TESTING HYPOTHESES IN RANDOMIZED FACTORIAL EXPERIMENTS[1]

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0. Introduction. In the present paper we study the problem of testing the significance of a subgroup of 2^s pre-assigned parameters in an $n/2^{m-s}$ fractional replicate of a 2^m factorial experiment ($s < m$). In our previous paper on Randomization and Factorial Experiments [3] we outlined ANOVA schemes for such tests of significance, which were shown to be appropriate for the two randomization procedures RP I and RP II studied there. The test statistics proposed are the familiar F-like ratios. The main difficulty in performing those tests is in choosing the critical level for a given level of significance. This is also the main problem in performing ANOVA tests in the non-randomized designs. The problem is due to the effects of the nuisance parameters, which may be excessive and yet not under our control.

To be more specific, as will be shown in the sequel, the conditional distribution of the F-like ratio test statistics, given the fractional replicate chosen, is like that of a double non-central $F[\nu_1, \nu_2; \lambda, \lambda^*]$. Here, ν_1 and ν_2 are the appropriate degrees of freedom, λ and λ^* the parameters of non-centrality, being functions of the fractional replicate chosen, and of the vector of unknown parameters. Even under the null hypotheses, that the pre-assigned parameters are zero, λ and λ^* might be quite large due to the effects of the nuisance parameters. In the classical fractional replication model the assumptions imply that, under the null hypotheses, $\lambda = \lambda^* = 0$. For such a model the proper test criterion for level of significance γ is the $(1 - \gamma)$th fractile of $F[\nu_1, \nu_2]$, i.e., $F_{1-\gamma}[\nu_1, \nu_2]$. This is not the case, however, when λ and λ^* are positive.

When the values of the nuisance parameters are known, the problem is solved by a simple adjustment of the test statistics. A similar adjustment may also yield locally optimal test procedures when certain information is available on the nuisance parameters (see K. Takeuchi [6]). However, no proper solution to the problem can be attained in the non-randomized case if the values of the nuisance parameters are unknown. The objective of the present article is to verify that under certain conditions on the nuisance parameters, if the fractional replicate is chosen according to the randomization procedures studied in [3], the distributions of the F-like test statistics are approximated by the distributions of central F statistics, and the test criterion $F_{1-\gamma}[\nu_1, \nu_2]$ yields approximately the required level of significance γ. We also prove that under the established conditions on the nuisance parameters, if the fractional replicate is of a sufficiently large size com-

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pared to the number of pre-assigned parameters, the Hotelling-T^2 statistic yields a uniformly most powerful invariant simultaneous test of the significance of all the pre-assigned parameters. This simultaneous test procedure is applicable in cases where the number of pre-assigned parameters is relatively small, since it requires that the fractional replicate will be of size $n/2^{m-s}$, where $n > 2^s$. To establish the required conditions on the nuisance parameters, we show that the conditional bias of the common least squares estimators of the subvector of pre-assigned parameters is a sum of terms which are, under the randomization procedure considered, uncorrelated random variables; and the partial sums of these terms constitute a martingale. Extension of Doob's theorem [2] on the asymptotic normality of such standardized sums yields the required conditions for these conditional bias functions to be asymptotically normal. This property implies that the distributions of the F-like test statistics are asymptotically like those of central F statistics.

In Section 1 we present the statistical model and the testing problem. We introduce certain required notions and describe briefly one randomization procedure, RP I to which we restrict attention in the present study. The results can be easily extended to the other randomization procedure, RP II. In Section 2 we study the asymptotic distribution of the common least squares estimator of the subvector of pre-assigned parameters. The asymptotic theory is formulated in terms of the number of uncorrelated summands in the conditional bias functions. We assume that the factorial model under consideration is of a very large size, while the number of pre-assigned parameters is small. In Section 3 we introduce the ANOVA scheme suggested in [3], and study the distributions of the F-like test statistics under RP I. In Section 4 we study the problem of determining the critical levels of the F-like ratio test statistics, which will insure a prescribed level of significance. The difficulty is discussed, and several possible solutions are suggested. In Section 5 we study the problem of the simultaneous testing of all the pre-assigned parameters. We show that under the conditions of Section 2, and whenever the fractional replicate is sufficiently large, the Hotelling-T^2 statistics yield uniformly most powerful invariant simultaneous tests, with respect to all possible testing procedures under RP I.

1. The statistical model and the testing problem. Consider a 2^m factorial system in which the main effects and interactions are represented by a vector $\boldsymbol{\beta}' = (\beta_0, \cdots, \beta_{N-1})$ of $N = 2^m$ parameters. We are concerned with a subgroup of $S = 2^s$ $(s < m)$ main effects and interactions. The corresponding parameters will be called *pre-assigned* parameters, and will be represented by a subvector $\boldsymbol{\alpha}$ of order S. The parameters not in $\boldsymbol{\alpha}$ are called *nuisance* parameters. Without loss of generality (see Ehrenfeld and Zacks [3]) we assume that $\boldsymbol{\alpha}$ consists of the first S parameters of $\boldsymbol{\beta}$.

The N treatment combinations are classified, according to the common confounding method, into $M = 2^{m-s}$ mutually disjoint and exhaustive blocks (fractional replicates). The defining parameters according to which these M blocks are constructed are not in $\boldsymbol{\alpha}$. According to Zacks [7], if $\mathbf{Y}_v$ $(v = 0, \cdots, M-1)$ designates a random vector of order S, representing the yields of the treatment combinations in the vth block, the statistical model specifies:

$$\mathbf{Y}_v = (C^{(S)})\boldsymbol{\alpha} + (H_v)\boldsymbol{\beta}^* + \boldsymbol{\varepsilon}_v \qquad (v = 0, \cdots, M-1) \tag{1.1}$$

where $\boldsymbol{\beta}^*$ is the subvector of nuisance parameters; $(C^{(S)})$ is a Hadamard matrix of order S, defined recursively by the direct Kronecker's multiplication as follows:

$$(C^{(S)}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \otimes (C^{(S/2)}) \tag{1.2}$$

and

$$(H_v) = (C_{v1}^{(M)}, \cdots, C_{v,M-1}^{(M)}) \otimes (C^{(S)}), \qquad v = 0, \cdots, M-1, \tag{1.3}$$

$C_{vu}^{(M)}$ $(v = 0, \cdots, M-1; u = 1, \cdots, M-1)$ are the elements of the matrix $(C^{(M)})$. It is further assumed that the error vector $\boldsymbol{\varepsilon}_v$ $(v = 0, \cdots, M-1)$ is distributed like $N(\mathbf{0}^{(S)}, \sigma^2 I^{(S)})$, $0 < \sigma^2 < \infty$. The conditional (unadjusted) least squares estimator of $\boldsymbol{\alpha}$ (see Zacks [7]) is

$$\hat{\alpha}_v = S^{-1}(C^{(S)})'\mathbf{Y}_v \qquad (v = 0, \cdots, M-1). \tag{1.4}$$

It is easy to verify that the components of $\hat{\boldsymbol{\alpha}}_v$ are given by,

$$\hat{\alpha}_{vi} = \alpha_i + \eta_{vi} + \epsilon_{vi}^* \qquad (v = 0, \cdots, M-1; i = 0, \cdots, S-1) \tag{1.5}$$

where

$$\eta_{vi} = \sum_{u=1}^{M-1} C_{vu}^{(M)} \beta_{i+uS} \qquad (v = 0, \cdots, M-1; i = 0, \cdots, S-1), \tag{1.6}$$

and $\epsilon_{vi}^* \sim N(0, \sigma^2/S)$ independently for all v and all i. The functions η_{vi} are linear functions of the nuisance parameters alias to α_i $(i = 0, \cdots, S-1)$.

