ON THE EXACT DISTRIBUTIONS OF THE CRITERION W FOR TESTING SPHERICITY IN A p-VARIATE NORMAL DISTRIBUTION

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1. Introduction. Let the p-component vectors $x_1, x_2, x_3, \cdots, x_n$ form a sample from $N(\mu, \Sigma)$. The hypothesis H that $\Sigma = \sigma^2 I$, where σ^2 is not specified, can be put either in the form that all the roots of

$$(1.1) \qquad |\sum - \phi I| = 0$$

are equal, or that the arithmetic mean of the roots $\phi_1, \phi_2, \cdots, \phi_p$ is equal to the geometric mean, i.e.

$$(1.2) \qquad \prod \phi_i^{1/p} / \{(\sum \phi_i)/p\} = |\Sigma|^{1/p} / \{(\text{tr } \Sigma)/p\} = 1.$$

Since the squares of the lengths of principal axes of ellipsoids of constant density are proportional to the roots ϕ_i, which are now equal, the hypothesis implies that the ellipsoids are spheres.

If the covariance matrix A, for the sample, be given by

$$(1.3) \qquad A = \sum_{\alpha=1}^{N} (x_\alpha - \bar{x})(x_\alpha - \bar{x})' = (a_{ij})$$

the criterion W for testing sphericity in the p-variate normal distribution can be defined by

$$(1.4) \qquad W = A / \{(\text{tr } A)/p\}^p$$

which resembles (1.2). Thus the criterion W is a power of the ratio of the geometric mean and the arithmetic mean of the roots $\theta_1, \theta_2, \cdots, \theta_p$ of $|A - \theta I| = 0$.

Mauchly [9] defined a significance test for finding the ellipticity in a harmonic dial. In a subsequent paper [10] he modified his test to define a criterion for determining the sphericity of a normal p-variate distribution and also obtained its moments under the null hypothesis. Girshick [6] obtained the distribution of the ellipticity statistic under some special conditions.

Hickman [7] has given an example for obtaining the confidence regions for the dispersion matrix if it is taken to be proportional to any given matrix. Ihm [8] has discussed a number of such criteria in the case of multivariate normal distributions.

Anderson [1] has given a nice exposition of these different criteria satisfying different needs, the moments of such criteria and their distributions and the asymptotic expansions of the distributions. The hth moment of the sphericity

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criterion W has been shown to be

$$(1.5)\quad E(W^h) = p^{ph}\{\Gamma(\tfrac{1}{2}pn)/\Gamma(\tfrac{1}{2}pn + ph)\} \cdot\prod_{i=1}^{p}[\Gamma\{\tfrac{1}{2}(n+1-i)+h\}/\Gamma\{\tfrac{1}{2}(n+1-i)\}].$$

Anderson [1] has also obtained the exact and cumulative distribution functions of W for the simple case $p = 2$. Consul [3] has given a method, based upon inversion theorem and operational calculus, to determine the exact and cumulative distribution functions of some likelihood ratio criteria. In this paper we use a modified form of that method to obtain the exact and cumulative distribution functions of the criterion W for $p = 2, 3, 4$, and 6.

2. Some preliminary results. We give here some known results and integrals, for ready reference at many places, from standard books and journals:

(i) Gauss and Legendre's multiplication theorem for gamma functions is

$$(2.1)\quad \prod_{r=0}^{n-1}\Gamma(z + r/n) = (2\pi)^{\frac{1}{2}(n-1)}n^{\frac{1}{2}-nz}\Gamma(nz).$$

(ii) We know that

$$(2.2)\quad (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} x^{-s}(s+a)^{-1}\cdot ds = x^a.$$

(iii) Consul [2] has obtained the inverse Mellin transform

$$(2.3)\quad (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(ps+a)\Gamma(ps+b)[\Gamma(ps+a+m)\Gamma(ps+b+n)]^{-1}\cdot ds$$
$$= x^{a/p}(1-x^{1/p})^{m+n-1}[p\Gamma(m+n)]^{-1}F(n, a+m-b; m+n; 1-x^{1/p}).$$

(iv) Consul [4] has also obtained the result

$$(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} x^{-s}\Gamma(s+a)\Gamma(s+b)\Gamma(s+c)$$
$$\cdot[\Gamma(s+a+m)\Gamma(s+b+n)\Gamma(s+c+p)]^{-1}\cdot ds$$
$$(2.4)\quad = x^a(1-x)^{m+n+p-1}[\Gamma(m+n+p)]^{-1}\sum_{r=0}^{\infty}(p)_r(b+n-c)_r$$
$$\cdot[r!\,(m+n+p)_r]^{-1}(1-x)^r$$
$$\cdot F(a+m-b, n+p+r; m+n+p+r; 1-x).$$

(v) Erdelyi and others [5] have given the result (22), (102),

$$(2.5)\quad (c-n)_n z^{c-n-1}F(a, b; c-n; z) = (d^n/dz^n)[z^{c-1}F(a, b; c; z)].$$

