SAMPLE FUNCTIONS OF GAUSSIAN RANDOM HOMOGENEOUS FIELDS ARE EITHER CONTINUOUS OR VERY IRREGULAR

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1. In [1] Yu. K. Belyaev proves the following is true of any stationary stochastic process $t \rightarrow x(t)$ over the real numbers, mean-square continuous as a function of the reals, and having Gaussian joint distributions: Either with probability one the paths of the process are continuous or with probability one thsey are totally unbounded in any time interval. In this paper we will establish in a impler way the analogous result for a left-homogeneous random field over a second-countable locally compact topological group; i.e., one whose topology contains a countable base and is locally compact. We may arrange and re-label Belyaev's results as follows:

Lemma 1.1. *If the map $t \rightarrow x(t)$ fails at t_0 to be continuous with probability one, then there exists $a > 0$ such that for all t_0 and every neighborhood N of t_0,*

$$\text{(1)} \qquad \sup_{t \in N} (x(t) - x(t_0)) > a \quad \textit{with probability one.}$$

Lemma 1.2. *If $a > 0$ exists such that (1) holds for all t_0 and all neighborhoods N of t_0, then for all t_0 and N $\sup_{t \in N} (x(t) - x(t_0)) > 2a$ with probability one.*

By induction, if condition (1) holds then given any positive M, for every N, $\sup_{t \in N} (x(t) - x(t_0)) > M$ with probability one. Belyaev's proofs of both lemmas depend on the fact that one can write $x(t) = \sum_{n \geqq 0} y_n(t)$, where for $i \neq j$, $y_i(s)$ and $y_j(t)$ are independent Gaussian random variables no matter what s and t, and where each summand $t \rightarrow y_n(t)$ is continuous with probability one.

Belyaev's proof of Lemma 1.1 is difficult and depends on the fact that each summand $t \rightarrow y_n(t)$ can be made stationary. This he does by regarding each $x(t)$ as a stochastic integral over the dual group of the reals, and by partioning the domain of integration into countably many compact subsets. Although the random variables of random homogeneous fields over many interesting groups can also be so regarded (i.e., as random integrals over the dual object of the group), we wish to avoid irrelevant and difficult questions such as: When is the support of a spectral measure small enough to ensure path continuity with probability one, and when can one partition the dual object into countably many such "small" subsets?

On the other hand, Belyaev's proof of Lemma 1.2 applies to arbitrary locally compact second-countable groups with only minor obvious changes of notation. The proof is given here as Lemmas 2.4 and 2.5. It is essentially a verbatim repeat of the corresponding part of Belyaev's proof.

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of Belyaev's result, and to Professors R. M. Blumenthal and N. D. Ylvisaker for suggesting a more elementary approach than Belyaev's.

2. For the remainder of the paper G will be a locally compact second-countable group. A *left-homogeneous random field* over G is a continuous map $x: G \to L_2(\Omega, P)$, a Hilbert space of real-valued random variables with finite mean-square norm, with $Ex(g) = 0$ for all $g \varepsilon G$ and with finite covariances $Ex(g)x(h)$ depending only on $h^{-1}g$. We regard elements ω of the sample space Ω as paths (sample functions) in the usual way. We shall assume that every finite family $x(g_1), \cdots, x(g_n)$ has a Gaussian joint distribution. By Theorem 2.4 in Chapter II of [3] (which considers the case where G is the real line but which is valid in the present context) we may assume with no loss of generality that x is separable; i.e., that there exists a countable dense subset $D \subseteq G$ such that for every closed set A of real numbers and every open subset $N \subseteq G$, the events $\{x(g) \varepsilon A \text{ for all } g \varepsilon N\}$ and $\{x(g) \varepsilon A \text{ for all } g \varepsilon N \cap D\}$ differ only by a P-null set. Let H be the closed subspace of $L_2(\Omega, P)$ generated by all $x(g)$. We assume the reader is familiar with the following result in case G is the real line. Our proof in our general context is the same; one uses continuity of the map $g \to x(g)$ and the existence of a countable dense subset of G. (Homogeneity (stationarity) is not needed, however.)

LEMMA 2.1. *There exists in H a sequence $e_1, e_2, \cdots$ of independent Gaussian random variables with mean zero and variance one, such that for all $g \varepsilon G$*

$$x(g) = \textstyle\sum_{n \geqq 1} f_n(g) e_n ; \tag{2}$$

each coefficient f_n is continuous as a function on G.

Henceforth $\mathfrak{N}$ will denote a countable totally ordered neighborhood base at the identity $e \varepsilon G$; each $N \varepsilon \mathfrak{N}$ is assumed to have compact closure. The following proof is essentially Dobrushin's proof for G the real line ([2]).