In a non-randomized procedure a block of treatment combinations (fractional replicate) is chosen in a fixed manner. A randomized procedure is one in which the vth block is chosen with probability ξ_v $(v = 0, \cdots, M-1)$. In RP I $\xi_v = 1/M$ for all $v = 0, \cdots, M-1$. In the present study we restrict attention to the case where n blocks are chosen independently according to RP I (sampling with replacement). Let $\hat{\boldsymbol{\alpha}}_{v_1}, \cdots, \hat{\boldsymbol{\alpha}}_{v_n}$ denote the estimators of $\boldsymbol{\alpha}$ according to (1.4) corresponding to the n chosen blocks. On the basis of the given values of $(\hat{\boldsymbol{\alpha}}_{v_1}, \cdots, \hat{\boldsymbol{\alpha}}_{v_n})$ we wish to test each of the S composite hypotheses:

(1.7) $H_0^{(i)}: \alpha_i = 0, \boldsymbol{\beta}^*$ arbitrary $(i = 0, \cdots, S-1)$ against the alternatives

$$H_1^{(i)}: \alpha_i \neq 0, \boldsymbol{\beta}^* \text{ arbitrary } \quad (i = 0, \cdots, S-1). \tag{1.8}$$

We shall also treat the problem of a simultaneous test of all the S hypotheses $H_0^{(i)}$. In most interesting cases we expect, however, that the simultaneous test will reject the hypothesis that all the components of α are zero. In this case we wish to test the significance of each component of α individually.

2. The asymptotic distribution of the estimator $\hat{\boldsymbol{\alpha}}_v$ under RP I. According to

(1.5) and (1.6), the conditional distribution of $\hat{\boldsymbol{\alpha}}_v$, given v, is like that of $N(\boldsymbol{\alpha} + \boldsymbol{\eta}_v, (\sigma^2/S)I^{(S)})$, when $\boldsymbol{\eta}_v' = (\eta_{v0}, \cdots, \eta_{v,s-1})$. Under RP I $\boldsymbol{\eta}_v$ is a random vector, being a linear combination of vectors consisting of the nuisance parameters, with random coefficients. Indeed, let $\boldsymbol{\beta}_{(1)}$ be a vector of order $S \times 1$, consisting of the first S elements of β, i.e., $\boldsymbol{\beta}_{(1)}' = (\beta_0, \cdots, \beta_{S-1})$. Let $\boldsymbol{\beta}_{(2)}' = (\beta_S, \cdots, \beta_{2S-1})$, and similarly, for each $j = 1, \cdots, M - 1$, let $\boldsymbol{\beta}_{(j)}' = (\beta_{(j-1)S}, \cdots, \beta_{jS-1})$. Define the $S \times (M - 1)$ matrix

$$B = (\boldsymbol{\beta}_{(1)}, \boldsymbol{\beta}_{(2)}, \cdots, \boldsymbol{\beta}_{(M-1)}).$$

Let $\mathbf{C}_v' = (C_{v,1}^{(M)}, \cdots, C_{v,M-1}^{(M)})$ where $\mathbf{C}_v'$ $(v = 0, \cdots, M - 1)$ is the vth row vector of $C^{(M)}$, short of the first element $C_{v,0}^{(M)} = 1$. Accordingly,

$$\boldsymbol{\eta}_v = B\mathbf{C}_v, \qquad v = 0, \cdots, M - 1. \tag{2.1}$$

We write $\boldsymbol{\eta}_v$ as a sum of $t = m - s$ random vectors $\mathbf{X}_k$ $(k = 1, \cdots, t)$ defined in the following manner: Partition the matrix B into t submatrices B_k of order $S \times 2^{k-1}$, where:

$$B_k = (\boldsymbol{\beta}_{(q)}, \cdots, \boldsymbol{\beta}_{(2q-1)}), \qquad q = 2^{k-1}, \qquad k = 1, \cdots, t. \tag{2.2}$$

Correspondingly, we partition the random vector $\mathbf{C}_v$ into t subvectors $\mathbf{z}_k$, where:

$$\mathbf{z}_k' = (C_{v,q}^{(M)}, \cdots, C_{v,2q-1}^{(M)}), \qquad q = 2^{k-1}, \qquad k = 1, \cdots, t. \tag{2.3}$$

According to the well known properties of the $C^{(M)}$ matrices (see [3]) we have:

$$\mathbf{z}_k' = C_{v,q}^{(M)}(\mathbf{z}_0', \cdots, \mathbf{z}_{k-1}'), \qquad q = 2^{k-1}, \qquad k = 1, \cdots, t, \tag{2.4}$$

where $\mathbf{z}_0 \equiv 1$. Furthermore, define for each $k = 1, \cdots, t$, $\mathbf{X}_k = B_k\mathbf{z}_k$. Then, it is simple to verify that

$$\boldsymbol{\eta}_v = \textstyle\sum_{k=1}^{t} \mathbf{X}_k. \tag{2.5}$$

We investigate in the present section the asymptotic distribution of $\boldsymbol{\eta}_v$ as $t \to \infty$.

For further development we present several properties of the sequence of random vectors $\{\mathbf{X}_1, \mathbf{X}_2, \cdots, \mathbf{X}_t, \cdots\}$. For the sake of simplifying notation, let $\mathbf{W}_k' = (\mathbf{z}_0', \cdots, \mathbf{z}_k')$, $k = 1, \cdots, t$. According to (2.4), given $\mathbf{W}_{k-1}$, $\mathbf{z}_k$ is determined up to a sign, and under RP I one has:

$$\begin{aligned} \mathbf{z}_k &= \mathbf{W}_{k-1}, && \text{with probability } \tfrac{1}{2}, \\ &= -\mathbf{W}_{k-1}, && \text{with probability } \tfrac{1}{2}, \end{aligned} \qquad k = 1, 2, \cdots, t. \tag{2.6}$$

Accordingly,

$$E\{\mathbf{X}_k\} = \mathbf{0}, \qquad \text{for all } k = 1, \cdots, t; \tag{2.7}$$

and

$$E\{\mathbf{X}_{k_1}\mathbf{X}_{k_2}'\} = 0, \qquad \text{for all } k_1 \neq k_2. \tag{2.8}$$