3. Distributions of the criterion W. By applying Mellin's inversion theorem on the hth moment, given by (1.5), the exact distribution function of the criterion W is given by

$$(3.1)\quad f(W) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} W^{-h-1}\cdot p^{ph}[\Gamma(\tfrac{1}{2}pn)/\Gamma(\tfrac{1}{2}pn+ph)] \cdot\prod_{i=1}^{p}[\Gamma\{\tfrac{1}{2}(n+1-i)+h\}/\Gamma\{\tfrac{1}{2}(n+1-i)\}]\cdot dh.$$

CASE I. For $p = 2$, by the use of duplication formula for gamma functions and by further simplication, the expression (3.1) can be reduced to

$$f(W) = (n-1)W^{-1}(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} W^{-h}(2h+n-1)\cdot dh$$

which, on evaluation of the integral with the help of (2.2) gives the density function of W as

$$f(W) = \tfrac{1}{2}(n-1)W^{\frac{1}{2}(n-3)}. \tag{3.2}$$

Obviously, the cumulative distribution function of W is

$$\Pr(W \leqq w) = w^{\frac{1}{2}(n-1)}. \tag{3.3}$$

The result (3.2) has been obtained by Anderson [1] by another method.

CASE II. For $p = 3$, by the use of Gauss and Legendre's multiplication theorem (2.1) on $\Gamma(\frac{3}{2}n + 3h)$, use of duplication formula for gamma functions and by simplification, the distribution function (3.1) can be transformed into

$$f(W) = \tfrac{3}{4}K(n)\cdot\pi^{\frac{1}{2}}W^{-1}(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} W^{-h}\cdot\Gamma(h+\tfrac{1}{2}n-1)\Gamma(h+\tfrac{1}{2}n-\tfrac{1}{2}) \cdot[\Gamma(h+\tfrac{1}{2}n+\tfrac{2}{3})\Gamma(h+\tfrac{1}{2}n+\tfrac{1}{3})]^{-1}\cdot dh$$

where

$$K(n) = 2^{n+1}\Gamma(\tfrac{3}{2}n)[\Gamma(n-1)\Gamma(\tfrac{1}{2}n-1)\cdot 3^{\frac{1}{2}(3n+1)}]^{-1}. \tag{3.4}$$

Now, by evaluating the integral with the help of Consul's transform (2.3) and by simplifyung it, the exact distribution function $f(W)$ becomes

$$f(W) = K(n)\cdot W^{\frac{1}{2}n-2}(1-W)^{\frac{3}{2}}\cdot F(\tfrac{5}{6}, \tfrac{7}{6}; \tfrac{5}{2}; 1-W) \tag{3.5}$$

for $0 \leqq W \leqq 1$, and $K(n)$ is given by (3.4).

The above expression can also be put in the form

$$f(W) = \tfrac{3}{2}K(n)\cdot W^{\frac{1}{2}n-2}\sum_{r=0}^{\infty}\Gamma(3r+\tfrac{3}{2}) \cdot[\Gamma(2r+1)\Gamma(r+\tfrac{5}{2})]^{-1}(4/27)^r(1-W)^{\frac{3}{2}+r} \tag{3.6}$$

and thus the cumulative distribution function of W is given by

$$\Pr(W \leqq w) = K(n)\cdot\Gamma(\tfrac{1}{2}n-1)\sum_{r=0}^{\infty}\Gamma(3r+\tfrac{3}{2}) \cdot[\Gamma(2r+1)\Gamma(\tfrac{1}{2}n+r+\tfrac{3}{2})](4/27)^r\cdot I_w(\tfrac{1}{2}n-1, r+\tfrac{5}{2}) \tag{3.7}$$

where $I_w(\frac{1}{2}n-1, r+\frac{5}{2})$ is the incomplete beta function tabulated by Pearson.

CASE III. For $p = 4$, the expression (3.1) can be modified by the repeated use of the duplication formula for gamma functions, into the form

$$f(W) = 2K(n)\cdot\Gamma(\tfrac{7}{2})W^{-1}(2\pi i)^{-1}\cdot\int_{c-i\infty}^{c+i\infty} W^{-h} \cdot\Gamma(2h+n-3)\Gamma(2h+n-1)[\Gamma(2h+n+\tfrac{1}{2})\Gamma(2h+n)]^{-1}\cdot dh$$

where

$$K(n) = (n-1)\Gamma(n+\tfrac{1}{2})/[2\Gamma(n-3)\Gamma(\tfrac{7}{2})]^{-1}. \tag{3.8}$$

Then, by putting the value of the integral with the help of Consul's transform (2.3) and by simplification, the exact distribution function becomes

$$(3.9)\qquad f(W) = \tfrac{2}{7}K(n)\cdot W^{\frac{1}{2}(n-5)}(1 - W^{\frac{1}{2}})^{7/2}\cdot F(1, \tfrac{3}{2}; \tfrac{9}{2}; 1 - W^{\frac{1}{2}})$$

for $0 \leqq W \leqq 1$, and $K(n)$ is given by (3.8).

