DISTRIBUTION OF THE LARGEST LATENT ROOT AND THE SMALLEST LATENT ROOT OF THE GENERALIZED B STATISTIC AND F STATISTIC IN MULTIVARIATE ANALYSIS

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1. Introduction and results. The cumulative distribution function (cdf) of the largest latent root and the smallest latent root of the generalized B statistic and the generalized F statistic in multivariate analysis has been studied by K. C. S. Pillai [8], [9], [10], [11], and so on. But, since the cdf is represented by the summation of $p!$ incomplete B functions, it is complicated, and we have many difficulties in calculating it. Recently, these general expressions in the following formula were obtained by T. Sugiyama and K. Fukutomi [15], namely the probability element of the largest latent root λ_1 and the smallest latent root λ_p of the generalized B statistic was given respectively by the following formula

$$C\lambda_1^{n_1 p/2-1}(1-\lambda_1)^{(n_2-p-1)/2} \cdot F((-n_2+p+1)/2, (n_1-1)/2; (n_1+p+1)/2; \lambda_1 I_{p-1})\, d\lambda_1$$

and

$$C'\lambda_p^{(n_1-p-1)/2}(1-\lambda_p)^{n_2 p/2-1} \cdot F((-n_1+p+1)/2, (n_2-1)/2; (n_2+p+1)/2; (1-\lambda_p) I_{p-1})\, d\lambda_p$$

where

$$C = \pi^{p/2} B_{p-1}((n_1-1)/2, (p+2)/2)/\Gamma(p/2) B_p(n_1/2, n_2/2),$$

and

$$C' = \pi^{p/2} B_{p-1}((n_2-1)/2, (p+2)/2)/\Gamma(p/2) B_p(n_1/2, n_2/2),$$

and also, the probability element of the largest latent root f_1 and the smallest latent root f_p of the generalized F statistic is given by substituting $f_1/(1+f_1)$ for λ_1 and $f_p/(1+f_p)$ for λ_p in above formula respectively. But, since the cdf is also represented by the series of incomplete B function, it is not easy to calculate it. The purpose of this paper is to find the simple general expression of the probability element of the largest latent root λ_1 and the smallest latent root λ_p of the generalized B statistic, and the simple general expression of the cdf of the largest latent root and the smallest latent root of the generalized B statistic and the generalized F statistic which is given by the following theorems, respectively.

Theorem 1. *Let U_1 and U_2 are two independent matrices having Wishart distribution $W(p, n_1, \Sigma)$ and $W(p, n_2, \Sigma)$ respectively, where $n_1, n_2 \geqq p$. Then the prob-*

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ability element of the largest latent root λ_1 *and the smallest latent root* λ_p *of the equation (namely generalized B statistic* $(U_1 + U_2)^{-\frac{1}{2}}U_1(U_1 + U_2)^{-\frac{1}{2}}$*)*

$$|U_1 - (U_1 + U_2)\lambda| = 0$$

is given respectively as follows:

$$\begin{aligned}(1.1)\quad &\{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2)/\Gamma_p(n_2/2)\Gamma_p((n_1+p+1)/2)\} \\ &\cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{(pn_1/2+k)((-n_2+p+1)/2)_\kappa(n_1/2)_\kappa \\ &\cdot ((n_1+p+1)/2)_\kappa^{-1}\}C_\kappa(I_p)k!^{-1}\lambda_1^{pn_1/2+k-1}\,d\lambda_1\end{aligned}$$

and

$$\begin{aligned}(1.2)\quad &\{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2)/\Gamma_p(n_1/2)\Gamma_p((n_2+p+1)/2)\} \\ &\cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{(pn_2/2+k)((-n_1+p+1)/2)_\kappa(n_2/2)_\kappa \\ &\cdot ((n_2+p+1)/2)_\kappa^{-1}C_\kappa(I_p)k!^{-1}(1-\lambda_p)^{pn_2/2+k-1}\,d\lambda_p\end{aligned}$$

where I_p *is the unit matrix* $(p \times p)$,

$$(1.3)\qquad C_\kappa(I_p) = 2^k(p/2)_\kappa,$$

κ *is the partition* $(k_1, \cdots, k_p)$, $k_1 \geqq \cdots \geqq k_p \geqq 0$, *of the integer k into not more than p parts,*

$$(1.4)\qquad (a)_\kappa = \textstyle\prod_{i=1}^{p}(a - \frac{1}{2}(i-1))_{k_i}, \qquad \kappa = (k_1, \cdots, k_p),$$

and, as usual, $(\alpha)_{k_i} = \alpha(\alpha+1)\cdots(\alpha+k_i-1)$, $(\alpha_0)_0 = 1$.