That is, the vectors $\{\mathbf{X}_k : k = 1, \cdots, t\}$ are mutually uncorrelated. Property (2.8) results from the fact that the inner product of two different column vectors of $C^{(M)}$ is zero; which implies that for every $q \neq q'$, $E\{C_{vq}^{(M)}C_{vq'}^{(M)}\} = (1/M)\sum_{W=0}^{M-1} C_{vq}^{(M)}C_{vq'}^{(M)} = 0$. The same orthogonality property yields, according to (2.3),

$$E\{\mathbf{z}_k\mathbf{z}_k'\} = I^{(q)}, \qquad q = 2^{k-1}, \qquad k = 1, \cdots, t. \tag{2.9}$$

Thus, the covariance matrix of $\mathbf{X}_k$ is:

$$E\{\mathbf{X}_k\mathbf{X}_k'\} = B_kB_k', \qquad k = 1, \cdots, t. \tag{2.10}$$

Let $\mathbf{T}_m$ $(m = 1, \cdots, t)$ be the vector of partial sums: $\mathbf{T}_m = \sum_{k=1}^{m} \mathbf{X}_k$. $\mathbf{T}_m$ is determined uniquely by the matrix B and the random vector $\mathbf{W}_m$. Let $\mathfrak{F}_m = \{(B, \mathbf{W}_m)\}$. Then, under RP I,

$$\begin{aligned} E\{\mathbf{T}_{m+1} \mid \mathfrak{F}_m\} &= \mathbf{T}_m + E\{B_{m+1}\mathbf{z}_{m+1} \mid \mathfrak{F}_m\} \\ &= T_m, \qquad m = 1, 2, \cdots. \end{aligned} \tag{2.11}$$

Thus, $\{(\mathbf{T}_m, \mathfrak{F}_m): m = 1, 2, \cdots\}$ is under RP I a martingale. The covariance matrix of $\mathbf{T}_m$ is, according to (2.8) and (2.10),

$$E\{\mathbf{T}_m\mathbf{T}_m'\} = \sum_{k=1}^{m} B_kB_k', \qquad m = 1, \cdots, t. \tag{2.12}$$

We assume that, for t sufficiently large $(t \geqq t')$, $\mathfrak{P}_t = \sum_{k=1}^{t} B_kB_k'$ is positive-definite. Thus, for every $t \geqq t'$, there exists a non-singular $S \times S$ matrix Q_t such that $\mathfrak{P}_t = Q_tQ_t'$. The following lemma is important for the proof of the main theorem.

Lemma 2.1. *Suppose that $\mathfrak{P}_t = \sum_{k=1}^{t} B_kB_k'$ is positive definite, and let $\mathfrak{P}_t = Q_tQ_t'$. Furthermore, let $\{\lambda_i^{(k)}: i = 0, \cdots, S-1\}$ be the characteristic roots of B_kB_k' $(k = 1, \cdots, t)$. Then, the condition:*

$$\sup_{0 \leqq i,j \leqq S-1} \sup_{1 \leqq k \leqq t} \left[\sum_{u=q}^{2q-1} |\beta_{j+us}| / \left(\sum_{k=1}^{t} \lambda_i^{(k)}\right)^{\frac{1}{2}}\right] = o(1), \qquad q = 2^{k-1}, \tag{2.13}$$

as $t \to \infty$, implies that

$$\sup_{1 \leqq k \leqq t} |Q_t^{-1}B_k\mathbf{W}_{k-1}| = o(1), \qquad \textit{as} \quad t \to \infty. \tag{2.14}$$

Proof. For every $k = 1, \cdots, t$, $B_kB_k' = \sum_{u=q}^{2q-1} \boldsymbol{\beta}_{(u)}\boldsymbol{\beta}_{(u)}'$, $q = 2^{k-1}$. Accordingly, for any k and $k' = 0, 1, \cdots, t$, if $q = 2^{k-1}$, $q' = 2^{k'-1}$,

$$\begin{aligned} B_kB_k'B_{k'}B_{k'}' &= \sum_{u=q}^{2q-1} \sum_{v=q'}^{2q'-1} \boldsymbol{\beta}_{(u)}\boldsymbol{\beta}_{(u)}'\boldsymbol{\beta}_{(v)}\boldsymbol{\beta}_{(u)}' \\ &= \sum_{u=q}^{2q-1} \sum_{v=q'}^{2q'-1} \varphi_{uv}\boldsymbol{\beta}_{(u)}\boldsymbol{\beta}_{(v)}', \end{aligned} \tag{2.15}$$

where $\varphi_{uv} = \boldsymbol{\beta}_{(u)}'\boldsymbol{\beta}_{(v)}$. The commutativity of the inner product, and the symmetry of $\boldsymbol{\beta}_{(u)}\boldsymbol{\beta}_{(v)}'$, imply that B_kB_k' and $B_{k'}B_{k'}'$ are commutative, for all $1 \leqq k, k' \leqq t$. Accordingly, there exists an orthogonal matrix P which simultaneously diagonalize each of the matrices B_kB_k' $(k = 1, \cdots, t)$. It follows that,

$$(2.16)\qquad P\Sigma_t P' = \begin{bmatrix} \sum_{k=1}^{t} \lambda_0^{(k)} & & 0 \\ & \ddots & \\ 0 & & \sum_{k=1}^{t} \lambda_{S-1}^{(k)} \end{bmatrix}.$$

P does not depend on t. Moreover, for every $k = 1, \cdots, t$,

$$(2.17)\qquad B_k' \Sigma_t^{-1} B_k = B_k' P' \begin{bmatrix} \sum_{k=1}^{t} \lambda_0^{(k)} & & 0 \\ & \ddots & \\ 0 & & \sum_{k=1}^{t} \lambda_{S-1}^{(k)} \end{bmatrix}^{-1} PB_k .$$

Hence,

$$(2.18)\qquad Q_t^{-1} B_k = \begin{bmatrix} \sum_{k=1}^{t} \lambda_0^{(k)} & & 0 \\ & \ddots & \\ 0 & & \sum_{k=1}^{t} \lambda_{S-1}^{(k)} \end{bmatrix}^{-\frac{1}{2}} PB_k , \qquad k = 1, \cdots, t.$$

Finally, according to (2.3), (2.4) and the definition of $\mathbf{W}_k$,

$$(2.19)\qquad Q_t^{-1} B_k \mathbf{W}_{k-1} = \Big(\zeta_0^{-\frac{1}{2}} \sum_{i=0}^{S-1} P_{0i} \sum_{u=0}^{q-1} \beta_{i+(q+u)S} C_{vu}^{(M)}, \cdots,$$
$$\zeta_{S-1}^{-\frac{1}{2}} \sum_{i=0}^{S-1} P_{S-1,i} \sum_{u=0}^{q-1} \beta_{i+(q+u)S} C_{vu}^{(M)}\Big), \quad q = 2^{k-1},$$

$k = 1, \cdots, t$; where $\zeta_i = \sum_{k=1}^{t} \lambda_i^{(k)}$ $(i = 0, \cdots, S-1)$; and $P = (P_{ij})$. Therefore, for every $k = 1, \cdots, t$,

$$(2.20)\qquad |Q_t^{-1} B_k \mathbf{W}_{k-1}| \leqq \sum_{i=0}^{S-1} \zeta_i^{-\frac{1}{2}} \sum_{k=0}^{S-1} |P_{ik}| \sum_{u=0}^{q-1} |\beta_{k+(q+u)S}|.$$

Hence (2.15) implies (2.14).

