A PERFECT MEASURABLE SPACE THAT IS NOT A LUSIN SPACE

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In connection with his study [1] of the measure-theoretic foundation for probability theory, Blackwell has introduced the notion of a Lusin space; i.e., a measurable space $(\Omega, \mathbf{B})$ such that: (a) $\mathbf{B}$ is a separable σ-algebra and (b) for every function f, real-valued and measurable $(\mathbf{B})$, the range of f is an analytic set. In the place cited, Blackwell observed that every probability measure P defined on $\mathbf{B}$ converts the Lusin space $(\Omega, \mathbf{B})$ into a perfect probability space $(\Omega, \mathbf{B}, P)$ and asked whether the Lusin spaces be the only measurable spaces having this property. In order to settle this question, there are presented below a separable σ-algebra $\mathbf{B}$ of subsets of a set Ω and a real-valued, measurable$(\mathbf{B})$ function f such that: (i) if P be any probability measure on $\mathbf{B}$, then $(\Omega, \mathbf{B}, P)$ is perfect, and (ii) the range of f is not an analytic set.

Let $\mathbf{B}_0$ be the class of all Borel subsets of the unit interval, and let Ω be an uncountable, universal null set contained in $[0, 1]$: i.e., if ν be any nonatomic probability measure on $\mathbf{B}_0$, then $\nu^*(\Omega) = 0$. (To conclude the existence of sets of this kind, it seems to be necessary to assume the continuum hypothesis, see, for example [3], 432–440.) Because an uncountable analytic set contains a perfect set and as a result, supports a nonatomic probability measure, Ω is necessarily nonanalytic. Let $\mathbf{B} = \mathbf{B}_0 \cap \Omega$.

Suppose that P is a probability measure on $\mathbf{B}$. For each B in $\mathbf{B}_0$, let $\mu(B) = P(B \cap \Omega)$. The set function thus engendered is obviously a measure on $\mathbf{B}_0$. Let μ_a and μ_c be the atomic and continuous parts of this measure. Because $\mu_c^*(\Omega) = 0$, it follows that μ_c is trivial, so that μ is totally atomic. Now the atoms of μ are singletons, and, by virtue of the definition of the measure, each of these must be a subset of Ω. Denote by X_μ the union of μ's atoms.

Let $g:\Omega \rightarrow R$ be measurable$(\mathbf{B})$. If $g^{-1}(A)$ belongs to $\mathbf{B}$, then $B = \{x \,\varepsilon\, A: g^{-1}(x) \cap X_\mu \neq \emptyset\}$ is a countable (Borel) subset of A, and $\mu(g^{-1}(B)) = \mu(g^{-1}(A))$; thus, $(\Omega, \mathbf{B}, \mu)$ is perfect.

On the other hand, let $f:\Omega \rightarrow R$ be the restriction to Ω of the identity function on $[0, 1]$. Certainly f is measurable$(\mathbf{B})$, but $f[\Omega]$ is nonanalytic.

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