This theorem is discussed in the following section.

THEOREM 2. *The* cdf *of the largest latent root* λ_1 *and the smallest latent root* λ_p *of the generalized B statistic is given respectively as follows:*

$$\begin{aligned}(1.5)\quad &P(\lambda_1 < x) \\ &= \{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2)/\Gamma_p(n_2/2)\Gamma_p((n_1+p+1)/2)\} \\ &\cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{((-n_2+p+1)/2)_\kappa(n_1/2)_\kappa((n_1+p+1)/2)_\kappa^{-1}\} \\ &\cdot C_\kappa(I_p)k!^{-1}x^{pn_1/2+k},\end{aligned}$$

and

$$\begin{aligned}(1.6)\quad P(\lambda_p < x) &= 1 - P(\lambda_p > x) \\ &= 1 - \{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2) \\ &\quad (\Gamma_p(n_1/2)\Gamma_p((n_2+p+1)/2))^{-1}\} \\ &\cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{((-n_1+p+1)/2)_\kappa(n_2/2)_\kappa((n_2+p+1)/2)_\kappa^{-1}\} \\ &\cdot C_\kappa(I_p)k!^{-1}(1-x)^{pn_2/2+k}.\end{aligned}$$

This theorem is obtained at once by integrating the equation (1.1) and (1.2) from 0 to x with respect to λ_1 and λ_p respectively.

THEOREM 3. *Let U_1 and U_2 are the two independent matrices having Wishart distribution $W(p, n_1, \Sigma)$ and $W(p, n_2, \Sigma)$ respectively, where $n_1, n_2 \geqq p$. Then the* cdf *of the largest latent root f_1 and the smallest latent root f_p of the equation (namely generalized F statistic $U_2^{-\frac{1}{2}}U_1U_2^{-\frac{1}{2}}$)*

$$|U_1 - U_2 f| = 0$$

is given respectively as follows:

$$\begin{aligned}
P(f_1 < y) &= P(\lambda_1 < y/(1+y)) \\
&= \{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2) \\
&\qquad (\Gamma_p(n_2/2)\Gamma_p((n_1+p+1)/2))^{-1}\} \\
&\qquad \cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{((-n_2+p+1)/2)_\kappa (n_1/2)_\kappa \\
&\qquad \cdot ((n_1+p+1)/2)_\kappa^{-1}\} C_\kappa(I_p) k!^{-1} (1/(1+y))^{pn_1/2+k}
\end{aligned} \tag{1.7}$$

and

$$\begin{aligned}
P(f_p < y) &= P(\lambda_p < y/(1+y)) = 1 - P(\lambda_p > y/(1+y)) \\
&= 1 - \{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2) \\
&\qquad (\Gamma_p(n_1/2)\Gamma_p((n_2+p+1)/2))^{-1}\} \\
&\qquad \cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{((-n_1+p+1)/2)_\kappa (n_2/2)_\kappa \\
&\qquad \cdot ((n_2+p+1)/2)_\kappa^{-1}\} C_\kappa(I_p) k!^{-1} (1/(1+y))^{pn_2/2+k}.
\end{aligned} \tag{1.8}$$

Since the roots $\lambda_1, \cdots, \lambda_p$ of the generalized B statistic are related to the roots $f_1, \cdots, f_p$ of the generalized F statistic as follows

$$\lambda_1 = f_1/(1+f_1), \cdots, \quad \lambda_p = f_p/(1+f_p),$$

this theorem will be obvious.

COROLLARY 1. *The probability element of the largest latent root λ_1 and the smallest latent root λ_{n_1} of the generalized B statistic of the degenerating case, namely $n_1 < p$, is given respectively as follows:*

$$\begin{aligned}
&\{\Gamma_{n_1}((n_1+1)/2)\Gamma_{n_1}((n_1+n_2)/2)/ \\
&\quad \Gamma_{n_1}((n_1+n_2-p)/2)\Gamma_{n_1}((p+n_1+1)/2)\} \\
&\quad \cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{(pn_1/2+k)((-n_2+p+1)/2)_\kappa (p/2)_\kappa \\
&\qquad\qquad \cdot ((n_1+p+1)/2)_\kappa^{-1}\} C_\kappa(I_{n_1}) k!^{-1} \lambda_1^{pn_1/2+k-1},
\end{aligned} \tag{1.9}$$

and

$$\begin{aligned}
&\{\Gamma_{n_1}((n_1+1)/2)\Gamma_{n_1}((n_1+n_2)/2) \\
&\quad (\Gamma_{n_1}(p/2)\Gamma_{n_1}((2n_1+n_2-p+1)/2))^{-1}\} \\
&\quad \cdot \textstyle\sum_{k=0}^{\infty}\sum_{\kappa}\{((n_1+n_2-p)n_1/2+k)((-p+n_1+1)/2)_\kappa \\
&\quad \cdot ((n_1+n_2-p)/2)_\kappa ((2n_1+n_2-p+1)/2)_\kappa^{-1} \\
&\qquad\qquad \cdot C_\kappa(I_{n_1}) k!^{-1} \lambda_{n_1}^{(n_1+n_2-p)n_1/2+k-1}.
\end{aligned} \tag{1.10}$$

The derivation of this corollary is mentioned in the following section.

