoting by $(\bar{S} - \bar{S}_k)\cdot \bar{S}_{e_1^*, e_2^*, x}$ and by $(\bar{S} - \bar{S}_k)\cdot \bar{S}_{e_1'', e_2'', x}$ the intersection sets of $(\bar{S} - \bar{S}_k)$ with $\bar{S}_{e_1^*, e_2^*, x}$ and $\bar{S}_{e_1'', e_2'', x}$ respectively, we have $(\bar{S} - \bar{S}_k)\cdot \bar{S}_{e_1'', e_2'', x} = (\bar{S} - \bar{S}_k)\cdot \bar{S}_{e_1^*, e_2^*, x}$. Therefore,

$$\sum_{s \varepsilon (\bar{S} - \bar{S}_k)\cdot \bar{S}_{e_1'', e_2'', x}} p(s) = \sum_{s \varepsilon (\bar{S} - \bar{S}_k)\cdot \bar{S}_{e_1^*, e_2^*, x}} p(s). \tag{94}$$

Combining (90), (92) and (94), we obtain that

$$\sum_{s \varepsilon \bar{S}_{e_1'', e_2'', x}} p(s) \geqq \sum_{s \varepsilon \bar{S}_{e_1^*, e_2^*, x}} p(s) \quad \text{for almost all} \quad x \varepsilon R_N, \tag{95}$$

and moreover both the inequalities viz.

$$\text{(a)} \quad h(s, x) = \bar{e}(s, x) - \bar{x}_s \neq 0 \quad \text{and} \quad \text{(b)} \quad v^*(s, x) > 0 \tag{96}$$

holds for at least one $s \varepsilon \bar{S}$, on a non-null $(\mu_N)$ subset $E$ of $R_N$.

We now complete the proof by proving in Lemma 4.1 below that the function $v^*(s, x)$ defined by (85), (86) and (93) satisfies the conditions (i), (ii) and (iii) of Remark 3.1. Assuming this lemma for the moment, by Corollary 3.1 read with Remark 3.1 the set $E$ of $R_N$ on which at each point, both the inequalities in (96) hold for at least one $s \varepsilon \bar{S}$, must be a null $(\mu_N)$ set. This establishes the contradiction and proves that our original assumption, that the subset $E_{N-k}^{\alpha}$ of $Q_{N-k}^{\alpha}$, defined by (78), has positive measure $(\mu_{N-k})$ must be false.

LEMMA 4.1. *The function* $v^*(s, x)$ *defined by* (85), (86) *and* (93) *satisfies conditions* (i), (ii) *and* (iii) *of Remark* 3.1.

PROOF. We prove the lemma by obtaining an explicit expression for $v^*(s, x)$ in terms of the coordinates of the point $x$. Let $s \varepsilon \bar{S}_k$, and let $x' \varepsilon Q_{N-k}^{\alpha'}$ be the point, which corresponds by (80) to the point $x \varepsilon Q_{N-k}^{\alpha}$. We then have, by (80),

$$x_r' = x_r + a, \quad \text{for} \quad r \varepsilon s, \quad r \leqq N - k. \tag{97}$$

Also by (83) and (84),

$$\begin{aligned} \bar{x}_s' &= [n(s)]^{-1}\{\textstyle\sum_{i \varepsilon s, i \leqq N-k} x_i' + \sum_{r=N-k+1}^{N} \alpha_r'\} \\ &= [n(s)]^{-1}\{\textstyle\sum_{i \varepsilon s, i \leqq N-k} x_i' + [n(s) - k]\cdot a + k\bar{\alpha}'\} \\ &= \bar{x}_s + a, \end{aligned} \tag{98}$$

and

$$\bar{\alpha}' - \bar{x}_s' = \bar{\alpha} - \bar{x}_s . \tag{99}$$

Hence by (86), (6) and (5),

$$
\begin{aligned}
[v^*(s, x')]^2 &= [v(s, x)]^2 \\
(100) \qquad &= [k_s^2/n(s)]\{\textstyle\sum_{i\varepsilon s, i \leqq N-k} (x_i - \bar{x}_s)^2 + \sum_{i=N-k+1}^{N} (\alpha_i - \bar{x}_s)^2\} \\
&= [k_s^2/n(s)]\{\textstyle\sum_{i\varepsilon s, i \leqq N-k} (x_i' - \bar{x}_s')^2 + \sum_{i=N-k+1}^{N} (\alpha_i - \bar{\alpha})^2 \\
&\quad + k\cdot(\bar{\alpha}' - x_s')^2\}
\end{aligned}
$$

after a little reduction, using (97), (98) and (99). Now,

$$
\begin{aligned}
(101) \quad [v(s, x')]^2 = [k_s^2/n(s)]\{&\textstyle\sum_{i\varepsilon s, i \leqq N-k} (\bar{x}_1' - \bar{x}_s')^2 \\
&+ \textstyle\sum_{i=N-k+1}^{N} (\bar{\alpha}_i - \bar{\alpha}')^2 + k\cdot(\bar{\alpha} - \bar{x}_s')^2\}.
\end{aligned}
$$

