ON THE EXACT DISTRIBUTIONS OF LIKELIHOOD RATIO CRITERIA FOR TESTING INDEPENDENCE OF SETS OF VARIATES UNDER THE NULL HYPOTHESIS

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1. Introduction. Let a p-component vector X, distributed according to $N(\mu, \Sigma)$ be partitioned into q sub-vectors with components $p_1, p_2, \cdots, p_q$ respectively. The vector of means μ and the covariance matrix Σ are also partitioned similarly; i.e.

$$X = \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(q)} \end{pmatrix}, \qquad \mu = \begin{pmatrix} \mu^{(1)} \\ \mu^{(2)} \\ \vdots \\ \mu^{(q)} \end{pmatrix} \tag{1.1}$$

and

$$\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1q} \\ \Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2q} \\ \vdots & \vdots & & \vdots \\ \Sigma_{q1} & \Sigma_{q2} & \cdots & \Sigma_{qq} \end{pmatrix}. \tag{1.2}$$

The null hypothesis H, to be tested, is whether the q sets are mutually independent, i.e. whether each variable in one set is uncorrelated with each variable in the others. Thus

$$H: n(x/\mu, \Sigma) = \prod_{i=1}^{q} n(x^{(i)}/\mu^{(i)}, \Sigma_{ii}). \tag{1.3}$$

Now, if $x_1, x_2, \cdots, x_N$ be a sample of N observations drawn from $N(\mu, \Sigma)$, where x_α, μ and Σ are partitioned as in (1.1) and (1.2), then Wilks (1935) has defined the likelihood ratio criterion V, that the q sets are mutually independent, by

$$V = \frac{|A|}{\prod_{i=1}^{q} |A_{ii}|} \tag{1.4}$$

where A is defined by

$$A = \sum_{\alpha=1}^{N} (x_\alpha - \bar{x})(x_\alpha - \bar{x})' \tag{1.5}$$

and is partitioned in the same manner as Σ in (1.2). The corresponding matrix A_{ii} is defined and partitioned similarly.

It has been shown by Anderson (1958) that the likelihood ratio criterion V can be expressed entirely in terms of sample correlation coefficients and that the test based upon the criterion is invariant with respect to linear transformations within each set.

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Daly (1940) and Narain (1950) have shown that, in the tests based on V, the probability of rejecting the null hypothesis is greater than the significance level if the hypothesis is not true; i.e. the tests based on the criterion V are strictly unbiased.

The moments of the criterion V under the null hypothesis are given by

$$(1.6)\quad M(V^h) = \prod_{i=p_q+1}^{p} [\Gamma\{(n+1-i)/2+h\}[\Gamma\{(n+1-i)/2\}]^{-1} \cdot \prod_{i=1}^{q-1} [\prod_{r=1}^{p_i} \Gamma\{(n+1-r)/2\}[\Gamma\{(n+1-r)/2+h\}]^{-1}]]$$

and since $0 \leqq V \leqq 1$, these moments uniquely determine the distributions of V. Also, the distributions of V are free from the nuisance parameters as the moments do not depend upon Σ_0.

Wilks (1935) could obtain the distributions of V in some special cases by considering it as integrals of joint densities of independent variables. Wilks has given exact distributions of V for (i) $q = 2$; $p_1 = 1, 2, 3$, $p_2 = 3$; $p_1 = 3$, $p_2 = 4$ and $p_1 = p_2 = 4$ and for (ii) $q = 3$; p_3, $p_1 = p_2 = 1$ and $p_1 = 1$, $p_2 = 2$, $p_3 = 2, 3, 4$ and $p_1 = p_2 = 2$, $p_3 = 3$. Anderson (1958) has considered only three cases which are covered by the above.

Wald and Brookner (1941) have given another method of deriving the distributions if not more than one p_i is odd, but exact distributions were not obtained for any new case. Box (1949) used the method of asymptotic expansions of gamma functions to obtain the distributions in asymptotic form in some cases.

In this paper we apply Mellin's inversion theorem on the expression of moments (1.6) and use operational calculus to obtain the exact distributions of the criterion V for a large number of more general cases viz.,

Section 3: $q = 2$, all values of p_2 and $p_1 = 1, 2, 3, 4, 5, 6$.

Section 4: $q = 3$, all values of p_3 and (i) $p_1 = p_2 = 1$, (ii) $p_1 = 1$, $p_2 = 2$, (iii) $p_1 = p_2 = 2$, (iv) $p_1 = 1$, $p_2 = 3$, (v) $p_1 = 2$, $p_2 = 3$, (vi) $p_1 = 2$, $p_2 = 4$.