COROLLARY 2. *The* hth *moment of the largest latent root* λ_1 *and the smallest latent root* λ_p *is given respectively as follows:*

$$\begin{aligned} E(\lambda_1^h) = \{&\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2) \\ &(\Gamma_p(n_2/2)\Gamma_p((n_1+p+1)/2))^{-1}\} \\ &\cdot \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} \{((-n_2+p+1)/2)_\kappa (n_1/2)_\kappa \\ &\cdot ((n_1+p+1)/2)_\kappa^{-1}\} \\ &\cdot C_\kappa(I_p) k!^{-1} (pn_1/2+k)/(pn_1/2+k+h), \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} E((1-\lambda_p)^h) = \{&\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2) \\ &(\Gamma_p(n_1/2)\Gamma_p((n_2+p+1)/2))^{-1}\} \\ &\cdot \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} \{((-n_1+p+1)/2)_\kappa (n_2/2)_\kappa \\ &\cdot ((n_2+p+1)/2)_\kappa^{-1}\} \\ &\cdot C_\kappa(I_p) k!^{-1} (pn_2/2+k)/(pn_2/2+k+h). \end{aligned} \tag{1.12}$$

Above each result are easily obtained by using each probability element of Theorem 1, and also, the hth moment of the degenerating case will be obtained by using Corollary 1.

COROLLARY 3. *The generalization of the Gauss formula*

$${}_2F_1(\alpha, \beta; \gamma; 1) = \Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)/\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)$$

is given by the following formula:

$${}_2F_1(\alpha, \beta; \gamma; I_p) = \Gamma_p(\gamma)\Gamma_p(\gamma-\alpha-\beta)/\Gamma_p(\gamma-\alpha)\Gamma_p(\gamma-\beta). \tag{1.13}$$

This corollary is mentioned in the following section.

COROLLARY 4. *The following equality holds:*

$$\begin{aligned} \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} \{(pn_1/2+k)((-n_2+p+1)/2)_\kappa (n_1/2)_\kappa ((n_1+p+1)/2)_\kappa^{-1}\} \\ \cdot C_\kappa(I_p) k!^{-1} = 0. \end{aligned}$$

This corollary will be obtained at once by letting $\lambda_1 = 1$ in (1.1).

2. The derivation of the largest latent root and the smallest latent root of the generalized B statistic. Let U_1 and U_2 be two independent matrices having Wishart distributions $W(p, n_1, \Sigma)$ and $W(p, n_2, \Sigma)$ respectively, where $n_1, n_2 \geqq p$. Then, the matrix $U = (U_1+U_2)^{-\frac{1}{2}} U_1 (U_1+U_2)^{-\frac{1}{2}}$ is called the generalized B statistic and the latent roots $\lambda_1, \cdots, \lambda_p$ of the matrix has the following probability element:

$$\text{Const } |\Lambda|^{(n_1-p-1)/2} |I-\Lambda|^{(n_2-p-1)/2} \textstyle\prod_{i<j}(\lambda_i-\lambda_j) \prod_{i=1}^{p} d\lambda_i \tag{2.1}$$

where

$$\text{(2.2)} \qquad \text{Const} = \pi^{p^2/2}\Gamma_p((n_1+n_2)/2)/\Gamma_p(p/2)\Gamma_p(n_1/2)\Gamma_p(n_2/2)$$

and Λ is the diagonal matrix

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \lambda_p \end{bmatrix}$$

with $1 > \lambda_1 > \cdots > \lambda_p > 0$.