Combining (100) and (101), we get

$$
\begin{aligned}
(102) \quad [v^*(s, x')]^2 = [v(s, x')]^2 &+ [k_s^2/n(s)] \\
&\cdot\{\textstyle\sum_{i=N-k+1}^{N} (\alpha_i - \bar{\alpha})^2 - \sum_{i=N-k+1}^{N} (\alpha_i' - \bar{\alpha}')^2\}.
\end{aligned}
$$

In the right hand side of (102), the first term in the curly bracket is a known constant. The second term depends on $x'$ only through $\alpha_i'$, i.e. only through those $x_i'$ for which $i \varepsilon s$. $v^*(s, x)$ is thus an estimate. It is also necessarily non-negative as $v^*(s, x') = v(s, x)$. Further under the assumed prior distribution on $R_N$, the term $\sum_{i=N-k+1}^{N} (\alpha_i' - \bar{\alpha}')^2$ is distributed independently of $\theta$ and $\bar{\alpha}'$, and since its distribution is independent of $x_i'$ for all $i \leqq N - k$, it is distributed independently of $\bar{x}_s'$ also. As these conditions are obviously satisfied by the first term $[v(s, x')]^2$ in the right hand side of (102), it follows that the function $v^*(s, x')$ for $s \varepsilon \bar{S}_k$, satisfies all the conditions (i), (ii) and (iii) of Remark 3.1. Also for $s \varepsilon (\bar{S} - \bar{S}_k)$, these conditions are obviously satisfied since for such $s$, $v^*(s, x) = v(s, x)$ for all $x \varepsilon R_N$. Thus the Lemma 4.1 is proved.

**5. Strict admissibility.** We now come to the final part of the argument. The argument here is similar to that in the proof of Theorem 5.1 of a previous paper [(1965), III]. The proof was however given there for a fixed sample size design, while here we have a varying sample size design. The proof being rather long, we shall first outline its main steps.

Outline of the Proof. Suppose the set of confidence intervals $[e_1(s, x), e_2(s, x)]$, defined by (6), is not strictly admissible. Then there exists a set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$, satisfying Definition 2.2. We define $e_1''(s, x)$, $e_2''(s, x)$ and $\bar{e}(s, x)$ as in (9). The strict ienquality in (13) must now hold at least at one point of $R_N$. Let $E$ be the set of all those points $x \varepsilon R_N$, at which $h(s, x) = \bar{e}(s, x) - \bar{x}_s \neq 0$ for at least one $s \varepsilon \bar{S}$. We have to show that the set $E$ must be empty. We first show, using the definition of strict admissibility, that if at a point $a$, for a particular sample $s_0$, $h(s_0, a) \neq 0$, then $v(s_0, a)$ cannot be $= 0$. Hence if $E$ is not empty, there exists at least one point $a \varepsilon R_N$, and a sample $s_0 \varepsilon \bar{S}$, such that (a) $h(s_0, a) \neq 0$ and (b) $v(s_0, a) > 0$. Let $m$ be the sample size of $s_0$. Then starting from the point $a$, we determine a set $E_{N-m}^a$ of infinite measure $(\mu_{N-m})$, such that at each point of $E_{N-m}^a$, both the inequalities, (a)

$h(s, x) \neq 0$ and (b) $v(s, x) > 0$ hold for at least one $s \varepsilon \bar{S}$, such that $s \neq s_0$. By partitioning the set $E^a_{N-m}$ according to samples $s \varepsilon \bar{S}$, we obtain a set $L^{a,s_1}_{N-m}$ with positive measure $(\mu_{N-m})$, such that for each $x \varepsilon L^{a,s_1}_{N-m}$, the inequalities (a) and (b) hold for a particular sample $s_1$. Then for each point $x \varepsilon L^{a,s_1}_{N-m}$, we determine as before a set on which both inequalities (a) and (b) hold for at least one $s \varepsilon \bar{S}$, $s \neq s_1$. All these sets together determine a set $E^a_{N-k}$, with $k < m$, such that $E^a_{N-k}$ has infinite measure $(\mu_{N-k})$, and both the inequalities (a) and (b) hold for each $x \varepsilon E^a_{N-k}$, for at least one $s \varepsilon \bar{S}$. The set $E^a_{N-k}$ is again partitioned by samples $s$, and the same process repeated. It is shown that the process can terminate only when we reach, either $(A')$ a set $E_N$ of infinite measure $(\mu_N)$, such that the inequalities (a) and (b) hold for at least one $s \varepsilon \bar{S}$, for each $x \varepsilon E_N$, or, (B$'$) we reach a subset $E^a_{N-j}$ of a hyperplane $P^a_{N-j}$, such that $E^a_{N-j}$ has infinite measure $(\mu_{N-j})$, and the inequalities (a) and (b) hold for each $x \varepsilon E^a_{N-j}$, for at least one $s \varepsilon \bar{S}$. But (A$'$) contradicts Theorem 3.1 and (B$'$) contradicts Theorem 4.1. The set $E$ must therefore be empty. Strict admissibility follows from this.

We shall now state and give the detailed proof of our main result, viz.