2. Some preliminary integrals and results. We list here four integrals and two other results, obtained by Consul (1965), (1966), for ready reference as they are used at many places in the following sections:

For $0 \leqq x \leqq 1$, the inverse Mellin transforms

$$(2.1)\quad (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} \cdot \Gamma(ps+a)\Gamma(ps+b) \cdot [\Gamma(ps+a+m)\Gamma(ps+b+n)]^{-1}\, ds$$
$$= x^{a/p}(1-x^{1/p})^{m+n-1}[p \cdot \Gamma(m+n)]^{-1} \cdot F(n, a+m-b; m+n; 1-x^{1/p});$$

$$(2\pi i)^{-1}\{ \int_{c-i\infty}^{c+i\infty} x^{-s} \cdot \Gamma(2s+a)\Gamma(s+b)[\Gamma(2s+a+m)\Gamma(s+b+n)]^{-1}\, ds$$
$$(2.2)\quad = x^{a/2}(1-x)^n[2 \cdot \Gamma(m)\Gamma(n+1)]^{-1} \sum_{i=0}^{m-1} \binom{m-1}{i} (-x^{\frac{1}{2}})^i \cdot F(n, 1-b+(a+i)/2; n+1; 1-x);$$

$$(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} \cdot \Gamma(qs+a)\Gamma(qs+b)\Gamma(rs+c)$$

$$\cdot[\Gamma(qs + a + m)\Gamma(qs + b + n)\Gamma(rs + c + p)]^{-1}\, ds$$

$$(2.3)\quad = x^{(a-1)/q}(1 - x^{1/q})^{m+n}[r\Gamma(p)]^{-1}\cdot \sum_{i=0}^{p-1} \binom{p-1}{i}(-1)^i$$

$$\cdot \sum_{j=0}^{\infty} (n)_j\cdot(a + m - b)_j[j!\,\Gamma(m + n + j + 1)]^{-1}\cdot(1 - x^{1/q})^j$$

$$\cdot F\{1,\, 1 - a + (c + i)q/r;\, m + n + j + 1;\, -(1 - x^{1/q})x^{-1/q}\};$$

$$(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s}\cdot\Gamma(s + a)\Gamma(s + b)\Gamma(s + c)$$

$$\cdot[\Gamma(s + a + m)\Gamma(s + b + n)\Gamma(s + c + p)]^{-1}\, ds$$

$$(2.4)\quad = x^{b-m}(1 - x)^{m+n+p-1}[\Gamma(m + n + p)]^{-1}$$

$$\cdot \sum_{r=0}^{\infty} (p)_r(b + n - c)_r[r!\,(m + n + p)_r]^{-1}\cdot(1 - x)^r$$

$$\cdot F(m,\, a + m - b;\, m + n + p + r;\, -(1 - x)x^{-1});$$

and further the results in hypergeometric functions are

$$(2.5)\quad F(a, b; c; x) = (c - m)_m(c - a - m)_m^{-1}\cdot x^{-m}\cdot \sum_{i=0}^{m} \binom{m}{i}(-1)^i(1 - x)^i \cdot F(a, b - m + i; c - m; x)$$

and

$$F(2, 1 + b; 3; x) = 2(1 - x)^{-b}[b(b - 1)]^{-1}\cdot x^{-2}[(1 - x)^b + bx - 1], \quad \text{for } b \neq 0, 1,$$

$$(2.6)\quad = -2x^{-2}[x + \log(1 - x)], \quad \text{if } b = 0,$$

$$= 2x^{-2}[x(1 - x)^{-1} + \log(1 - x)], \quad \text{if } b = 1.$$

3. Exact distributions of the criteria V for $q = 2$. By simplifying the expression of moments (1.6) for $q = 2$ and then by applying Mellin's inversion theorem on it, the exact distribution function of the criteria V becomes

$$f(V) = \prod_{r=1}^{p_1} [\Gamma\{(n + 1 - r)/2\}[\Gamma\{(n + 1 - p_2 - r)/2\}]^{-1}\cdot(2\pi i)^{-1} \int_{c'-i\infty}^{c'+i\infty} V^{-h-1}$$

$$\cdot \prod_{r=1}^{p_1} \Gamma\{(n + 1 - p_2 - r)/2 + h\}[\Gamma\{(n + 1 - r)/2 + h\}]^{-1}\, dh$$

which, on setting $h + (n + 1 - p_1 - p_2)/2 = t$ and on simplification, takes the form

$$(3.1)\quad f(V) = \prod_{r=1}^{p_1} [\Gamma\{(n + 1 - r)/2\}[\Gamma\{(n + 1 - p_2 - r)/2\}]^{-1}]\cdot V^{(n-p_1-p_2-1)/2}$$

$$\cdot(2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} V^{-t}\cdot \prod_{r=1}^{p_1} \Gamma\{t + (p_1 - r)/2\}[\Gamma\{t + (p_1 + p_2 - r)/2\}]^{-1}\, dt.$$