From the expansion of the generalized binomial series, $|I - \Lambda|^{(n_2-p-1)/2}$ is written as follows:

$$\text{(2.3)} \qquad \sum_{k=0}^{\infty} \sum_{\kappa} ((-n_2+p+1)/2)_\kappa [C_\kappa(\Lambda)/k!],$$

where $C_\kappa(\Lambda)$ is the zonal polynomials defined each partition $\kappa(k_1, \cdots, k_p)$ of the integer k into not more than p parts as certain homogeneous and symmetric kth degree polynomials of $\lambda_1, \cdots, \lambda_p$. A detailed discussion of zonal polynomials may be found in A. T. James [5], and [6]. Using (2.3), (2.1) is rewritten as follows:

$$\text{(2.4)} \quad \text{Const} \sum_{k=0}^{\infty} \sum_{\kappa} \{((-n_2+p+1)/2)_\kappa/k!\} \cdot |\Lambda|^{(n_1-p-1)/2} C_\kappa(\Lambda) \prod_{i<j} (\lambda_i - \lambda_j) \prod_{i=1}^{p} d\lambda_i .$$

Let $l_i = \lambda_i/\lambda_1$, $i = 2, \cdots, p$, then we can write (2.4) as follows:

$$\text{(2.5)} \quad \text{Const} \sum_{k=0}^{\infty} \sum_{\kappa} \{((-n_2+p+1)/2)_\kappa/k!\} \lambda_1^{pn_1/2+k-1}\, d\lambda_1 \cdot |\Lambda_l|^{(n_1-p-1)/2} C_\kappa({}^1\Lambda_l) \prod_{i=2}^{p} (1-l_i) \prod_{i<j} (l_i - l_j) \prod_{i=2}^{p} dl_i ,$$

where $1 > \lambda_1 > 0$, $1 > l_2 > \cdots > l_p > 0$, and

$$\Lambda_l = \begin{bmatrix} l_2 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & l_p \end{bmatrix}, \qquad {}^1\Lambda_l = \begin{bmatrix} 1 & 0 \\ 0 & \Lambda_l \end{bmatrix}.$$

Therefore, integrating (2.5) with respect to $l_2, \cdots, l_p$, we will have the probability element of λ_1, namely to calculate the following

$$\text{(2.6)} \quad \int_{1>l_2>\cdots>l_p>0} |\Lambda_l|^{(n_1-p-1)/2} C({}^1\Lambda_l) \prod_{i=2}^{p} (1-l_i) \prod_{i<j} (l_i - l_j) \prod_{i=2}^{p} dl_i$$

is equal to obtain the probability element of λ_1. The following result which is shown in [13] will be useful to get the value of the integral (2.6):

$$\text{(2.7)} \quad \int_{1>\lambda_1>\cdots>\lambda_p>0} |\Lambda|^{t-(p+1)/2} |I-\Lambda|^{u-(p+1)/2} C_\kappa(\Lambda) \prod_{i<j}(\lambda_i - \lambda_j) \prod_{i=1}^{p} d\lambda_i = (\Gamma_p(p/2)/\pi^{p^2/2})(\Gamma_p(t,\kappa)\Gamma_p(u)\,[\Gamma_p(t+u,\kappa)]^{-1}) C_\kappa(I_p).$$

Now, let $u = (p+1)/2$ and $l_i = \lambda_i/\lambda_1$, $i = 2, \cdots, p$, then the left hand side (2.7) is written as follows:

$$(2.8)\quad \int_0^1 \lambda_1^{pt+k-1}\, d\lambda_1 \cdot \int_{1>l_2>\cdots>l_p>0} |\Lambda_l|^{t-(p+1)/2} \cdot C_\kappa({}^1\Lambda_l) \prod_{i=2}^p (1-l_i) \prod_{i<j} (l_i - l_j) \prod_{i=2}^p dl_i,$$

where κ is a partition of k. Thus we have the following:

$$\int_{1>l_2>\cdots>l_p>0} |\Lambda_l|^{t-(p+1)/2} C_\kappa({}^1\Lambda_l) \prod_{i=2}^p (1-l_i) \prod_{i<j} (l_i - l_j) \prod_{i=2}^p dl_i$$
$$(2.9)\quad = (pt+k)(\Gamma_p(p/2)/\pi^{p^2/2})(\Gamma_p(t,\kappa)\Gamma_p((p+1)/2) [\Gamma_p(t+(p+1)/2,\kappa)]^{-1}) C_\kappa(I_p).$$

Therefore, by letting $t = n_1/2$ in (2.9), from (2.2) and (2.5) we have the probability element of λ_1,

$$\{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2)/\Gamma_p(n_1/2)\Gamma_p(n_2/2)\}$$
$$(2.10)\quad \cdot \sum_{k=0}^\infty \sum_\kappa \{(pn_1/2 + k)((-n_2+p+1)/2)_\kappa \Gamma_p(n_1/2,\kappa) \cdot [\Gamma_p((n_1+p+1)/2,\kappa)]^{-1}\} C_\kappa(I_p) k!^{-1} \lambda_1^{pn_1/2+k-1}\, d\lambda_1 .$$