THEOREM 5.1. *The set of confidence intervals* $[e_1(s, x), e_2(s, x)]$, *defined by* (6), *is strictly admissible.*

PROOF. Suppose it is not strictly admissible; then by Definition 2.2 there exists another set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$, satisfying (i) and (ii) of Definition 2.2. Then defining $e_1''(s, x)$ and $e_2''(s, x)$ as in (9), and proceeding as from (9) to (13), we obtain in place of (13),

$$\sum_{s \varepsilon \bar{S}_{e_1'', e_2'', x}} p(s) \geqq \sum_{s \varepsilon \bar{S}_{e_1, e_2, x}} p(s) \tag{103}$$

where the strict inequality holds for at least one $x \varepsilon R_N$.

Let $E$ be the subset of $R_N$ defined by

$$\text{(104)} \quad x \varepsilon E, \quad \text{if and only if,} \quad h(s, x) = \bar{e}(s, x) - \bar{x}_s \neq 0, \quad \text{for at least one} \quad s \varepsilon \bar{S}.$$

Here $\bar{e}(s, x)$ is as in (9). We prove the theorem by showing that the set $E$ must be empty. Suppose it is not empty. Then, there exists at least one point $x = a = (a_1, a_2, \cdots, a_N)$, and a sample $s_0 \varepsilon \bar{S}$, such that $h(s_0, a) = h_0 \neq 0$. We first show that $v(s_0, a)$ must be $>0$. For suppose $v(s_0, a) = 0$. Then, since $v(s_0, a)$ is a multiple of $v'(s_0, a)$ defined in (5), we must have,

$$a_i = \text{some constant } c \quad \text{for all} \quad i \varepsilon s_0.$$

Let $a'$ be the point obtained by equating all the remaining variates $x_i$, $i \not\varepsilon s_0$, to $c$, i.e., $a'$ is the point such that $x_i = c$, $i = 1, 2, \cdots, N$. Then at $a'$, $\bar{X}_N = c$ and $\bar{x}_s = c$ for every sample $s$. Hence for every sample $s \varepsilon \bar{S}$, $s \varepsilon \bar{S}_{e_1, e_2, a'}$. Now by the definition of an estimate (Definition 2.1), and by (9),

$$v(s_0, a') = v(s_0, a) = 0;$$

$$\bar{e}(s_0, a') = \bar{e}(s_0, a) = \bar{x}_{s_0}(a) + h(s_0, a) = c + h_0;$$
$$e_1''(s_0, a') = \bar{e}(s_0, a') - v(s_0, a') = c + h_0;$$
$$e_2''(s_0, a') = \bar{e}(s_0, a') + v(s_0, a') = c + h_0.$$

Since at $a'$, $\bar{X}_N = c$, and $h_0 \neq 0$, $\{e_1''(s_0, a') \leqq \bar{X}_N \leqq e_2''(s_0, a')\}$ does not hold. Hence $s_0 \not\in \bar{S}_{e_1'', e_2'', a'}$. But since $s_0 \varepsilon \bar{S}$, $p(s_0) > 0$. Therefore at the point $x = a'$, the right hand side of (103) would exceed its left hand side. The supposition that $v(s_0, a) = 0$ thus leads to a contradiction. Hence we must have $v(s_0, a) > 0$.

The result may be stated more generally as: if at any point $x' \varepsilon R_N$, for a particular sample $s_0 \varepsilon \bar{S}$, $h(s_0, x) \neq 0$, then (13) implies that,

(105) $$v(s_0, x') > 0.$$

Thus if $E$ is non-empty, there exists at least one point $a = (a_1, \cdots, a_N)$ and a sample $s_0$ such that

$$\text{(a)} \quad h(s_0, a) = h_0 \neq 0 \quad \text{and} \quad \text{(b)} \quad v(s_0, a) > 0.$$

Without loss of generality we may suppose the sample $s_0$ to consist of the first $m$ units $u_1, u_2, \cdots, u_m$. Consider the $(N - m)$ dimensional hyperplane $P_{N-m}^a$ as defined by,

(106) $$x \varepsilon P_{N-m}^a \quad \text{if, and only if,} \quad x_i = a_i, \quad i = 1, 2, \cdots, m.$$

For every $x \varepsilon P_{N-m}^a$,

$$\bar{x}_{s_0} = m^{-1}\sum_{i=1}^m a_i = \bar{a}_0, \quad \text{say;}$$
$$\bar{e}(s_0, x) = \bar{a}_0 + h_0;$$
$$v(s_0, x) = v(s_0, a);$$
$$\bar{X}_N = m\bar{a}_0/N + N^{-1}\sum_{i=m+1}^N x_i.$$

Hence for $x \varepsilon P_{N-m}^a$, $\bar{x}_{s_0} - v(s_0, x) \leqq \bar{X}_N \leqq \bar{x}_{s_0} + v(s_0, x)$, holds, if and only if,

(107) $$\bar{a}_0(1 - m/N) - v(s_0, a) \leqq N^{-1}\sum_{i=m+1}^N x_i \leqq \bar{a}_0(1 - m/N) + v(s_0, a).$$

Similarly for $x \varepsilon P_{N-m}^a$, $e_1''(s, x) = \bar{e}(s, x) - v(s, x) \leqq \bar{X}_N \leqq \bar{e}(s, x) + v(s, x) = e_2''(s, x)$ holds, if and only if,