The above integral can be easily expressed in the form of Meijer's G-function, so that the exact distribution of V becomes

$$(3.2)\quad f(V) = \prod_{r=1}^{p_1} [\Gamma\{(n + 1 - r)/2\}[\Gamma\{(n + 1 - p_2 - r)/2\}]^{-1}]$$

$$\cdot V^{(n-p_1-p_2-1)/2}\cdot G_{p_1 p_1}^{p_1 0}\left(V \,\middle|\, {}^{p_2/2,(p_2+1)/2,\cdots,(p_2+p_1-1)/2}_{0,\frac{1}{2},1,\cdots,(p_1-1)/2}\right).$$

Since the properties of a large number of G-functions are not well known yet, we shall change them into better known forms. The expression (3.1) is of the same form as the one, obtained by Consul (1966), for the exact distribution functions of $U_{p,m,n}$ with the following relations between the parameters:

Former paper	Above expression
n	$n - p_2$
p	p_1
$q_1 = m$	p_2

Thus the exact distributions of the criteria V for $q = 2$ are the same as the distributions of the criteria $U_{p_1,p_2,n-p_2}$, which have been evaluated for $p_1 = 1, 2, 3, 4$ and for all values of n and p_2.

The respective exact distribution functions $f(V)$ are being listed below, for the sake of convenience, for $p_1 = 1, 2, 3, 4$ and are being determined for $p_1 = 5$ and 6.

Case I. When $p_1 = 1$,

$$f(V) = \Gamma(n/2)[\Gamma(p_2/2)\cdot\Gamma[(n - p_2)/2]]^{-1}\cdot V^{(n-p_2-2)/2} \cdot(1 - V)^{p_2/2-1}, \quad 0 \leqq V \leqq 1. \tag{3.3}$$

Case II. When $p_1 = 2$,

$$f(V) = \Gamma(n - 1)[2\Gamma(n - p_2 - 1)\Gamma(p_2)]^{-1}\cdot V^{(n-p_2-3)/2}\cdot(1 - V^{\frac{1}{2}})^{p_2-1}. \tag{3.4}$$

Case III. When $p_1 = 3$,

$$\begin{aligned} f(V) = {} & \Gamma(n - 1)\Gamma(n/2 - 1) \\ & \cdot[\Gamma(n - p_2 - 1)\Gamma[(n - p_2)/2 - 1]\cdot 2\Gamma(p_2)\Gamma(p_2/2 + 1)]^{-1} \\ & \cdot V^{(n-p_2-4)/2}\cdot(1 - V)^{p_2/2}\cdot \textstyle\sum_{i=0}^{p_2-1} \binom{p_2-1}{i}(-V^{\frac{1}{2}})^i \\ & \cdot F(p_2/2, i/2; p_2/2 + 1; 1 - V). \end{aligned} \tag{3.5}$$

Case IV. When $p_1 = 4$,

$$f(V) = \Gamma(n - 3)\Gamma(n - 1)[2\Gamma(2p_2)\Gamma(n - p_2 - 1)\Gamma(n - p_2 - 3)]^{-1} \cdot V^{(n-p_2-5)/2}\cdot(1 - V^{\frac{1}{2}})^{2p_2-1}\cdot F(p_2 - 2, p_2; 2p_2; 1 - V^{\frac{1}{2}}). \tag{3.6}$$

Consul (1966) has also shown that by using the special functions (2.5) and (2.6), the hypergeometric functions in Case III and IV can be transformed into algebraic forms for all particular values of p_2. Some of these algebraic forms of the distribution functions as well as their cumulative distribution functions have also been given in that paper.

Case V. When $p_1 = 5$, the expression (3.1), by the successive use of Legendre's duplication formula, takes the form

$$\begin{aligned} f(V) = {} & \Gamma(n - 1)\Gamma(n - 3)\Gamma(n/2 - 2) \\ & \cdot[\Gamma(n - p_2 - 1)\Gamma(n - p_2 - 3)\Gamma[(n - p_2)/2 - 2]]^{-1}\cdot V^{(n-p_2-6)/2} \\ & \cdot(2\pi i)^{-1} \textstyle\int_{c-i\infty}^{c+i\infty} V^{-t}\cdot\Gamma(2t)\Gamma(2t + 2)\Gamma(t + 2) \\ & \cdot[\Gamma(2t + p_2)\Gamma(2t + p_2 + 2)\Gamma(t + p_2/2 + 2)]^{-1}\, dt. \end{aligned}$$

Now, by applying a particular case of Consul's inverse Mellin transform (2.3) to the above integral, we find that the exact distribution function of V is given by

$$f(V) = \Gamma(n-1)\Gamma(n-3)\Gamma(n/2-2)$$
$$\cdot[\Gamma(n-p_2-1)\Gamma(n-p_2-3)\Gamma[(n-p_2)/2-2]\Gamma(p_2/2)]^{-1}\cdot V^{(n-p_2-7)/2}$$
$$\cdot(1-V^{\frac{1}{2}})^{2p_2}\cdot\sum_{i=0}^{p_2/2-1}\binom{p_2/2-1}{i}(-1)^i\cdot\sum_{j=0}^{\infty}(p_2)_j(p_2-2)_j$$
$$\cdot[j!\,\Gamma(2p_2+j+1)]^{-1}.(1-V^{\frac{1}{2}})^j$$
$$\cdot F(1, 2i+5; 2p_2+1+j; -(1-V^{\frac{1}{2}})V^{-\frac{1}{2}}).$$