THEOREM 2.2. *The following three conditions imply that the distribution-law of* $Q_t^{-1}\mathbf{T}_t$ approaches $N(\mathbf{0}, I^{(S)})$ *as* $t \to \infty$:

(i) *for* t *sufficiently large* Σ_t *is positive-definite*;

(ii) *condition* (2.13) *of Lemma* 2.1.

(iii) *the fractional replicate is chosen according to randomization procedure* RP I.

PROOF. Let $\Psi_t(\mathbf{u})$ denote the characteristic function of $Q_t^{-1}\mathbf{T}_t$. We show that the above three conditions imply that $\lim_{t\to\infty} \Psi_t(\mathbf{u}) = \exp\{-\frac{1}{2}\mathbf{u}'\mathbf{u}\}$, which is the characteristic function of $N(\mathbf{0}, I^{(S)})$. The proof follows Doob [2] p. 386.

Write,

$$(2.21)\qquad \Psi_t(\mathbf{u}) = E\{\exp\{i\mathbf{u}'Q_t^{-1}\mathbf{T}_{t-1}\} \exp\{i\mathbf{u}'Q_t^{-1}\mathbf{X}_t\}\}.$$

Expanding $\exp\{i\mathbf{u}'Q_t^{-1}\mathbf{X}_t\}$ we obtain:

$$(2.22)\qquad \exp\{i\mathbf{u}'Q_t^{-1}\mathbf{X}_t\} = 1 + i\mathbf{u}'Q_t^{-1}\mathbf{X}_t - \tfrac{1}{2}(\mathbf{u}'Q_t^{-1}\mathbf{X}_t)^2$$
$$+ \tfrac{1}{6}\delta\, |\mathbf{u}|^3 \, |Q_t^{-1}\mathbf{X}_t|^3, \qquad |\delta| \leqq 1.$$

Furthermore, under RP I, $E\{\mathbf{X}_t \mid \mathfrak{F}_{t-1}\} = \mathbf{0}$.
Let,

$$\mathfrak{P}_{k,k-1} = E\{Q_t^{-1}\mathbf{X}_k\mathbf{X}_k'(Q_t^{-1})' \mid \mathfrak{F}_{k-1}\}, \qquad k = 1, \cdots, t.$$

Then,

$$E\{\exp\{i\mathbf{u}'Q_t^{-1}\mathbf{X}_t\} \mid \mathfrak{F}_{t-1}\} = 1 - \tfrac{1}{2}\mathbf{u}'\mathfrak{P}_{t,t-1}\mathbf{u} + \tfrac{1}{6}\delta\,|\mathbf{u}|^3 E\{|Q_t^{-1}\mathbf{X}_t|^3 \mid \mathfrak{F}_{t-1}\}. \tag{2.23}$$

However, $E\{|Q_t^{-1}\mathbf{X}_t|^3 \mid \mathfrak{F}_{t-1}\} = |Q_t^{-1}B_t\mathbf{W}_{t-1}|^3 = o(1)$, as $t \to \infty$, according to condition (ii) and Lemma 2.1. Thus,

$$\begin{aligned} \Psi_t(\mathbf{u}) &= \{\exp\{i\mathbf{u}'Q_t^{-1}\mathbf{T}_{t-1}\}E\{\exp\{i\mathbf{u}'Q_t^{-1}\mathbf{X}_t\} \mid \mathfrak{F}_{t-1}\}\} \\ &= E\{\exp\{i\mathbf{u}'Q_t^{-1}T_{t-1}\}(1 - \tfrac{1}{2}\mathbf{u}'\mathfrak{P}_{t,t-1}\mathbf{u} + o(1))\} \\ & \qquad \text{as} \quad t \to \infty. \end{aligned} \tag{2.24}$$

Condition (ii) implies that for every $k = 1, \cdots, t$,

$$\begin{aligned} \mathbf{u}'\mathfrak{P}_{k,k-1}\mathbf{u} &= E\{\mathbf{u}'Q_t^{-1}B_k\mathbf{W}_{k-1}\mathbf{W}_{k-1}'B_k'(Q_t^{-1})'\mathbf{u} \mid \mathfrak{F}_{k-1}\} \\ &\leqq |\mathbf{u}|^2 E\{|Q_t^{-1}B_k\mathbf{W}_{k-1}|^2 \mid \mathfrak{F}_{k-1}\} = |\mathbf{u}|^2 |Q_t^{-1}B_k\mathbf{W}_{k-1}|^2 \\ &= o(1), \qquad \text{as} \quad t \to \infty. \end{aligned} \tag{2.25}$$

Therefore, for all $m = 1, \cdots, t$, if $\mathbf{T}_m = \sum_{k=1}^m \mathbf{X}_k$,

$$\begin{aligned} \Psi_m(\mathbf{u}) &= E\{\exp\{i\mathbf{u}'Q_t^{-1}\mathbf{T}_m\}\} \\ &= E\{\exp\{i\mathbf{u}'Q_t^{-1}\mathbf{T}_{m-1} - \tfrac{1}{2}\mathbf{u}'\mathfrak{P}_{m,m-1}\mathbf{u}(1 + o(1))\}\}, \end{aligned} \tag{2.26}$$

as $t \to \infty$, where $\mathbf{T}_0 \equiv \mathbf{0}$, and $\mathfrak{P}_{1,0} \equiv \mathfrak{P}_1$. Thus, for every $m = 1, \cdots, t$ the following inequality holds as $t \to \infty$,

$$\begin{aligned} &|\Psi_m(\mathbf{u}) - E\{\exp\{i\mathbf{u}'Q_t^{-1}\mathbf{T}_{m-1} - (1/2t)\mathbf{u}'\mathbf{u}\}\}| \\ &\quad = |E\{\exp[i\mathbf{u}'Q_t^{-1}\mathbf{T}_{m-1} - (1/2t)\mathbf{u}'\mathbf{u}] \\ &\qquad \cdot(\exp\{-\tfrac{1}{2}\mathbf{u}'[\mathfrak{P}_{m,m-1}(1 + o(1)) - (1/t)I^{(S)}]\mathbf{u}\} - 1)\}| \\ &\quad \leqq E\{|\exp\{+\tfrac{1}{2}\mathbf{u}'[(1/t)I^{(S)} - \mathfrak{P}_{m,m-1}]\mathbf{u} + \mathbf{u}'\mathfrak{P}_{m,m-1}\mathbf{u}\cdot o(1)\} \\ &\qquad - 1|\} \leqq O(1)[E\{\tfrac{1}{2}\mathbf{u}'[(1/t)I^{(S)} - \mathfrak{P}_{m,m-1}]\mathbf{u}\} + o(1)/t]. \end{aligned} \tag{2.27}$$