THEOREM 2.2. *Either $P\{x(h)$ is continuous at every $h \varepsilon G\} = 1$ or for some $\beta > 0$*

$$P\{\limsup_{g \to h} x(g) - \liminf_{g \to h} x(g) \geqq \beta\} = 1 \quad \textit{for every} \quad h \varepsilon G. \tag{3}$$

PROOF. If $P\{x(h)$ is continuous at every $h \varepsilon G\} \neq 1$ then for some $\beta > 0$, some $h \varepsilon G$ and $N \varepsilon \mathfrak{N}$, $PA(\beta, h, N) > 0$, where $A(\beta, h, N) = \{\limsup_{g \to g_0} x(g) - \liminf_{g \to g_0} x(g) \geqq \beta$ for some $g_0 \varepsilon D \cap hN\}$. (This set is an event, since we may take lim sup and lim inf with g restricted to the countable set D.) Since at every $\omega \varepsilon \Omega$ the summands $f_n(g)e_n(w)$ in (2) are continuous functions of $g \varepsilon G$, $A(\beta, h, N)$ is a "tail" event belonging to the Borel fields $\mathfrak{B}\{e_n \mid n \geqq m\}$ for every m. By the zero-one law ([4] page 228) $PA(\beta, h, N) = 1$, since it is non-zero. From invariance of $Ex(g)x(h)$ under left translation it follows that the probability of any event generated by a subclass of $\{x(g) \mid g \varepsilon G\}$ is invariant under left-translation of the subclass; thus $PA(\beta, h, N)$ does not depend on h. From $PA(\beta, e, N) = 1$ it follows that, given any $M \varepsilon \mathfrak{N}$, $PA(\beta, e, M) = 1$: Otherwise $PA(\beta, e, M) = 0$ by the zero-one law, but this is impossible, since if $h_1, \cdots, h_n \varepsilon G$ are such that $\bigcup_{i=1}^n h_i M \supseteq N$, then the condition $PA(\beta, h_i, M) = 0$ $(i = 1, \cdots, n)$ would imply $PA(\beta, e, N) = 0$.

Now if $h \varepsilon G$ is arbitrary,

$$
\begin{aligned}
P\{\limsup_{g\to h} x(g) - \liminf_{g\to h} x(g) \geqq \beta\} \\
&= P\{\lim_{M\to e}(\sup_{g\varepsilon hM} x(g) - \inf_{g\varepsilon hM} x(g)) \geqq \beta\} \\
&= \lim_{M\to e} P\{\sup_{g\varepsilon hM} x(g) - \inf_{g\varepsilon hM} x(g) \geqq \beta\} \\
&\geqq \lim_{M\to e} PA(\beta, h, M) = 1, \quad \text{proving the Theorem.}
\end{aligned}
$$

We need a slight strengthening of Dobrushin's theorem:

LEMMA 2.3. *Either* $P\{x(h)$ *is continuous at every* $h \varepsilon G\} = 1$, *or for some* $b > 0$

$$(4) \qquad P\{\sup_{g\varepsilon hN}(x(g) - x(h)) > b\} = 1$$

for all h and every neighborhood N of the identity.

PROOF. Fix h and N. If $P\{x(h)$ is continuous at every $h \varepsilon G\} \neq 1$ let $\beta > 0$ be such that (3) holds. Then for every $M \varepsilon \mathfrak{N}, P\{\sup_{g\varepsilon hM} |x(g) - x(h)| \geqq \beta/2\} = 1$. We may replace the $\geqq$ here with $>$, by taking β slightly smaller. Further, we may approximate this probability within any $\epsilon > 0$ by replacing $\sup_{g\varepsilon hM}$ with a sup over merely finitely many g's, say $g_1, g_2, \cdots, g_n$. Due to symmetry of the joint distribution of $x(g_1), \cdots, x(g_n)$, $2P\{\max_{g_i} x(g_i) - x(h) \geqq \beta/2\} \geqq P\{\max_{g_i} |x(g_i) - x(h)| \geqq \beta/2\}$. By letting $\epsilon \to 0$ we find $P\{\sup_{g\varepsilon hM} x(g) - x(h) \geqq \beta/2\} \geqq \frac{1}{2}$. Letting $M \to e$, $P\{\lim_{g\to h} x(g) - x(h) \geqq \beta/2\} \geqq \frac{1}{2}$. Since the summands in (2) are continuous at all g when evaluated at any ω, $\{\limsup_{g\to h} x(g) - x(h) \geqq \beta/2\}$ is a "tail" event, so that by the zero-one law it has probability one as promised.

LEMMA 2.4. *If the equation in* (4) *holds for a given* $N \varepsilon \mathfrak{N}$, *then for any* $\epsilon > 0$ *and* $\delta > 0$ *one can write* $x = y + z$ *and find a finite subset* $\{g_1, \cdots, g_n\} \subseteq N$ *such that the following conditions are true of all sufficiently small* $M \varepsilon \mathfrak{N}$:

(i) *For all g and h,* $y(g)$ *and* $z(h)$ *are independent elements of* H,

(ii) $\inf_{g\varepsilon hM} P\{z(g) - z(h) > -\delta\} > 1 - \epsilon$, *and*

(iii) $P \bigcup_{i=1}^{n} \{\inf_{g\varepsilon g_i hM} y(g) - y(h) > b\} \geqq 1 - \epsilon$.