CASE VI. When $p_1 = 6$, the expression (3.1) can be reduced, by the successive use of Legendre's duplication formula, to

$$f(V) = \Gamma(n-1)\Gamma(n-3)\Gamma(n-5)$$
$$\cdot[\Gamma(n-p_2-1)\Gamma(n-p_2-3)\Gamma(n-p_2-5)]^{-1}\cdot V^{(n-p_2-7)/2}$$
$$\cdot(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty}V^{-t}\cdot\Gamma(2t)\Gamma(2t+2)\Gamma(2t+4)$$
$$\cdot[\Gamma(2t+p_2)\Gamma(2t+p_2+2)\Gamma(2t+p_2+4)]^{-1}\cdot dt$$

which, by the application of Consul's inverse Mellin transform (2.4), gives the exact distribution of V as

$$f(V) = \Gamma(n-1)\Gamma(n-3)\Gamma(n-5)$$
$$\cdot[\Gamma(n-p_2-1)\Gamma(n-p_2-3)\Gamma(n-p_2-5)\cdot 2\Gamma(3p_2)]^{-1}\cdot V^{(n-2p_2-7)/2}$$
$$\cdot(1-V^{\frac{1}{2}})^{3p_2-1}\cdot\sum_{i=0}^{\infty}(p_2)_i(p_2-4)_i[i!\,(3p_2)_i]^{-1}$$
$$\cdot(1-V^{\frac{1}{2}})^i\cdot F(p_2+2, p_2; 3p_2+i; -(1-V^{\frac{1}{2}})V^{-\frac{1}{2}}).$$

4. Exact probability distributions of V for $q = 3$. When $q = 3$, by simplifying the expression of moments (1.6) and then by applying Mellin's inversion theorem to it, the exact distribution functions of the criteria V are obtained in the form

$$f(V) = K(n)\cdot(2\pi i)^{-1}\cdot\int_{c'-i\infty}^{c'+i\infty}V^{-h-1}\cdot\prod_{r=1}^{p_1+p_2}\Gamma[(n-p_3+1-r)/2+h]$$
$$\cdot\{\prod_{r=1}^{p_1}\Gamma[(n+1-r)/2+h]\cdot\prod_{r=1}^{p_2}\Gamma[(n+1-r)/2+h]\}^{-1}\cdot dh$$

where

$$K(n) = \prod_{i=1}^{p_1}\Gamma[(n+1-i)/2]\cdot\prod_{i=1}^{p_2}\Gamma[(n+1-i)/2]\cdot\{\prod_{i=1}^{p_1+p_2}\Gamma[(n-p_3+1-i)/2]\}^{-1}. \tag{4.0}$$

Now, by setting $h + (n - p_1 - p_2 - p_3 + 1)/2 = t$ and on further simplification, the exact distribution of V becomes

$$f(V) = K(n)\cdot V^{\frac{1}{2}(n-p_1-p_2-p_3-1)}\cdot(2\pi i)^{-1}$$
$$\cdot\int_{c-i\infty}^{c+i\infty}V^{-t}\prod_{r=1}^{p_1+p_2}\Gamma\{t+\tfrac{1}{2}(p_1+p_2-r)\} \tag{4.01}$$
$$\cdot\{\prod_{r=1}^{p_1}\prod_{r=1}^{p_2}\Gamma\{t+\tfrac{1}{2}(p_1+p_2+p_3-r)\}\}^{-1}dt.$$

The above representation of the distribution as an inverse Mellin transform is of much interest as the distribution splits into a factor depending on n and an integral which depends upon the values of p_1 and p_2 only and is independent of n and p_3. So the expression for the distribution of V shall be valid for all values of n and p_3. The above integral can be easily put in terms of Meijer's G-function so that the exact distribution function $f(V)$ becomes

$$(4.02) \qquad f(V) = K(n) \cdot V^{\frac{1}{2}(n-p_1-p_2-p_3-1)} \cdot G^{p_1+p_2,0}_{p_1+p_2,p_1+p_2}\left(V \,\middle|\, {}^{a_r}_{b_s}\right)$$

where $K(n)$ is given by (4.0) and, b_s stands for $0, \frac{1}{2}, 1, \cdots, \frac{1}{2}(p_1 + p_2 - 1)$ and a_r stands for $\frac{1}{2}(p_3 + p_1), \frac{1}{2}(p_3 + p_1 + 1), \cdots, \frac{1}{2}(p_3 + p_1 + p_2 - 1)$, $\frac{1}{2}(p_3 + p_2 + 1), \frac{1}{2}(p_3 + p_2 + 2), \cdots, \frac{1}{2}(p_3 + p_2 + p_1 - 1)$. However, the values of G-functions are neither tabulated nor known for a large number of cases. So we evaluate the integral of the expression (4.01) for determining the distributions of the criteria V for six different cases as mentioned in the introduction.