Moreover,

$$\begin{aligned} |\Psi_t(\mathbf{u})e^{\frac{1}{2}\mathbf{u}'\mathbf{u}} - 1| &= |\textstyle\sum_{m=1}^t (\Psi_m(\mathbf{u})e^{(m/2t)\mathbf{u}'\mathbf{u}} - \Psi_{m-1}(\mathbf{u})e^{((m-1)/2t)\mathbf{u}'\mathbf{u}})| \\ &\leqq \textstyle\sum_{m=1}^t |\Psi_m(\mathbf{u}) - \Psi_{m-1}(\mathbf{u})e^{-(1/2t)\mathbf{u}'\mathbf{u}}| \\ &\leqq O(1)\textstyle\sum_{m=1}^t E\{(1/2t)\mathbf{u}'\mathbf{u} - \tfrac{1}{2}\mathbf{u}'\mathfrak{P}_{m,m-1}\mathbf{u}\} + o(1), \qquad \text{as} \quad t \to \infty \end{aligned} \tag{2.28}$$

Finally, for every t,

$$\begin{aligned} \textstyle\sum_{m=1}^t E\{\mathfrak{P}_{m,m-1}\} &= Q_t^{-1}(\textstyle\sum_{m=1}^t E\{\mathbf{X}_m\mathbf{X}_m'\})(Q_t^{-1})' \\ &= I^{(S)}. \end{aligned} \tag{2.29}$$

Hence

$$\sum_{m=1}^{t} E\{(1/2t)\mathbf{u}'\mathbf{u} - \tfrac{1}{2}\mathbf{u}'\mathfrak{T}_{m,m-1}\mathbf{u}\} = 0, \qquad \text{for all } \mathbf{u}. \tag{2.30}$$

We have thus obtained,

$$|\Psi_t(\mathbf{u})e^{\frac{1}{2}\mathbf{u}'\mathbf{u}} - 1| = o(1), \qquad \text{as } t \to \infty, \tag{2.31}$$

which implies that

$$\lim_{t\to\infty} \Psi_t(\mathbf{u}) = \exp\{-\tfrac{1}{2}\mathbf{u}'\mathbf{u}\}. \tag{2.31a}$$

This proves the theorem.

In the next section we present an ANOVA scheme in which individual tests of significance are performed for each component of $\boldsymbol{\alpha}$. For these tests we shall be concerned with the conditions under which all the marginal distributions of the components of $\hat{\boldsymbol{\alpha}}_v$ approach normal distributions. The condition is weaker than (2.13), since we are not concerned with the joint distribution of all the components of $\hat{\boldsymbol{\alpha}}_v$. From Theorem 2.2 we obtain

COROLLARY 2.3. *Under the randomization procedure* RP I *if the nuisance parameters satisfy the condition*

$$\sup_{1\leq k\leq t}\left[\sum_{u=q}^{2q-1}|\beta_{i+uS}|\Big/\left(\sum_{k=1}^{t}\sum_{u=q}^{2q-1}\beta_{i+uS}^2\right)^{\frac{1}{2}}\right] = o(1), \qquad q = 2^{k-1}, \quad as \quad t \to \infty, \tag{2.32}$$

for each $i = 0, \cdots, S-1$; *the marginal distribution laws of the components of* $\hat{\boldsymbol{\alpha}}_v$ *are asymptotically, as* $t \to \infty$, $N(\alpha_i, \sigma^2/S + D_{i,t}^2)$, *where*:

$$D_{i,t}^2 = \sum_{k=1}^{t}\sum_{u=q}^{2q-1}\beta_{i+uS}^2, \qquad q = 2^{k-1}, \qquad i = 0, \cdots, S-1 \tag{2.33}$$

The proof of Corollary 2.3 is obtained from Theorem 2.2 by considering one-dimensional vectors. In this particular case, letting in (2.13) $i = j$ be indices which can assume only one value, (2.13) is reduced to (2.32). Under this condition $\eta_{vi}/D_{i,t}$ is distributed asymptotically like $N(0, 1)$, $i = 0, \cdots, S-1$. Finally, the conditional distribution of $\hat{\alpha}_{vi}$ ($i = 0, \cdots, S-1$), given v, is like that of $N(\alpha_i + \eta_{vi}, \sigma^2/S)$. Hence, the asymptotic distribution of $\hat{\alpha}_{vi}$ as $t \to \infty$, under RP I and condition (2.32), is like that of $N(\alpha_i, \sigma^2/S + D_{i,t}^2)$.

The following is a trivial example of a sequence of constants $\{\beta_j : j = 0, 1, \cdots\}$, which for every $i = 0, \cdots, S-1$ satisfy condition (2.32). This sequence is defined as follows:

$$\begin{aligned} \beta_{i+uS} &= 1/k, \qquad \text{for} \quad u = q, q+1, \cdots, q+k-1, q = 2^{k-1}, \\ &= 0, \qquad \text{for} \quad u = q+k, \cdots, 2q-1, \end{aligned}$$

where $k = 1, 2, \cdots t$; $t = 1, 2, \cdots, i = 0, 1, \cdots, S-1$. For this sequence we have

$$\sum_{u=q}^{2q-1}\beta_{i+uS}^2 = 1/k, \qquad q = 2^{k-1}, \quad k = 1, 2, \cdots,$$

and $\sum_{u=q}^{2q-1}|\beta_{i+uS}| = 1$. Thus,

$$\sup_{1\leq k\leq t}\sum_{u=q}^{2q-1}|\beta_{i+uS}|\Big/\sum_{k=1}^{t}\sum_{u=q}^{2q-1}\beta_{i+uS}^2 = 1\Big/\sum_{k=1}^{t}k^{-1} \to 0, \quad \text{as} \quad t \to \infty,$$

for each $i = 0, \cdots, S-1$. Thus, condition (2.32) is satisfied.

This example shows that the number of non-zero nuisance parameters in each row of B_k ($k = 1, 2, \cdots, t$) should not grow too fast. For large values of k (as $t \to \infty$) most of the nuisance parameters should have a "negligible" effect. Condition (2.32) requires that all the diagonal elements of $\mathfrak{T}_t = \sum_{k=1}^t B_k B_k'$ will approach infinity as $t \to \infty$. This means that as k grows there must be some contribution to the growth of D_{it} from elements in B_k.