PROOF. Let $\{g_1, \cdots, g_n\} \subseteq D \cap N$ be such that $P\{\max_i x(hg_i) - x(h) > b\} > 1 - \epsilon/2$. Let m be an integer and write $y(g) = \sum_{n\leqq m} f_n(g)e_n$ (see (2)); let $z(g) = x(g) - y(g)$. Since the map $g \to z(g)$ into $H \subseteq L_2(\Omega, P)$ is continuous, and since N has compact closure, it follows from Chebyshev's inequality and from symmetry of the distributions of all elements of H, that m may be taken large enough to satisfy both $\inf_{g\varepsilon hN} P\{z(g) - z(h) > -\delta\} > 1 - \epsilon$ and

$$(5) \qquad P\{\max_i y(hg_i) - y(h) > b\} > 1 - \epsilon/2.$$

For each $k = 1, \cdots, n$ let $A_k = \{y(hg_i) - y(h) > b \text{ for } i = k \text{ but not } i < k\}$. If $PA_k > 0$ let P_k be the conditional P-probability given A_k. Since the map $g \to y(g)$ is continuous at every ω, $\lim_{g\to hg_k} y(g) - y(h) > b$ a.e. wrt P_k; thus for $k = 2, \cdots, n$ $P\{\inf_{g\varepsilon hg_k M} y(g) - y(h) > b\} \cap A_k > (1 - \epsilon/2)PA_k$ for all sufficiently small $M \varepsilon \mathfrak{N}$. (If for some k, $PA_k = 0$, one trivially obtains equality here.) Taking unions over k and applying (5), one obtains condition (iii).

LEMMA 2.5. *If condition* (4) *holds then for any* $\epsilon > 0$ *and* $\delta > 0$

$$(6) \qquad P\{\sup_{g\varepsilon hN} x(g) - x(h) > 2b - \delta\} \geqq 1 - 2\epsilon \quad \textit{for all} \quad N \varepsilon \mathfrak{N}.$$

PROOF. Let y, z and $\{g_1, \cdots, g_n\}$ be as in Lemma 2.4. Since $g \to y(g)$ is continuous on G when evaluated at any $\omega \varepsilon \Omega$, for all h the events $\{\limsup_{g\to h} x(g) - x(h) > b\}$ and $\{\limsup_{g\to h} z(h) - z(h) > b\}$ coincide. By (4) then, $P\{\sup_{g\varepsilon hg_iM} z(g) - z(h) > b\} = 1$ for $i = 1, \cdots, n$ and $M \varepsilon \mathfrak{N}$. Fix $M \varepsilon \mathfrak{N}$, let $B_k = \{\sup_{g\varepsilon hg_kM} z(g) - z(h) > b - \delta\}$. Since $B_k \supseteqq \{\sup_{g\varepsilon hg_kM} z(g) - z(g_k) > b\} \cap \{z(g_k) - z(h) \geqq -\delta\}$, condition (ii) of Lemma 2.4 implies $PB_k \geqq 1 - \epsilon$. For each k let $C_k = \{\inf_{g\varepsilon g_ihM} y(g) - y(h) > b$ for $i = k$ but not for $i < k\}$, so that $\bigcup_{k=1}^{n} C_k$ is the event in condition (iii) of Lemma 2.4. Condition (6) follows immediately from $\{\sup_{g\varepsilon hN} x(g) - x(h) > 2b - \delta\} \supseteqq \bigcup_{k=1}^{n} (B_k \cap C_k)$ and from condition (iii) of Lemma 2.4, since $PB_k \cap C_k = PB_kPC_k \geqq (1 - \epsilon) PC_k$ for each k.

THEOREM 2.6. (Belyaev's alternatives). *Let x be a Gaussian left-homogeneous random field over a locally compact second-countable group G. Then either $P\{x(h)$ is continuous at every $h \varepsilon G\} = 1$ or for every open set N, $P\{x(h)$ is totally unbounded for $h \varepsilon N\} = 1$.*

PROOF. If the first alternative condition fails then condition (4) implies that (6) holds for every $\epsilon > 0$ and $\delta > 0$. Since δ and ϵ are arbitrary $P\{\sup_{g\varepsilon hN} x(g) - x(h) \geqq 2b\} = 1$ for all $h \varepsilon G$ and all $N \varepsilon \mathfrak{N}$; this also holds if we replace $\geqq$ with $>$. By induction, for all $n = 1, 2, \cdots$, $P\{\sup_{g\varepsilon hN} x(g) - x(h) > 2^n b\} = 1$.

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