4.1. *Exact distributions of V for $p_1 = p_2 = 1$ and all values of p_3.* When $p_1 = p_2 = 1$, the expression (4.01) gives

$$f(V) = [\{\Gamma(\tfrac{1}{2}n)\}^2/\Gamma(\tfrac{1}{2}n - \tfrac{1}{2}p_3)\Gamma(\tfrac{1}{2}n - \tfrac{1}{2}p_3 - \tfrac{1}{2})] \cdot V^{\frac{1}{2}(n-p_3-3)}(2\pi i)^{-1} \cdot \int_{c-i\infty}^{c+i\infty} V^{-t} \cdot \Gamma(t)\Gamma(t + \tfrac{1}{2})\{\Gamma(t + \tfrac{1}{2}p_3 + \tfrac{1}{2})\}^{-2}\, dt$$

which, on applying Consul's inverse Mellin transform (2.1) and on simplification, gives the exact distribution of V in the hypergeometric form

$$(4.1.1) \quad f(V) = C \cdot V^{\frac{1}{2}(n-p_3-3)}(1 - V)^{p_3-\frac{1}{2}} \cdot F(\tfrac{1}{2}p_3, \tfrac{1}{2}p_3; p_3 + \tfrac{1}{2}; 1 - V)$$

where C is given by

$$(4.1.2) \quad C = \{\Gamma(\tfrac{1}{2}n)\}^2/[\Gamma(\tfrac{1}{2}n - \tfrac{1}{2}p_3)\Gamma\{\tfrac{1}{2}(n - p_3 - 1)\}\Gamma(p_3 + \tfrac{1}{2})].$$

To find the probability that $V \leqq v\ (\leqq 1)$ we integrate the expression (4.1.1) by parts $\frac{1}{2}(p_3 + 1)$ times, when p_3 is odd, between the limits 0 to $v\ (\leqq 1)$ and use the relation

$$(4.1.3) \quad (d^n/dz^n)[z^{c-1} \cdot F(a, b; c; z)] = (c - n)_n z^{c-1-n} \cdot F(a, b; c - n; z)$$

given by Erdelyi and others, each time and thus we obtain the cumulative distribution of V as

$$(4.1.4) \quad \Pr(V \leqq v) = I_v(\tfrac{1}{2}n - \tfrac{1}{2}p_3, \tfrac{1}{2}p_3) + Cv^{\frac{1}{2}(n-p_3-1)} \cdot \sum_{r=0}^{\frac{1}{2}(p_3-1)} (p_3 - r + \tfrac{1}{2})_r \{\tfrac{1}{2}(n - p_3 - 1)\}_{r+1}^{-1} v^r (1 - v)^{p_3-\frac{1}{2}-r} \cdot F(\tfrac{1}{2}p_3, \tfrac{1}{2}p_3; p_3 + \tfrac{1}{2} - r; 1 - v)$$

where $I_v(a, b)$ is the incomplete beta function tabulated by Pearson.

When p_3 is even, $F(\frac{1}{2}p_3, \frac{1}{2}p_3; p_3 + \frac{1}{2}; 1 - V)$ can be changed to $V^{\frac{1}{2}} \cdot F(\frac{1}{2}p_3 + \frac{1}{2}, \frac{1}{2}p_3 + \frac{1}{2}; p_3 + \frac{1}{2}; 1 - V)$ and then (4.1.1) can be integrated by parts $\frac{1}{2}p_3$ times between the limits 0 to $v\ (\leqq 1)$ and by using the same relation

(4.1.3) the cumulative distribution function becomes

$$\Pr(V \leqq v) = I_v(\tfrac{1}{2}n - \tfrac{1}{2}p_3 - \tfrac{1}{2}; \tfrac{1}{2}p_3 + \tfrac{1}{2})$$

$$(4.1.5) \quad + Cv^{\frac{1}{2}(n-p_3)} \textstyle\sum_{r=0}^{\frac{1}{2}p_3-1} (p_3 + \tfrac{1}{2} - r)_r(\tfrac{1}{2}n - \tfrac{1}{2}p_3)_{r+1}^{-1} v^r (1 - v)^{p_3 - \frac{1}{2} - r}$$

$$\cdot F(\tfrac{1}{2}p_3 + \tfrac{1}{2}, \tfrac{1}{2}p_3 + \tfrac{1}{2}; p_3 + \tfrac{1}{2} - r; 1 - v).$$

For particular values of p_3, the above hypergeometric functions in $f(V)$ and $\Pr(V \leqq v)$ can be transformed into algebraic forms by the use of the special functions (2.5) and (2.6) and some other results given by Consul in [4].