There are examples of sequences of nuisance parameters which do not satisfy (2.32), and *a fortiori* do not satisfy (2.13), but nevertheless the fourth central moment of $\eta_{vi}/D_{it} \to 3$ as $t \to \infty$, which is the fourth central moment of $N(0, 1)$. One example of such a sequence is: $\beta_{i+uS} = k^{\frac{1}{2}}/2^{k-1}$, for all $i = 0, \cdots, S-1$, and all $u = 2^{k-1}, \cdots, 2^k - 1$, where $k = 1, 2, \cdots, t, \cdots$. For this sequence,

$$D_{i,t}^2 = \sum_{k=1}^t k/2^{k-1} = 4(1 - (t+1)2^{-t} + t2^{-(t+1)}) \to 4$$

as $t \to \infty$; for all $i = 0, \cdots, S-1$. On the other hand,

$$\sum_{u=q}^{2q-1} |\beta_{i+uS}| = k^{\frac{1}{2}}, \qquad q = 2^{k-1}, \qquad \text{for all} \quad i = 0, \cdots, S-1.$$

Hence

$$\sup_{1 \leq k \leq t} \sum_{u=q}^{2q-1} |\beta_{i+uS}| = t^{\frac{1}{2}} \to \infty.$$

Thus, condition (2.32) is not satisfied. Nevertheless, one can show that for such a sequence, since under RP I

$$(2.34) \quad E\Big(\sum_{k=1}^t X_{ki}\Big)^4 = \sum_{k=1}^t E\{X_{ki}^4\} + 4 \sum\sum_{1 \leq k_1 < k_2 \leq t} E\{X_{k_1 i}^2 X_{k_2 i}^2\},$$

where X_{ki} denotes the ith components of $\mathbf{X}_k$, $\sum_{k=1}^t E\{X_{ki}^4\} \to 16$ as $t \to \infty$, and $\sum_{1 \leq k_1 < k_2 \leq t} E\{X_{k_1 i}^2 X_{k_2 i}^2\} \to 8$ as $t \to \infty$. Therefore,

$$\lim_{t \to \infty} E\Big(\sum_{k=1}^t X_{ki}\Big)^4 / D_{i,t}^4 = 3.$$

This example shows that we can expect a fairly good approximation to the asymptotic distribution of $\eta_{vi}/D_{i,t}$ by that of $N(0, 1)$ even if condition (2.32) is somewhat violated. If the contribution of the alias (nuisance) parameters is too small or too large the asymptotic approximation to a normal distribution will not hold.

3. The ANOVA scheme and the distribution of test statistics for RP I. As shown in [2], the hypotheses (1.7) can be tested on the basis of n independent estimates of $\boldsymbol{\alpha}$, $\hat{\boldsymbol{\alpha}}_{v_1}, \cdots \hat{\boldsymbol{\alpha}}_{v_n}$ say, obtained by RP I, by an ANOVA scheme in which the following statistics are determined for each component α_i of $\boldsymbol{\alpha}$ ($i = 0, \cdots, S-1$).

$$(3.1) \quad Q(\hat{\alpha}_{v_1 i}, \cdots, \hat{\alpha}_{v_n i}) = S \sum_{j=1}^n (\hat{\alpha}_{v_j i} - \hat{\alpha}_{\cdot i})^2 \qquad (i = 0, \cdots, S-1)$$

where $\hat{\alpha}_{\cdot i} = (1/n) \sum_{j=1}^n \hat{\alpha}_{v_j i}$. We compute also

$$(3.2) \qquad Q^*(\hat{\alpha}_{\cdot i}) = nS\hat{\alpha}_{\cdot i}^2 \qquad (i = 0, \cdots, S-1).$$

The conditional distribution law of $\hat{\alpha}_{vi}$ given v is $N(\hat{\alpha}_i + n_{vi}, \sigma^2/S)$. Thus, under RP I, the distribution law of $\hat{\alpha}_{vi}$ is the mixture $M^{-1}\sum_{v=0}^{M-1} N(\alpha_i + n_{vi}, \sigma^2/S)$; for which $E\hat{\alpha}_{vi} = \alpha_i$, and $\text{Var}\{\hat{\alpha}_{vi}\} = \sigma^2/S + \sum_{u=1}^{M-1} \beta_{i+uS}^2$ $(i = 0, \cdots, S - 1)$. This results from properties established previously. It follows that,

$$(3.3) \quad E\{Q(\hat{\alpha}_{v_1 i}, \cdots, \hat{\alpha}_{v_n i})\} = \sigma^2 + S\sum_{u=1}^{M-1} \beta_{i+uS}^2, \quad i = 0, \cdots, S - 1.$$

Similarly, the conditional distribution law of $\hat{\alpha}_{\cdot i}$, given $(v_1, \cdots, v_n)$, is $N(\alpha_i + \bar{n}_{\cdot i}, \sigma^2/nS)$ where $\bar{n}_{\cdot i} = (1/n)\sum_{j=1}^{n} n_{v_j i}$. Hence,

$$\text{Var}\{\hat{\alpha}_{\cdot i}\} = \sigma^2/nS + (1/n)\sum_{u=1}^{M-1} \beta_{i+uS}^2,$$

and

$$(3.4) \quad E\{Q^*(\hat{\alpha}_{\cdot i})\} = \sigma^2 + S\sum_{u=1}^{M-1} \beta_{i+uS}^2 + nS\alpha_i^2; \quad i = 0, \cdots, S - 1.$$

Comparing (3.3) and (3.4) we conclude that the F like ratios,

$$(3.5) \quad F^{(i)} = n\hat{\alpha}_{\cdot i}^2/(n - 1)^{-1}\sum_{j=1}^{M-1} (\hat{\alpha}_{v_j i} - \hat{\alpha}_{\cdot i})^2 \qquad (i = 0, \cdots, S - 1),$$

are proper test statistics for testing $H_0^{(i)}$ $(i = 0, \cdots, S - 1)$. One can ask whether a more sensitive test of $H_0^{(i)}$ can be attained by pooling the "within" mean-square-errors, which appear in the denominators of (3.5) for the various values of i. In other words, is

$$(3.6) \quad F^{*(i)} = n\hat{\alpha}_{\cdot i}^2/(n - 1)^{-1}\sum_{i=0}^{S-1}\sum_{j=1}^{n} (\hat{\alpha}_{v_j i} - \hat{\alpha}_{\cdot i})^2, \quad i = 0, \cdots, S - 1,$$

a more powerful test statistic than $F^{(i)}$, for each $i = 0, \cdots, S - 1$? This question will be discussed in the sequel. We shall see that under RP I the distributions of $F^{*(i)}$ might be considerably more complicated than those of $F^{(i)}$. This complication does not appear in RP II (see ANOVA scheme for RP II in [3]). The comparison of power functions of the test statistics in the ANOVA schemes for RP I and for RP II is of great practical importance, and will be presented elsewhere.

We turn to the study of the distribution of the test statistic $F^{(i)}$ (3.5) under RP I. For the sake of simplification we delete the subscript i.