(108) $$\bar{a}_0(1 - m/N) - v(s_0, a) + h_0 \leqq N^{-1}\sum_{i=m+1}^N x_i \leqq \bar{a}_0(1 - m/N) + v(s_0, a) + h_0.$$

We now determine a subset $T_{N-m}^a$ of $P_{N-m}^a$ such that, for $x \varepsilon T_{N-m}^a$, (107) holds, but (108) does not, as follows: $x \varepsilon T_{N-m}^a$, if and only if, $x \varepsilon P_{N-m}^a$, and,

if $h_0 > 0$,

(109) $$\bar{a}_0(1 - m/N) - v(s_0, a) \leqq N^{-1}\sum_{i=m+1}^N x_i < \min\{[\bar{a}_0(1 - m/N) + v(s_0, a)], [\bar{a}_0(1 - m/N) - v(s_0, a) + h_0]\};$$

if $h_0 < 0$,

$$\max\{[\bar{a}_0(1 - m/N) + v(s_0, a) + h_0], [\bar{a}_0(1 - m/N) - v(s_0, a)]\}$$
$$< N^{-1}\textstyle\sum_{i=m+1}^{N} x_i \leqq \bar{a}_0(1 - m/N) + v(s_0, a).$$

Since $h_0 \neq 0$, and $v(s_0, a) > 0$, $T_{N-m}^a$ is always determined and has infinite measure $(\mu_{N-m})$. Then by (109), for every point $x \,\varepsilon\, T_{N-m}^a$, $s_0 \,\varepsilon\, \bar{S}_{e_1,e_2,x}$, and $s_0 \not\varepsilon\, \bar{S}_{e_1'',e_2'',x}$; and hence at every such point, there must be at least one other sample $s \,\varepsilon\, \bar{S}$, for which $h(s, x) = \bar{e}(s, x) - \bar{x}_s \neq 0$ as otherwise at this point the right hand side of (103) will exceed its left hand side. Furthermore, by (105), for the sample $s$ for which $h(s, x) \neq 0$, we must have $v(s, x) > 0$.

Let $E_{N-m}^a$ be the subset of all the points $x \,\varepsilon\, P_{N-m}^a$, for which $h(s, x) \neq 0$ for at least one $s \,\varepsilon\, \bar{S}$. Obviously.

$$(110) \qquad T_{N-m}^a \subset E_{N-m}^a.$$

Since $T_{N-m}^a$ is of infinite measure $(\mu_{N-m})$, $E_{N-m}^a$ is also of infinite measure $(\mu_{N-m})$. We now partition the set $E_{N-m}^a$ into (not necessarily) disjoint subsets indexed by the samples $s \,\varepsilon\, \bar{S}$. Let for a specified sample $s$, $L_{N-m}^{a,s}$ be the subset consisting of all those points $x \,\varepsilon\, E_{N-m}^a$ for which $h(s, x) \neq 0$, i.e., $x \,\varepsilon\, L_{N-m}^{a,s}$ if, and only if, $x \,\varepsilon\, E_{N-m}^a$ and $h(s, x) \neq 0$. Then from the definition of $E_{N-m}^a$ it follows that,

$$(111) \qquad E_{N-m}^a = \bigcup_{s\varepsilon\bar{S}} L_{N-m}^{a,s}.$$

Since $E_{N-m}^a$ has infinite measure $(\mu_{N-m})$, there must be at least one non-null $(\mu_{N-m})$ set in the right hand side of (111).

Now there are two possible cases:

(A) For every non-null $(\mu_{N-m})$ set $L_{N-m}^{a,s}$ in the right hand side of (111), the sample $s \neq s_0$, contains all the units $u_1, u_2, \cdots, u_m$, with in addition some other units;

(B) There exists at least one non-null $(\mu_{N-m})$ set $L_{N-m}^{a,s}$ in the right hand side of (111), for which the sample $s$, contains only some $k$, $(0 \leqq k < m)$ out of the first $m$ units.

We shall first consider case (B), in which there exist one or more non-null sets $L_{N-m}^{a,s}$ in the right hand side of (111), for which the sample $s$ does not contain all the first $m$ units; if there are more than one such set we select one of them arbitrarily. Let $L_{N-m}^{a,s_1}$ be the set selected, and suppose the sample $s_1$ is of sample size $m_1$, and that it contains $k$, $(0 \leqq k < m)$ out of the first $m$ units, the remaining $(m_1 - k)$ units being from the last $(N - m)$ units $u_{m+1}, u_{m+2}, \cdots, u_N$. Then take any point $a' \,\varepsilon\, L_{N-m}^{a,s_1}$, Since $a' \,\varepsilon\, P_{N-m}^a$, we have,

$$(112) \qquad a' = (a_1, a_2, \cdots, a_{m+1}', a_{m+2}', \cdots, a_N').$$

Then for the point $a'$ we define, as in (106), an $(N - m_1)$ dimensional hyperplane $P_{N-m_1}^{a'}$ by:

$$(113) \qquad \begin{aligned} x_i &= a_i \quad \text{for} \quad i \,\varepsilon\, s_1, \quad i \leqq m, \\ &= a_i' \quad \text{for} \quad i \,\varepsilon\, s_1, \quad i > m. \end{aligned}$$