4.2. *Exact distributions of V for $p_1 = 1$, $p_2 = 2$ and all values of p_3.* When $p_1 = 1$ and $p_2 = 2$, the expression (4.01) becomes

$$f(V) = \{\Gamma(\tfrac{1}{2}n)\}^2\Gamma(\tfrac{1}{2}n - \tfrac{1}{2}) \cdot V^{\frac{1}{2}(n-p_3)-2}$$

$$\cdot [\Gamma(\tfrac{1}{2}n - \tfrac{1}{2}p_3)\Gamma\{\tfrac{1}{2}(n - p_3 - 1)\}\Gamma\{\tfrac{1}{2}(n - p_3 - 2)\}]^{-1}(2\pi i)^{-1}$$

$$\cdot \textstyle\int_{c-i\infty}^{c+i\infty} V^{-t}\Gamma(t)\Gamma(t + \tfrac{1}{2})\Gamma(t + 1)$$

$$\cdot [\{\Gamma(t + 1 + \tfrac{1}{2}p_3)\}^2\Gamma(t + \tfrac{1}{2}p_3 + \tfrac{1}{2})]^{-1}\, dt$$

which can be reduced by the successive use of Legendre's duplication formula into

$$f(V) = \Gamma(\tfrac{1}{2}n)\Gamma(n - 1)[\Gamma(\tfrac{1}{2}n - \tfrac{1}{2}p_3 - 1)\Gamma(n - p_3 - 1)]^{-1}V^{\frac{1}{2}(n-p_3-4)}$$

$$\cdot (2\pi i)^{-1} \textstyle\int_{c-i\infty}^{c+i\infty} V^{-t}\Gamma(2t + 1)\Gamma(t)[\Gamma(2t + p_3 + 1)\Gamma(t + \tfrac{1}{2}p_3 + 1)]^{-1}\, dt.$$

Now, by evaluating the integral by the use of Consul's integral transform (2.2), the exact distribution of the criteria V is obtained in the form

$$(4.2.1) \quad f(V) = \tfrac{1}{2}C \cdot V^{\frac{1}{2}(n-p_3-3)}(1 - V)^{\frac{1}{2}p_3+1}$$

$$\cdot \textstyle\sum_{r=0}^{p_3-1} \binom{p_3 - 1}{r} (-V^{\frac{1}{2}})^r F(\tfrac{1}{2}p_3 + 1, \tfrac{1}{2}r + \tfrac{3}{2}; \tfrac{1}{2}p_3 + 2; 1 - V)$$

where

$$C = \Gamma(n - 1)\Gamma(\tfrac{1}{2}n)/[\Gamma(n - p_3 - 1)\Gamma(\tfrac{1}{2}n - \tfrac{1}{2}p_3 - 1)\Gamma(p_3)\Gamma(\tfrac{1}{2}p_3 + 2)].$$

To determine the probability that $V \leqq v$ ($\leqq 1$), we integrate the distribution (4.2.1) between the limits 0 to v and then use the relation (4.1.3) and thus, on simplification, we obtain the cumulative distribution function of V as

$$\Pr(V \leqq v) = I_v[\tfrac{1}{2}(n - p_3) - 1, \tfrac{1}{2}p_3 + 1] + Cv^{\frac{1}{2}(n-p_3-2)}(1 - v)^{\frac{1}{2}p_3+1}$$

$$(4.2.2) \quad \cdot \textstyle\sum_{r=0}^{p_3-1} \binom{p_3 - 1}{r} (-1)^r(n - p_3 + r - 1)^{-1}$$

$$\cdot F(1, \tfrac{1}{2}p_3 - \tfrac{1}{2}r + \tfrac{1}{2}; \tfrac{1}{2}p_3 + 2; 1 - v).$$

By the use of our formula (2.6) and some other special functions, obtained by Consul (1966), the hypergeometric functions in $f(V)$ and $\Pr(V \leqq v)$ can be transformed for particular values of p_3 into algebraic functions also.

4.3. *Exact distributions of V for $p_1 = p_2 = 2$ and all values of p_3.* When $p_1 = p_2 = 2$, the expression (4.01), on reduction by the successive use of Legendre's duplication formula and on simplification, becomes

$$f(V) = \{\Gamma(n-1)\}^2[\Gamma(n-p_3-1)\Gamma(n-p_3-3)]^{-1}\cdot V^{\frac{1}{2}(n-p_3-5)}$$
$$\cdot(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} V^{-t}\Gamma(2t)\Gamma(2t+2)\{\Gamma(2t+p_3+2)\}^{-2}\cdot dt.$$

Now, by putting the value of the integral in the above expression by our result (2.1), the exact distribution function $f(V)$ takes the form

$$f(V) = \tfrac{1}{2}C\cdot V^{\frac{1}{2}(n-p_3-5)}(1-V^{\frac{1}{2}})^{2p_3+1}\cdot F(p_3, p_3; 2p_3+2; 1-V^{\frac{1}{2}}) \tag{4.3.1}$$

where C is given by (4.3.3).