Let a_j be a scalar component of $\hat{\alpha}_{v_j}$ $(j = 1, \cdots, n)$. The conditional distribution of the quadratic form $Q(a_1, \cdots, a_n)$, given $(v_1, \cdots, v_n) \equiv \mathbf{v}_n'$, is like that of $(\sigma^2/S)\chi_1^2[n - 1; \lambda(\mathbf{v}_n; \boldsymbol{\beta})]$, where the parameter of noncentrality is:

$$(3.7) \qquad \lambda(\mathbf{v}_n; \boldsymbol{\beta}) = (S/2\sigma^2)\sum_{j=1}^{n} (\eta_j - \bar{\eta})^2.$$

η_j designates the component of $\mathbf{n}_{v_j}$, corresponding to a_j and $\bar{\eta} = \sum_{j=1}^{n} \eta_j/n$. As specified in (1.6), each value of η_j is a linear function of the nuisance parameters which are alias to the component of $\boldsymbol{\alpha}$ under consideration. Similarly, the conditional distribution of $\bar{a} = \sum_{j=1}^{n} a_j/n$, given $\mathbf{v}_n$, is like that of $(\sigma^2/nS)\cdot\chi_2^2[1; \lambda^*(\mathbf{v}_n; \boldsymbol{\beta})]$, where

$$(3.8) \qquad \lambda^*(\mathbf{v}_n; \boldsymbol{\beta}) = nS(\alpha + \bar{\eta})^2/2\sigma^2.$$

$\chi_1^2[n - 1; \lambda]$ and $\chi_2^2[1; \lambda^*]$ are independent. Accordingly, the conditional distribution of the F-like statistic (3.5), given $\mathbf{v}_n$, is like that of a double-non-central

$F[1, n-1; \lambda^*, \lambda]$; whose distribution can be represented by a mixture of central F variates according to:

$$\begin{aligned} F[1, n-1; \lambda^*, \lambda] &\sim (n-1)\chi_2^2[1, \lambda^*]/\chi_1^2[n-1; \lambda] \\ &\sim (n-1)\chi_2^2[1+2J]/\chi_1^2[n-1+2M] \\ &\sim [(1+2J)(1+2M/(n-1))^{-1}] \\ &\quad \cdot F[1+2J, n-1+2M], \end{aligned} \tag{3.9}$$

where J and M are two independent random variables, having Poisson distributions with parameters (expectations) λ^* and λ, respectively.

Let $H(x \mid \nu_1, \nu_2; \alpha, \beta)$ denote the distribution function of the double non-central $F[\nu_1, \nu_2; \alpha, \beta]$, and let $H(x \mid \nu_1, \nu_2)$ denote the distribution function of $F[\nu_1, \nu_2]$. If $G(x \mid 1, n-1)$ denotes the distribution function of the F-like ratio (3.5), then

$$\begin{aligned} G_n(x \mid 1, n-1) &= E_{(\mathbf{v}_n)}\{H(x \mid 1, n-1; \lambda^*(\mathbf{v}_n, \boldsymbol{\beta}), \lambda(\mathbf{v}_n; \beta))\} \\ &= E_{(\mathbf{v}_n)}\{E_{(J,M)}\{H(x[1+2M/(n-1)] \\ &\quad (1+2J)^{-1} \mid 1+2J, n-1+2M)\}\}. \end{aligned} \tag{3.10}$$

In Section 2 we established conditions under which the asymptotic distribution of η is normal. As is readily proven, if the distribution of η is $N(0, D^2)$, $0 < D^2 < \infty$, the distribution of the F-like ratio is like that of a non-central $F[1, n-1; \xi]$. Indeed, if $\eta \sim N(0, D^2)$ then $a_j \sim N(\alpha, \sigma^2/S + D^2)$ identically for every $j = 1, \cdots, n$. Thus, in this case $Q(a_1, \cdots, a_n) \sim (\sigma^2 + SD^2)\chi_1^2[n-1]$ and $Q^*(\bar{a}) \sim (\sigma^2 + SD^2)\chi_2^2[1, \xi]$. Therefore,

$$F = nSQ^*(\bar{a})/(S/(n-1))Q(a_1, \cdots, a_n) \sim F[1, n-1; \xi],$$

where

$$\xi = nS\alpha^2/2(\sigma^2 + SD^2). \tag{3.11}$$

4. Determination of critical levels. In all cases where the distribution of η is expected to be reasonably approximated by a normal $N(0, D^2)$ distribution, the $(1-\gamma)$th fractile of the central $F[1, n-1]$, as a critical level for a test which rejects when the F-like ratio is too large, will attain a level of significance (size of the test) $\tilde{\gamma}$, close to the value γ aimed at. The actual size of the test $\tilde{\gamma}$ is:

$$\begin{aligned} \tilde{\gamma} &= E_{(\mathbf{v}_n)}\{P\{F[1, n-1; \lambda^*, \lambda] \geqq F_{1-\gamma}[1, n-1]\}\} \\ &= E_{(\mathbf{v}_n)}\{E_{(J,M)}\{P[F[1+2J, n-1+2M]] \\ &\quad \geqq F_{1-\gamma}[1, n-1][1+2M/(n-1)/(1+2J) \mid J, M]\}\}. \end{aligned} \tag{4.1}$$

In the case of a non-randomized fractional replicate the attained size of the test depends on the fractional replicate chosen and on the nuisance parameters. In the non-randomized case we have:

$$(4.2)\quad \tilde{\gamma}(\mathbf{v}_n) = E_{(J,M)}\{P[F[1 + 2J, n - 1 + 2M] \geqq F_{1-\gamma}[1, n - 1][1 + 2M/(n - 1)]/(1 + 2J) \mid J, M]\}.$$

A check of the tables of the fractiles of the central F distribution shows that for every $\gamma \leqq 0.10$ and all $n \geqq 4$,

$$(4.3)\quad F_{1-\gamma}[1, n - 1] \geqq F_{1-\gamma}[1 + 2J, n - 1 + 2M], \quad \text{for all } J, M = 0, 1, \cdots.$$

Thus, if the values of $\lambda^*(\mathbf{v}_n ; \boldsymbol{\beta})$ and $\lambda(\mathbf{v}_n ; \beta)$ yield a high probability that the Poisson random variables (J, M) satisfy the relationship $M \geqq (n - 1)J$, we may expect that with a high probability

$$P\{F[1 + 2J, n - 1 + 2M] \geqq F_{1-\gamma}[1, n - 1][1 + 2M/(n - 1)]/(1 + 2J) \mid (J, M)\} \leqq \gamma,$$

and that the resulting $\tilde{\gamma}(\mathbf{v}_n)$ does not exceed γ. Such a situation is insured when λ^* is small compared to λ. Examples can be constructed, when λ^* is large compared to λ, in which the attained level of significance $\tilde{\gamma}(\mathbf{v}_n)$ is substantially greater than the value γ aimed at. It seems that better control on the level of significance is provided by employing the randomization procedure RP I rather than a non-randomized procedure.