Next putting

$$\bar{a}_1 = m_1^{-1}[\sum_{i\varepsilon s_1, i\leqq m} a_i + \sum_{i\varepsilon s_1, i>m} a_i'],$$

we define as in (109) a subset $T^{a'}_{N-m_1} \subset P^{a'}_{N-m_1}$ by,

$$(114) \qquad x \,\varepsilon\, T^{a'}_{N-m_1} \quad \text{if, and only if,} \quad x \,\varepsilon\, P^{a'}_{N-m_1}$$

and; if $h_1 = h(s_1, a') > 0$,

$$\bar{a}_1(1 - m_1/N) - v(s_1, a')$$
$$\leqq N^{-1}\sum_{i\not\varepsilon s_1} x_i$$
$$< \min\{[\bar{a}_1(1 - m_1/N) + v(s_1, a')], [\bar{a}_1(1 - m_1/N) - v(s_1, a') + h_1]\} ;$$

if $h_1 < 0$,

$$\max\{[\bar{a}_1(1 - m_1/N) + v(s_1, a') + h_1], [\bar{a}_1(1 - m_1/N) - v(s_1, a')]\}$$
$$< N^{-1}\sum_{i=m+1}^{N} x_i \leqq \bar{a}_1(1 - m_1/N) + v(s_1, a').$$

Furthermore, assign to all co-ordinates $x_i$ in (114) for $i \not\varepsilon s_1$ and $i > m$, fixed values equal to the corresponding co-ordinates of the point $a'$, ie.., for $i$ satisfying $i \not\varepsilon s_1$, $i > m$, $x_i = a'_i$. Thus in the inequalities in (114), we replace the middle term by

$$(115) \qquad N^{-1}[\sum_{i\not\varepsilon s_1, i>m} a'_i + \sum_{i\not\varepsilon s_1, i\leqq m} x_i].$$

Since $s_1$ contains $k$ out of the first units, there are $(m - k)$ values of $i$ for which $i \not\varepsilon s_1$, and $i \leqq m$. Hence the inequalities in (114) on replacing the middle term by (115), define an $(m - k)$ dimensional subset $T^{a'}_{m-k} \subset P^{a'}_{m-k}$. Since by (105), $v(s_1, a') > 0$ the set $T^{a'}_{m-k}$ has infinite measure $(\mu_{m-k})$. The hyperplane $P^{a'}_{m-k}$ is defined by,

$$(116) \qquad x_i = a_i \quad \text{for} \quad i \,\varepsilon\, s_1, \quad i \leqq m;$$
$$x_i = a'_i \quad \text{for} \quad i > m.$$

The hyperplane $P^{a'}_{m-k}$ is orthogonal to $P^a_{N-m}$, and hence the set $L^{a,s_1}_{N-m}$, with the set $T^{a'}_{m-k}$ defined for each $a' \,\varepsilon\, L^{a,s_1}_{N-m}$, by substituting (115) for the middle term in (114), determine a set $D^a_{N-k} \subset P^a_{N-k}$, where, $P^a_{N-k}$ is the hyperplane defined by $x_i = a_i$ for each $i$, such that $i \,\varepsilon\, s_1$, $i \leqq m$. Combining (114) and (115), the explicit definition of the set $D^a_{N-k}$ is

$$x = (x_1, x_2, \cdots, x_n) \,\varepsilon\, D^a_{N-k} \quad \text{if, and only if,} \quad x \,\varepsilon\, T^{a'}_{m-k} \quad \text{for some} \quad a' \,\varepsilon\, L^{a,s_1}_{N-m}.$$

Since $L^{a,s_1}_{N-m}$ is of positive measure $(\mu_{N-m})$, and for each $a' \,\varepsilon\, L^{a,s_1}_{N-m}$, the set $T^{a'}_{m-k}$ has infinite measure $(\mu_{m,k})$, the set $D^a_{N-k}$ has infinite measure $(\mu_{N-k})$. Here $0 \leqq k < m$.

Now let $E^a_{N-k}$ be the set consisting of all those points $x \,\varepsilon\, P^a_{N-k}$ for which $h(s, x) \neq 0$ for at least one $s \,\varepsilon\, \bar{S}$. Then,

$$(117) \qquad D^a_{N-k} \subset E^a_{N-k} ;$$

hence the set $E^a_{N-k}$ is of infinite measure $(\mu_{N-k})$. We now again partition the set $E^a_{N-k}$ into subsets by,

$$E^a_{N-k} = \bigcup_{s \varepsilon \bar{S}} L^{a,s}_{N-k}, \tag{118}$$

where the subsets $L^{a,s}_{N-k}$ are defined for each specified $s \varepsilon \bar{S}$ by $x \varepsilon L^{a,s}_{N-k}$, if and only if, $x \varepsilon E^a_{N-k}$ and $h(s, x) \neq 0$. Again at least one of the subsets in the right hand side of (118) must be non-null $(\mu_{N-k})$. If there is only one non-null $(\mu_{N-k})$ set $L^{a,s}_{N-k}$, such that $s$ does not include each of the $k$ units $u_i$ with $i \varepsilon s_1$, $i \leqq m$, we select it; if there are more than one such subset, we select one of them arbitrarily. Let $L^{a,s_2}_{N-k}$ be the subset selected and suppose the sample $s_2$ contains $j$, $(0 \leqq j < k)$ out of the $k$ units $u_i$, given by $i \varepsilon s_1$, $i \leqq m$. Then again proceeding as from (114) to (117) we reach a set $E^a_{N-j}$ of infinite measure $(\mu_{N-j})$ such that for every $x \varepsilon E^a_{N-j}$, $h(s, x) \neq 0$, and hence by (105), $v(s, x) > 0$, for at least one $s \varepsilon \bar{S}$.