By integrating the above function by parts (p_3+2) times between the limits 0 to v $(\leqq 1)$ and by using the formula (4.1.3) during each integration and on simplification, we find that the cumulative distribution function $\Pr(V \leqq v)$ is given by

$$\Pr(V \leqq v) = I_{v^{\frac{1}{2}}}(n-p_3-1, p_3) + Cv^{\frac{1}{2}(n-p_3-3)}$$
$$\cdot\sum_{r=0}^{p_3+1}(2p_3+2-r)_r(n-p_3-3)_{r+1}^{-1}v^{\frac{1}{2}r}(1-v^{\frac{1}{2}})^{2p_3+1-r} \tag{4.3.2}$$
$$\cdot F(p_3, p_3; 2p_3+2-r; 1-v^{\frac{1}{2}})$$

where

$$C = \{\Gamma(n-1)\}^2/\{\Gamma(n-p_3-1)\Gamma(n-p_3-3)\Gamma(2p_3+2)\}. \tag{4.3.3}$$

By the use of (2.5), (2.6) and other special functions given by Consul [4], the forms of $f(V)$ in (4.3.1) and of $\Pr(V \leqq v)$ in (4.3.2) can be changed into algebraic expressions for particular values of p_3.

4.4. *Exact distributions of V for $p_1 = 1$, $p_2 = 3$, and all values of p_3.* When $p_1 = 1$ and $p_2 = 3$, the expression (4.01) can be reduced by successive use of Legendre's duplication formula and simplified into

$$f(V) = C\cdot\Gamma(p_3)(2V^{\frac{1}{2}})^{n-p_3-4}V^{-\frac{1}{2}}$$
$$\cdot(2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} V^{-t}\Gamma(t)\Gamma(t+\tfrac{3}{2})\Gamma(2t+1)$$
$$\cdot[\{\Gamma(t+\tfrac{1}{2}p_3+\tfrac{3}{2})\}^2\Gamma(2t+p_3+1)]^{-1}\,dt$$

where $C = \Gamma(\frac{1}{2}n)\Gamma(\frac{1}{2}n-1)\Gamma(n-1)/\{\Gamma(n-p_3-3)\Gamma(n-p_3-1)\cdot\Gamma(p_3)\cdot\Gamma(\frac{1}{2})\}$.

The evaluation of the integral in the above expression by the help of our result (2.4) gives the exact distribution function of V in the hypergeometric form

$$f(V) = \frac{1}{2}C(2V^{\frac{1}{2}})^{n-p_3-4}(1-V)^{p_3+\frac{3}{2}}$$
$$\cdot\sum_{r=0}^{\infty}\left\{\left(\frac{1}{2}p_3+\frac{3}{2}\right)_r\right\}^2\left[r!\,\Gamma\left(p_3+r+\frac{5}{2}\right)\right]^{-1}(1-V)^r \tag{4.4.1}$$
$$\cdot\sum_{i=0}^{p_3-1}\binom{p_3-1}{i}(-1)^iF\left(1,\frac{1}{2}i; p_3+r+\frac{5}{2}; 1-V^{-1}\right).$$

By integrating $f(V)$ by parts between the limits 0 to v ($\leqq 1$) and by simplifying it with the use of (4.1.3) we obtain the cumulative distribution function as

$$\Pr(V \leqq v) = C \cdot 2^{n-p_3-4} \cdot \Gamma(p_3)\Gamma(n-p_3-2)\left[\Gamma(n-2)\Gamma\left(p_3+\frac{3}{2}\right)\right]^{-1}$$

$$\cdot \int_0^v V^{\frac{1}{2}(n-p_3-2)}(1-V)^{p_3+\frac{1}{2}} F\left(\frac{1}{2}p_3+\frac{3}{2}, \frac{1}{2}p_3+\frac{3}{2}; p_3+\frac{3}{2}; 1-V\right) dV$$

$$(4.4.2) \quad + \frac{1}{4} C(2v^{\frac{1}{2}})^{n-p_3-2}(1-v)^{p_3+\frac{3}{2}} \cdot \sum_{r=0}^{\infty} \left\{\left(\frac{1}{2}p_3+\frac{3}{2}\right)_r\right\}^2 \left[r!\,\Gamma\left(p_3+r+\frac{5}{2}\right)\right]^{-1}$$

$$\cdot (1-v)^r \sum_{i=0}^{p_3-1} \binom{p_3-1}{i} \{(-v^{\frac{1}{2}})^i/(n-p_3+i-2)\}$$

$$\cdot F\left(p_3+r+\frac{3}{2}, \frac{1}{2}i; p_3+r+\frac{5}{2}; 1-v\right)$$

where the value of the integral can be written as in (4.1).