When the assumption concerning the approximation to normality of the distribution of η is unwarranted, but there is a prior knowledge concerning the magnitudes of λ^* and λ, one can try to adjust the critical level accordingly. For example, a first order approximation to the distribution of $G_n(x \mid 1, n - 1)$ can be obtained by the function:

$$(4.4)\quad G^*(x \mid 1, n - 1) = H(x \mid 1, n - 1; \Lambda^*(\boldsymbol{\beta}), \Lambda(\boldsymbol{\beta}));$$

where $\Lambda^*(\boldsymbol{\beta})$ and $\Lambda(\boldsymbol{\beta})$ are the expectations under RP I of $\lambda^*(\mathbf{v}_n ; \boldsymbol{\beta})$ and $\lambda(\mathbf{v}_n ; \boldsymbol{\beta})$ respectively. This approximation requires the prior knowledge of $\Lambda^*(\boldsymbol{\beta})$ and $\Lambda(\boldsymbol{\beta})$. The root of the equation $G^*(x \mid 1, n - 1) \equiv 1 - \gamma$, say, $G^*_{1-\gamma}(1, n - 1; \Lambda^*(\boldsymbol{\beta}), \Lambda(\boldsymbol{\beta}))$ determined for the conditions of the null hypothesis, may serve as an adjusted critical level. If the variances of $\lambda^*(\mathbf{v}_n ; \boldsymbol{\beta})$ and $\lambda(\mathbf{v}_n ; \boldsymbol{\beta})$ can also be assumed, a second order approximation can be tempted. A numerical analysis of the goodness of such approximations will be given elsewhere.

A final remark in this connection is that from a Bayesian point of view, if the prior distribution assumed for the nuisance parameters is $N(0, \tau^2 I)$, $0 < \tau^2 < \infty$, the uniformly most powerful invariant test of the null hypothesis, based only on the n independent estimates of the component of $\boldsymbol{\alpha}$ under consideration, is the test which rejects the null hypothesis whenever the value of the F-like ratio (3.5) exceeds $F_{1-\gamma}[1, n - 1]$. This shows that among all invariant test procedures whose level of significance does not exceed γ, and which are based only on the n independent estimates of the component of α under consideration, the F-like ratio compared with $F_{1-\gamma}[1, n - 1]$ is an admissible test. We have mentioned twice in the present discussion that the optimality of the individual F-tests, under the normality conditions, is restricted to the class of tests based only on the n independent estimates of each individual component of $\boldsymbol{\alpha}$. Actually, the estimates of the different components of $\boldsymbol{\alpha}$ are dependent, and it seems reasonable to consider simultaneous test procedures, or individual test procedures that take into account the estimates of all the S components of $\boldsymbol{\alpha}$. In the next section we discuss such a test.

5. Simultaneous testing under RP I. In cases where the size of the subgroup of pre-assigned parameters is not too large, so that the number of blocks of treatment combination $n > S = 2^s$, a simultaneous test procedure of the hypothesis:

$$H_0^*: \boldsymbol{\alpha} = 0, \boldsymbol{\beta} \text{ arbitrary}; \quad \text{versus} \quad H_1^*: \boldsymbol{\alpha} \neq 0, \boldsymbol{\beta} \text{ arbitrary},$$

is attained by the Hotelling-T^2 statistic in the following manner:

Let $\{\mathbf{a}_1, \cdots, \mathbf{a}_n\}$ be independent and identically distributed least-squares estimates of $\boldsymbol{\alpha}$. Define the $S \times n$ matrix: $A = (\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n)$. Furthermore, let

$$\hat{V} = A(I_n - (1/n)J_n)A', \tag{5.1}$$

where $J_n = \mathbf{1}_n\mathbf{1}_n'$ is an $n \times n$ matrix with all elements equal to $\mathbf{1}$. We consider the Hotelling-T^2 statistic

$$T^2 = n(n-1)\mathbf{y}'\hat{V}^{-1}\mathbf{y}, \tag{5.2}$$

where $\mathbf{y}$ is the mean vector: $\mathbf{y} = (1/n)\sum_{j=1}^{n} \mathbf{a}_j$; and reject H_0^* whenever T^2 is larger than a certain critical level.

THEOREM 5.1. *Under the conditions of Theorem 2.2, the test procedure: reject H_0^* whenever $T^2 \geqq [(n-1)S/(n-S)]F_{1-\gamma}[S, n-S]$ is asymptotically of level of significance γ; and asymptotically uniformly most powerful invariant.*

REMARK. As in Section 2, the asymptoteness of the result relates to the number of alias parameters for each pre-assigned parameter.

PROOF. Under the conditions of Theorem 2.2, the distribution of $\mathbf{a}_j$ $(j = 1, \cdots, n)$ is asymptotically like that of $N(\boldsymbol{\alpha}, (\sigma^2/S)I^{(S)} + \mathfrak{P}_t)$, as $t \to \infty$, where $\mathfrak{P}_t = \sum_{k=1}^{t} B_kB_k'$. Let $V_t = (\sigma^2/S)I^S + \mathfrak{P}_t$. As is known (see T. W. Anderson [1], Chapter 5), if $\mathbf{X}_1, \cdots, \mathbf{X}_n$ are independent, identically distributed like $N(\boldsymbol{\mu}, V)$, of all tests of $\boldsymbol{\mu} = 0$, which are invariant with respect to a non-singular linear transformation, the above T^2-test is uniformly most powerful. Furthermore, under the assumptions of Theorem 2.2, the asymptotic distribution of T^2 given by (5.2) is like that of $[(n-1)S/(n-S)]F[S, n-S]$. This proves the theorem.

In the ANOVA scheme, given in [3], we indicated how one can test simultaneously whether $\boldsymbol{\eta} = 0$. There is no special difficulty in performing such a test, since under the null hypothesis the distribution of the proper test statistic is like that of a central F. If this test accepts the hypothesis $\boldsymbol{\eta} = 0$, while the Hotelling-T^2 test rejects the hypothesis $\boldsymbol{\alpha} = 0$, a test of significance of the individual components of $\boldsymbol{\alpha}$ can be performed by $F^{*(i)}$ $(i = 0, \cdots, S-1)$, defined in (3.6). If $\boldsymbol{\eta} = 0$, the distribution of $F^{*(i)}$, under the null hypothesis, is like that of $F[1, S(n-1)]$, and the test which rejects whenever $F^{*(i)} \geqq F_{1-\gamma}[1, S(n-1)]$ is, obviously, more powerful than the test based on $F^{(i)}$ given in (3.5). On the other hand, if $\mathbf{n} \neq 0$ there are again distributional complications. Even if the conditions of Theorem 2.2 are fulfilled, the asymptotic distribution of $F^{*(i)}$ is like that of:

$$(\sigma^2 + SD_i^2)\chi^2[1; \xi]/\sum_{i=0}^{s-1} (\sigma^2 + SD_i^2)\chi^2[n - 1],$$

where all the χ^2's are independent, and ξ is given in (3.11). If the values of D_i $(i = 0, \cdots, S - 1)$ are unknown we cannot control the level of significance by a proper choice of a critical level. The question of devising a test procedure more powerful than the test statistics $F^{(i)}$, for testing the significance of individual components of $\boldsymbol{\alpha}$ or some linear functions of $\boldsymbol{\alpha}$, is still open for further research.

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