By integrating the above expression by parts three times between the limits 0 to $w(\leqq 1)$ and with the repeated use of (2.5), the cumulative distribution function can be obtained in the form

$$(3.10)\qquad \Pr(W \leqq w) = I_{w^{\frac{1}{2}}}(n - 1, \tfrac{3}{2}) + \tfrac{4}{7}K(n)\cdot w^{\frac{1}{2}(n-3)} \cdot \textstyle\sum_{r=0}^{2} (\tfrac{9}{2} - r)_r (n - 3)_{r+1}^{-1} w^{\frac{1}{2}r}(1 - w^{\frac{1}{2}})^{7/2-r}\cdot F(1, \tfrac{3}{2}; \tfrac{9}{2} - r; 1 - w^{\frac{1}{2}}).$$

By the use of special functions the distributions, given by (3.9) and (3.10), can be expressed in terms of the following algebraic functions also:

$$(3.11)\qquad f(W) = \tfrac{1}{2}K(n)\cdot W^{\frac{1}{2}(n-5)}[\tfrac{15}{2}\log\{W^{-\frac{1}{4}} + W^{-\frac{1}{4}}(1 - W^{\frac{1}{2}})^{\frac{1}{2}}\} - \tfrac{5}{2}(1 - W^{\frac{1}{2}})^{\frac{1}{2}}(W^{\frac{1}{4}} + 2W)]$$

and

$$(3.12)\qquad \begin{aligned}\Pr(W \leqq w) = {} & I_{w^{\frac{1}{2}}}(n - 1, \tfrac{3}{2}) + K(n)\cdot w^{\frac{1}{2}(n-3)}[(1 - w^{\frac{1}{2}})^{\frac{5}{2}}/(n - 3) \\ & - \tfrac{15}{2}w(1 - w^{\frac{1}{2}})^{\frac{1}{2}}/(n - 1) - \tfrac{5}{2}(n - 4)w^{\frac{1}{2}}(1 - w^{\frac{1}{2}})^{\frac{3}{2}}/(n - 2)(n - 3) \\ & + \tfrac{15}{2}w(n - 1)^{-1}\log\{w^{-\frac{1}{4}} + w^{-\frac{1}{4}}(1 - w^{\frac{1}{2}})^{\frac{1}{2}}\}].\end{aligned}$$

Case IV. For $p = 6$, the expression (3.1) can, by the successive use of duplication formula for gamma functions and by factorising $\Gamma(3n + 6h)$ by Gauss and Legendre's multiplication theorem (2.1), be reduced and simplified to

$$f(W) = 2K(n)\cdot W^{-1}(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} W^{-h} \cdot\Gamma(2h + n - 1)\Gamma(2h + n - 3)\Gamma(2h + n - 5) \cdot[\Gamma(2h + n)\Gamma(2h + n + \tfrac{1}{3})\Gamma(2h + n + \tfrac{2}{3})]^{-1}\cdot dh$$

where

$$(3.13)\qquad K(n) = \pi\cdot 3^{\frac{1}{2}-3n}\Gamma(3n)/[\Gamma(n - 1)\Gamma(n - 3)\Gamma(n - 5)].$$

By evaluating the integral with the help of Consul's integral transform (2.4), the exact distribution of W becomes

$$(3.14)\qquad f(W) = \{2K(n)/9!\}W^{\frac{1}{2}(n-7)}(1 - W^{\frac{1}{2}})^9 \textstyle\sum_{r=0}^{\infty} (3 + \tfrac{2}{3})_r(3 + \tfrac{1}{3})_r \cdot[r!(10)_r]^{-1}(1 - W^{\frac{1}{2}})^r F(1, 5 + r; 10 + r; 1 - W^{\frac{1}{2}})$$

where $0 \leqq W \leqq 1$ and $K(n)$ is given by (3.13).

By integrating (3.14) by parts five times between the limits 0 to $w(\leqq 1)$ and by using the result (2.5) in each integration and on simplification, the cumulative distribution function is found to be

$$\begin{aligned}\Pr(W \leqq w) &= \{2K(n)/9!\} \textstyle\sum_{r=0}^{\infty} (3 + \frac{2}{3})_r (3 + \frac{1}{3})_r [r!(10)_r]^{-1} \\ &\cdot \{\Gamma(n-1)\Gamma(11+r)[\Gamma(n+r+4)\cdot(n-5)_5]^{-1} \\ &\cdot I_{w^{\frac{1}{2}}}(n-1, r+5) + \textstyle\sum_{i=0}^{4} [(10+r-i)_i (n-5)_{i+1}^{-1}] \\ &\cdot w^{\frac{1}{2}(n-5+i)} (1 - w^{\frac{1}{2}})^{9+r-i} \cdot F(1, 5+r; 10+r-i; 1-w^{\frac{1}{2}})\}.\end{aligned} \tag{3.15}$$

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