4.5. *Distributions of V for $p_1 = 2$, $p_2 = 3$ and for even integral values of p_3.* When $p_1 = 2$, and $p_2 = 3$, the expression (4.01) can be reduced by the successive use of Legendre's duplication formula, to the form

$$f(V) = C\Gamma(\tfrac{1}{2}p_3) \cdot (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} V^{-t} \cdot \Gamma(2t)\Gamma(2t+3)\Gamma(t+1)$$

$$\cdot [\{\Gamma(2t+p_3+3)\}^2 \Gamma(t+\tfrac{1}{2}p_3+1)]^{-1}\, dt$$

where

$$C = \{\Gamma(n-1)\}^2 \Gamma(\tfrac{1}{2}n-1)/\{\Gamma(n-p_3-1)\Gamma(n-p_3-4)$$

$$\cdot \Gamma(\tfrac{1}{2}n - \tfrac{1}{2}p_3 - 1)\Gamma(\tfrac{1}{2}p_3)\}.$$

Now, by using Consul's inverse Mellin transform (2.3) to evaluate the above integral, the exact distribution function of V becomes

$$(4.5.1) \quad f(V) = C \cdot V^{\frac{1}{2}(n-p_3-4)}(1-V^{\frac{1}{2}})^{2p_3+3}$$

$$\cdot \sum_{r=0}^{\infty} (p_3+3)_r (p_3+3)_r\, [r!\,\Gamma(2p_3+r+4]^{-1} (1-V^{\frac{1}{2}})^r$$

$$\cdot \sum_{i=0}^{\frac{1}{2}p_3-1} \binom{\frac{1}{2}p_3-1}{i} (-1)^i F(1, 2i; 2p_3+4+r; 1-V^{-\frac{1}{2}}).$$

By integrating the above expression by parts between the limits 0 to v ($\leqq 1$), with the help of (4.1.3), and by simplifying, the cumulative distribution $\Pr(V \leqq v)$ becomes

$$\Pr(V \leqq v) = \frac{1}{2} C\Gamma\left(\frac{1}{2}p_3\right)\Gamma\left(\frac{1}{2}n - \frac{1}{2}p_3 - 1\right)\left[\Gamma\left(\frac{1}{2}n-2\right)\right]^{-1}$$

$$\cdot \int_0^v V^{\frac{1}{2}(n-p_3-3)}(1-V^{\frac{1}{2}})^{2p_3+2} F(p_3+3, p_3+3; 2p_3+3; 1-V^{\frac{1}{2}}) \cdot dV$$

$$(4.5.2)\qquad + 2Cv^{\frac{1}{2}(n-p_3-2)}(1 - v^{\frac{1}{2}})^{2p_3+3} \sum_{r=0}^{\infty} \{(p_3 + 3)_r\}^2[r!\Gamma(2p_3 + r + 4)]^{-1} \cdot(1 - v^{\frac{1}{2}})^r \sum_{i=0}^{\frac{1}{2}p_3-1} \binom{\frac{1}{2}p_3 - 1}{i} \{(-v)^i/(n - p_3 - 2 - 2i) \cdot F(2p_3 + r + 3, 2i; 2p_3 + r + 4; 1 - v^{\frac{1}{2}})$$

where the value of the integral can be easily written as in (4.3.2).

4.6. *Exact distributions of V for $p_1 = 2, p_2 = 4$ and all values of p_3*. When $p_1 = 2$ and $p_2 = 4$, the expression (4.01) can be first simplified and reduced by Legendre's duplication formula and then the integral can be evaluated with the help of formulae (2.4) and thus the exact distribution of V becomes

$$(4.6)\qquad f(V) = \tfrac{1}{2}C\cdot V^{\frac{1}{2}(n-p_3-7)}(1 - V^{\frac{1}{2}})^{3(p_3+1)} \cdot \textstyle\sum_{r=0}^{\infty} \{\Gamma(p_3 + r)\}^2[r!\Gamma(3p_3 + 4 + r)]^{-1}(1 - V^{\frac{1}{2}})^r \cdot F(p_3, 2p_3 + 2 + r; 3p_3 + 4 + r; 1 - V^{\frac{1}{2}})$$

where

$$C = \Gamma(n - 3)\{\Gamma(n - 1)\}^2/\{\Gamma(n - p_3 - 5)\Gamma(n - p_3 - 3) \cdot\Gamma(n - p_3 - 1)\Gamma(p_3)\Gamma(p_3)\}.$$

By integrating $f(V)$ by parts $(p_3 + 2)$ times between the limits 0 to v $(\leqq 1)$ and by using (4.1.3) each time and then by simplifying the resultant expression the cumulative distribution function $\Pr(V \leqq v)$ can also be determined.

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