Clearly the process can end only when, we either

(A′) reach a set $E_N \subset R_N$, such that $E_N$ has infinite measure $(\mu_N)$, and for every $x \varepsilon E_N$, (a) $h(s, x) \neq 0$, and (b) $v(s, x) > 0$ for some $s \varepsilon \bar{S}$; or

(B′) we reach a hyperplane $P^a_{N-j}$, defined by some $j$, $(0 < j \leqq m)$ out of the first $m$ variates having fixed values equal to the corresponding co-ordinates of the point $a$, i.e., $x_r = a_r$; for $r = i_1; i_2, \cdots, i_j$ where $i_1, i_2, \cdots, i_j \leqq m$, and a set $E^a_{N-j} \subset P^a_{N-j}$ such that $E^a_{N-j}$ has infinite measure $(\mu_{N-j})$, and for every $x \varepsilon E^a_{N-j}$, (a) $h(s, x) \neq 0$ and (b) $v(s, x) > 0$ for some $s \varepsilon \bar{S}$, and further such that for any sample $s \varepsilon \bar{S}$, which does not include each of the $j$ units $u_{i_1}, u_{i_2}, \cdots, u_{i_j}$, $h(s, x) = 0$ for almost all $(\mu_{N-j}) x \varepsilon P^a_{N-j}$.

Here we note that the case (A), (below (111)) is included in the case (B′), the corresponding value of $j$ is equal to $m$.

Now case (A′) is void because $E_N = E$ is a null $(\mu_N)$ set by Theorem 3.1. Case (B′) also leads to a contradiction. To show this let $\bar{S}_j$ be the subset of $\bar{S}$, consisting of all those samples $s \varepsilon \bar{S}$, which include each of the units $u_{i_1}, u_{i_2}, \cdots, u_{i_j}$. Then for every $s \varepsilon (\bar{S} - \bar{S}_j)$, the set of points $x \varepsilon P^a_{N-j}$ for which $h(s, x) \neq 0$ forms a null $(\mu_{N-j})$ set. Consequently introducing intersection sets as previously in (72), we have,

$$\sum_{s\varepsilon(\bar{S}-\bar{S}_j)\cdot \bar{S}_{e_1'', e_2'', x}} p(s) = \sum_{s\varepsilon(\bar{S}-\bar{S}_j)\cdot S_{e_1, e_2, x}} p(s) \tag{119}$$

for almost all $(\mu_{N-j})$, $x \varepsilon P^a_{N-j}$. (119) combined with (103) gives then,

$$\sum_{s\varepsilon \bar{S}_j \cdot \bar{S}_{e_1'', e_2'', x}} p(s) \geqq \sum_{s\varepsilon \bar{S}_j \cdot \bar{S}_{e_1, e_2, x}} p(s)$$

for almost all $(\mu_{N-j})$, $x \varepsilon P^a_{N-j}$.

But then, by Theorem 4.1, the set of points $x \varepsilon P^a_{N-j}$ for which (a) $h(s, x) \neq 0$, and (b) $v(s, x) > 0$ must be a null $(\mu_{N-j})$ set, while according to (B′) $E_{N-j}$ is of infinite measure $(\mu_{N-j})$.

Thus neither $(A')$ nor $(B')$ is possible. Hence no point $a \varepsilon E$ exists such that $h(s, a) \neq 0$ for some $s \varepsilon \bar{S}$ and thus the set $E$ is empty.

Consequently the sign of strict inequality in (103) does not hold for any point $z \varepsilon R_N$.

We shall here dispose of a case which was initially left out.It was assumed that for the initial sample $s_0$, the sample size $n(s_0) \leqq N - 1$. Now suppose the only pair $(a, s_0)$, for which $h(s_0, a) \neq 0$ is such that $n(s_0) = N$. Then for all $x \varepsilon R_N$ $\bar{x}_{s_0} = \bar{X}_N$. Hence $s_0 \varepsilon \bar{S}_{e_1,e_2,x}$. Also for every other sample $s \varepsilon \bar{S}$, $e_1''(s, x) = e_1(s, x)$, $e_2''(s, x) = e_2(s, x)$, and therefore $s \varepsilon \bar{S}_{e_1'',e_2'',x}$, if, and only if $s \varepsilon \bar{S}_{e_1,e_2,x}$. Hence the left hand side of (103) cannot exceed its right hand side for any $x \varepsilon R_N$. Thus the conclusion holds good for this case also.

Thus no set of confidence intervals $[e_1'(s, x), e_2'(s, x)]$ satisfying (i) and (ii) of Definition 2.2 exists. The set of confidence intervals $[e_1(s, x), e_2(s, x)]$ is therefore strictly admissible. This completes the proof.