As usual, if a is such that the gamma function are defined, then $(a)_\kappa = \Gamma_p(a,\kappa)/\ \Gamma_p(a)$. Using it, (2.10) is rewritten as follows:

$$\{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2)/\Gamma_p(n_2/2)\Gamma_p((n_1+p+1)/2)\}$$
$$(2.11)\quad \cdot \sum_{k=0}^\infty \sum_\kappa \{(pn_1/2 + k)((-n_2+p+1)/2)_\kappa (n_1/2)_\kappa \cdot ((n_1+p+1)/2)_\kappa^{-1}\} C_\kappa(I_p) k!^{-1} \lambda_1^{pn_1/2+k-1}\, d\lambda_1 .$$

Thus, we get the probability element of the largest latent root, namely the formula (1.1) of Theorem 1.

To get the probability element of the smallest latent root λ_p, let $\Lambda^* = I - \Lambda$ in (2.1), namely $\lambda_i^* = I - \lambda_i$, $i = 1, \cdots, p$. Then, since λ_p^* is the largest latent root, we have the probability element of λ_p^* from (2.11)

$$\{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2)/\Gamma_p(n_1/2)\Gamma_p((n_2+p+1)/2)\}$$
$$(2.12)\quad \cdot \sum_{k=0}^\infty \sum_\kappa \{(pn_2/2 + k)((-n_1+p+1)/2)_\kappa (n_2/2)_\kappa \cdot ((n_2+p+1)/2)_\kappa^{-1}\} C_\kappa(I_p) k!^{-1} (\lambda_p^*)^{pn_2/2+k-1}\, d\lambda_p^*.$$

Therefore, by substituting $1-\lambda_p$ for λ_p^*, we have the probability element of the smallest latent root λ_p, namely the formula (1.2) of the Theorem 1. And also, since the joint probability element of the latent roots $\lambda_1, \cdots, \lambda_{n_1}$ of the degenerating case, i.e. $n_1 < p$, is given as follows:

$$(2.13)\quad \{\pi^{n_1^2/2}\Gamma_{n_1}((n_1+n_2)/2)/\Gamma_{n_1}(p/2)\Gamma_{n_1}((n_1+n_2-p)/2)\Gamma_{n_1}(n_1/2)\} \cdot |\Lambda|^{(p-n_1-1)/2}|I-\Lambda|^{(n_2-p-1)/2} \prod_{i<j}(\lambda_i-\lambda_j) \prod_{i=1}^{n_1} d\lambda_i$$

we will have the results of Corollary 1 by the same calculation.

A. G. Constantine [2] has defined as the series of zonal polynomials the generalized hypergeometric function ${}_pF_q\ (Z)$ of a complex symmetric matrix Z defined by C. S. Herz [3] by means of a multidimensional form of the Laplace transform. Namely

$$
\begin{aligned}
{}_pF_q\ (a_1,\ \cdots,\ a_p\ ;b_1,\ \cdots,\ b_q\ ;Z) \\
&= \textstyle\sum_{k=0}^{\infty} \sum_{\kappa} \{ (a_1)_\kappa \cdots (a_p)_\kappa C_\kappa(Z)/(b_1)_\kappa \cdots (b_q)_\kappa\, k!\}.
\end{aligned}
$$

If we use the above expression, (1.5) and (1.6) of Theorem 2 will be rewritten respectively as follows:

$$
\begin{aligned}
P(\lambda_1 < x) &= \{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2) \\
&\quad (\Gamma_p(n_2/2)\Gamma_p((n_1+p+1)/2))^{-1}\} \cdot {}_2F_1\ ((-n_2+p+1)/2, \\
&\quad n_1/2;\ (n_1+p+1)/2; xI_p)x^{pn_1/2},
\end{aligned}
\tag{2.14}
$$

and

$$
\begin{aligned}
P(\lambda_p < x) &= 1 - \{\Gamma_p((p+1)/2)\Gamma_p((n_1+n_2)/2) \\
&\quad (\Gamma_p(n_1/2)\Gamma_p((n_2+p+1)/2))^{-1}\} \cdot {}_2F_1\ ((1-n_1+p+1)/2, n_2/2; \\
&\quad (n_2+p+1)/2;\ (1-x)I_p) \cdot (1-x)^{pn_2/2}.
\end{aligned}
\tag{2.15}
$$

Let $x = 1$ in (2.14), and also $\alpha = (-n_2 + p + 1)/2$, $\beta = n_1/2$, $\gamma = (n_1 + p + 1)/2$ then we will obtain Corollary 3.

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