REMARKS 5.1. The above proof holds only subject to a measurability restriction viz. that the function $\bar{e}(s, x)$ must be measurable in $x$. This is in contrast with the result in a previous paper [(1965), II] where the admissibility of the sample mean as an esimator of the population mean, with the squared error as loss function, was proved without any measurability restrictions. However the measurability restrictions are not practically important as discussed in a previous paper [(1966), IV, Section 2].

The whole argument holds good also for the confidence intervals of fixed length $[\bar{x}_s - k_s, \bar{x}_s + k_s]$. These confidence intervals though, are not of much practical use.

**6. Confidence intervals based on ratio estimate.** The whole of the foregoing argument can be easily generalized and shown to hold for confidence intervals based on a ratio estimate and a generalized version of the sample standard deviation.

Let $y_i > 0$, $i = 1, 2, \cdots, N$, be arbitrary positive numbers. We replace the definitions in (1), (4), (5) and (17) by more general definitions as follows: Let,

$$\begin{aligned} y(s) &= \textstyle\sum_{i \varepsilon s} y_i ; \\ Y &= \textstyle\sum_{i=1}^{N} y_i ; \\ \bar{x}_s &= [y(s)]^{-1} \textstyle\sum_{i \varepsilon s} x_i ; \\ \bar{X}_N &= Y^{-1} \textstyle\sum_{i=1}^{N} x_i , \\ \bar{X}_{N-n(s)} &= [Y - y(s)]^{-1} \textstyle\sum_{i \varepsilon s} x_i ; \\ v'(s, x) &= [[y(s)]^{-1} \textstyle\sum_{i \varepsilon s} y_i(x_i/y_i - \bar{x}_s)^2]^{\frac{1}{2}}. \end{aligned} \tag{120}$$

Then $e_1(s, x)$, $e_2(s, x)$ are defined as in (6). Note that for convenience we have retained the same symbols $\bar{x}_s$, $\bar{X}_N$, etc. as in (1), (4), (5) and (17) to denote the more general functions defined in (120). In the following the above definitions should be deemed to apply in all the equations in Sections 3 to 5.

In place of the prior distribution on $R$ assumed in Section 3, we now assume that $x_i$, $i = 1, 2, \cdots, N$, are distributed independently, and normally with mean $\theta y_i$, and variance $y_i$. It is easily verified that $\bar{x}_s$, $\bar{X}_{N-n(s)}$ as defined in (120) are sufficient for $\theta$, under this prior distribution, so that in place of (18) we get,

$$(121) \quad f(x, \theta) = L[x_i, i \varepsilon s \mid \bar{x}_s] \cdot p(\bar{x}_s - \theta) \cdot L[\bar{x}_i, i \not\varepsilon s \mid \bar{X}_{N-n(s)}] \cdot q(\bar{X}_{N-n(s)} - \theta),$$

where,

$$p(\bar{x}_s - \theta) = [y(s)/2\pi]^{\frac{1}{2}} \exp[-\tfrac{1}{2}y(s)(\bar{x}_s - \theta)^2];$$

$$q(\bar{X}_{N-n(s)} - \theta) \doteq \{[Y - y(s)]/2\pi\}^{\frac{1}{2}} \exp\{\theta\tfrac{1}{2}[Y - y(s)](\bar{X}_{N-n(s)} - \theta)^2].$$

In place of (25) we put

$$(122) \qquad g = 1 + (Y\tau^2)^{-1}, \qquad g_s = 1 + [y(s)\tau^2]^{-1}.$$

Then proceeding as from (18) to (27), we obtain in place of the formulae in (27),

$$F_1 = (2\pi g_s)^{-\frac{1}{2}}\tau^{-1} \exp(-\bar{x}_s^2/2g_s);$$

$$(123) \quad F_2 = \{[Y - y(s)]/2\pi Y g_s\}^{\frac{1}{2}}[y(s)g_s]^{\frac{1}{2}} \cdot \exp\{-\tfrac{1}{2}\{[Y - y(s)] \cdot y(s)g_s/Yg\} \cdot (\bar{X}_{N-n(s)} - \bar{x}_s/g_s)^2\}$$

$$F_3 = (Yg/2\pi)^{\frac{1}{2}} \exp\{-\tfrac{1}{2}Yg[\theta - [y(s)\bar{x}_s + [Y - y(s)]X_{N-n(s)}](Yg)^{-1}]^2\}.$$

Similarly in place of (30), we get,

$$(124) \qquad \bar{X}_N = [Y - y(s)]Y^{-1}\bar{X}_{N-n(s)} + y(s)Y^{-1}\bar{x}_s.$$

It is seen in short, that with the revised definitions in (120), all the equations in Sections 3 to Section 5, remains valid simply on substituting everywhere.

$$(125) \qquad \begin{aligned} &y(s) \quad \text{for} \quad n(s), \\ &Y \qquad \text{for} \quad N \end{aligned}$$

and consequently, $Y - y(s)$ for $N - n(s)$.

Obviously the substitution for $N$ is made only where $N$ occurs as an independent term, and not as a suffix or a summation limit.

The whole argument in Section 3 (I) remains unchanged and we get in place of (45),

$$(126) \qquad \begin{aligned} \beta(s, v) &< \alpha(s, v) \cdot [1 + 1/2y(s)\tau^2] \\ &\leqq \alpha(s, v) \cdot [1 + 1/2w\tau^2] \end{aligned}$$

where $w = \min[y_i, i = 1, 2, \cdots, N]$. By assumption $w > 0$. Thus in place of (48), we get $1/2w\tau^2$ as the upper bound for the improvement in the inclusion probability. The whole subsequent argument in Section 3 (II), leading to Theorem 3.1, Corollary 3.1 and Remark 3.1, remains valid.

Next in the proof of Theorem 4.1, theonly change is that the 1-1 correspondence is now fixed in place of (80), by

(127) $$\bar{x}'_s = x_r + ay_r,$$

$a$ being fixed (uniquely) so as to satisfy (81). As shown in a previous paper [(1966), IV, Section 5]

(128) $$a = \bar{\alpha}' - \bar{\alpha},$$

where, $\bar{\alpha}' = \sum_{i=N-k+1}^{N} \alpha'/\sum_{N-k+1}^{N} y_i$, and $\bar{\alpha} = \sum_{i=N-k+1}^{N} \alpha_i/\sum_{i=N-k+1}^{N} y_i$.

The remaining argument in Section 4, then remains applicable. By a calculation similar to that in Lemma 4.1, (102) is also seen to remain valid with $\bar{\alpha}'$ and $\bar{\alpha}$ as in (128) and $n(s)$ replaced by $y(s)$. Thus Lemma 4.1 and Theorem 4.1 remain valid.

Next in Section 5, (105) remains valid. For by (120), if at the point $a$, $v'(s_0, a) = 0$, we have $x_i/y_i = c$ for all $i \varepsilon s_0$. Then by taking the point $a'$ to be such that $x_i/y_i = c$, for $i = 1, 2, \cdots, N$, the whole argument underlying (105) holds. Then in (106), we put

(129) $$\bar{a}_0 = [y(s_0)]^{-1} \sum_{i\varepsilon s} a_i,$$

where by (120), $y(s_0) = \sum_{i=1}^{m} y_i$. Each formula from (107) to (109) then remains valid simply on replacing the factor $(1 - m/N)$ by $(1 - y(s_0)/Y)$. The whole subsequent proof of Theorem 5.1 thus remains valid except for obvious modifications. The set of confidence intervals $[e_1(s, x), e_2(s, x)]$ defined by (6) with (120), is thus strictly admissible for $\bar{X}_N$ which is the weighted population mean, as in (120). The population arithmetic mean is $YN^{-1}\bar{X}_N$.

Hence putting,

$$e_1^*(s, x) = YN^{-1}\bar{x}_s - v(s, x),$$
$$e_2^*(s, x) = YN^{-1}\bar{x}_s + v(s, x),$$
$$v(s, x) = k_s \cdot v'(s, x),$$

where $\bar{x}_s$, $v'(s, x)$ are as in (120), the set of confidence intervals $[e_1^*(s, x), e_2^*(s, x)]$ is strictly admissible for the population mean.

REMARK 6.1. The function $v'(s, x')$ in (120) has a significance. The ratio estimate $\bar{x}_s$ in (120) is superior to the sample mean as estimate of the population mean, if prior knowledge about the distribution of the $x_i$ exists that,

(130) $$x_i \text{ is } N(\theta y_i, \sigma^2 \cdot y_i), \quad i = 1, 2, \cdots, N,$$

all $x_i$ being distributed independently.

Then $(\bar{X}_{N-n(s)} - \bar{x}_s)$ is $N(0, \sigma^2 \cdot Y/[Y - y(s)] \cdot y(s))$. Hence the best confidence intervals, for $\bar{X}_{N-n(s)}$, i.e. equivalently for $\bar{X}_N$, are based on a function which gives an estimate of $\sigma^2$. It is seen that the estimate of $\sigma^2$ is given, not by the sample standard deviation but by the function $v'(s, x)$ in (120), as $E_\theta\{v'(s, x)\}^2 \cdot \alpha\sigma^2$ for all $\theta$.

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## REFERENCES

[1] Godambe, V. P. (1961). Bayesian shortest confidence intervals and admissibility. *Int. Inst. de Stat.* 33rd session Parts.

[2] Godambe, V. P. (1966). Empirical Bayes shortest confidence intervals for the mean of a finite population, (Abstract). *Ann. Math. Statist.* **37** 772–773

[3] Godambe, V. P. and Joshi, V. M. (1965). Admissibility and Bayes estimation in sampling finite populations. I. *Ann. Math. Statist.* **36** 1707–1722.

[4] Joshi, V. M. (1965). Admissibility and Bayes estimation in sampling finite populations. II. *Ann. Math. Statist.* **36** 1723–1729.

[5] Joshi, V. M. (1965). Admissibility and Bayes estimation in sampling finite populations. III. *Ann. Math. Statist.* **36** 1730–1742.

[6] Joshi, V. M. (1966). Admissibility and Bayes estimation in sampling finite populations. IV. *Ann. Math. Statist.* **37** 1658–1670

[7] Joshi, V. M. (1966). Admissibility of confidence intervals, *Ann. Math. Statist.* **37** 629–638.

[8] Loève, Michel (1960). *Probability Theory* (second edition). Princeton.