ON A STOPPING RULE AND THE CENTRAL LIMIT THEOREM[1]

By Richard F. Gundy and David Siegmund[2]

Rutgers—The State University and Columbia University

Let $S_n = \sum_{k=1}^{n} x_k$, $n \geqq 1$, where x_k, $k \geqq 1$, are independent, orthonormal (i.o.) random variables, i.e., $Ex_k = 0$, $Ex_k^2 = 1$, $k \geqq 1$. For $c > 0$, let t_c be the smallest positive integer n such that $|S_n| > cn^{\frac{1}{2}}(= \infty$ if no such n exists). Our principal aim is to prove the following:

Theorem 1. *If*

$$\lim_{n\to\infty} P(S_n/n^{\frac{1}{2}} \leqq x) = \Phi(x), \tag{1}$$

where $\Phi(x)$ is the standard normal distribution, then

$$E(t_c) < \infty \quad \text{if} \quad 0 \leqq c < 1; \tag{2}$$

$$E(t_c) = \infty \quad \text{if} \quad c \geqq 1. \tag{3}$$

Special cases of this theorem have appeared previously. Blackwell and Freedman [1] have treated the "coin-tossing" case in which x_k, $k \geqq 1$, are symmetric and assume the values ± 1. Chow, Robbins, and Teicher [3] have proved that (2) holds for i.o. sequences that are uniformly bounded and that (3) holds for *arbitrary* i.o. sequences. When the variables are identically distributed and $E|x_1|^3 < \infty$, one may conclude that (2) holds, at least for c sufficiently small, by using estimates of tail probabilities, due to Breiman [2].

Our computations rely on the well-known Lindeberg-Feller theorem: *Condition* (1) *is equivalent to*

$$\lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} \int_{\{x_k^2 > \epsilon n\}} x_k^2 \, dP = 0 \tag{4}$$

for every $\epsilon > 0$.

Since Chow *et al.* have shown that (3) holds for every sequence of i.o. variables, it suffices to show that $E(t_c) < \infty$ for $0 < c < 1$. We now fix c and so may dispense with it as a subscript in the sequel. In what follows, we deal with the sequence of stopping rules $\tau = \tau(n) = \min(t, n)$, $n \geqq 1$. Then by a variant of Wald's lemma $ES_\tau^2 = E\tau$, $n \geqq 1$, so that

$$\text{Et} = \lim_{n\to\infty} E\tau = \lim_{n\to\infty} ES_\tau^2 \leqq \infty.$$

(See Section 2 of [3].) The following lemma says, in effect, that instead of considering the expectation of the square of the entire random sum, it is sufficient for our purposes to consider only Ex_τ^2, the expectation of the square of the final term.

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[2] Now at Stanford University.

LEMMA. *If* x_k, $k \geqq 1$, *is a sequence of* i.o. *variables, then*

$$0 < (1 - c)^2 E\tau \leqq Ex_\tau^2 \leqq E\tau, \qquad n = 1, 2, \cdots.$$

PROOF. By the variant of Wald's lemma mentioned above and the definition of t, we have for $n = 1, 2, \cdots$,

$$E\tau = ES_\tau^2 = ES_{\tau-1}^2 + 2E(S_{\tau-1}x_\tau) + Ex_\tau^2$$
$$\leqq c^2 E\tau + 2c(E\tau)^{\frac{1}{2}}(Ex_\tau^2)^{\frac{1}{2}} + Ex_\tau^2.$$

Putting $r^2 = Ex_\tau^2/E\tau$, we have $0 \leqq (c^2 - 1) + 2cr + r^2$. An examination of this expression as a quadratic form in r leads to the inequality $(1 - c)^2 \leqq Ex_\tau^2/E\tau$, $n \geqq 1$, which together with $Ex_\tau^2 \leqq E(\sum_{k=1}^{\tau} x_k^2) = E\tau$, $n \geqq 1$, gives the desired conclusion.

PROOF OF THE THEOREM. Suppose that $\lim_{n\to\infty} E\tau = Et = \infty$; we contradict the conclusion of the lemma by showing that this implies $Ex_\tau^2 = o(E\tau)$.

Let $I\{\cdot\}$ denote the indicator function of the set in braces.

For any $\epsilon > 0$,

$$(5) \qquad Ex_\tau^2 = \int I\{x_\tau^2 \leqq \epsilon\tau\}x_\tau^2\, dP + \int I\{x_\tau^2 > \epsilon\tau\}x_\tau^2\, dP$$
$$\leqq \epsilon E\tau + E\left(\sum_{k=1}^{\tau} I\{x_k^2 > \epsilon k\}x_k^2\right).$$

By taking conditional expectations under the summation, we find that

$$(6) \qquad E\left(\sum_{k=1}^{\tau} I\{x_k^2 > \epsilon k\}x_k^2\right) = E\left(\sum_{k=1}^{\tau} \int I\{x_k^2 > \epsilon k\}x_k^2\, dP\right).$$

To estimate (6), note that for all n sufficiently large, the Lindeberg condition (4) implies

$$\sum_{k=1}^{n} \int I\{x_k^2 > \epsilon k\}x_k^2\, dP \leqq \sum_{k=1}^{[\epsilon n]} 1 + \sum_{k=[\epsilon n]+1}^{n} \int I\{x_k^2 > \epsilon^2 n\}x_k^2\, dP \leqq 2\epsilon n.$$

Therefore,

$$(7) \qquad E\left(\sum_{k=1}^{\tau} \int I\{x_k^2 > \epsilon k\}x_k^2\, dP\right) \leqq \text{Const} + 2\epsilon E\tau.$$

Since $\epsilon > 0$ is arbitrary, we may combine (5), (6), and (7) to obtain $E(x_\tau^2) \leqq 3\epsilon E\tau + \text{Const} = o(E\tau)$ as $E\tau \to \infty$. This, however, contradicts the conclusion of the lemma so that the theorem is proved.

REMARKS. Slight modifications of the above proof show that if $t_c(m) =$ first $n \geqq m$ such that $|S_n| > cn^{\frac{1}{2}}$, $m = 1, 2, \cdots$, then (4) implies that $Et_c(m) < \infty$, all $m \geqq 1$, $c < 1$, and it is easy to construct counter examples to show that some condition such as (4) is necessary in order to obtain this conclusion. We do not know if (4) is itself necessary in the sense that if for some $\epsilon > 0$ (4) does not hold then for some $m = 1, 2, \cdots, 0 < c < 1$, $Et_c(m) = \infty$.

Combining the above methods of proof with a truncation argument, we can obtain similar results in the case of infinite variances. Suppose that $x_1, x_2, \cdots$ are independent random variables. For each $a > 0$, $n = 1, 2, \cdots$, define

$$(8) \qquad y_n = y_n(a) = x_n I\{|x_n| \leqq an^{\frac{1}{2}}\}, \qquad T_n = y_1 + \cdots + y_n,$$
$$\beta_n = (ET_n)^2, \qquad B_n = \text{Var}(T_n).$$

THEOREM 2. *Let* $0 < c < \infty$ *and suppose that for some* $a > 2c$

$$\lim_{n\to\infty} n^{-1} B_n = \infty. \tag{9}$$

If either

$$\limsup (\beta_n / B_n) < 1 \tag{10}$$

or

$$\liminf (\beta_n / B_n) > 1, \tag{11}$$

then $Et_c < \infty$.

Condition (10) is satisfied if the x_k are symmetrically distributed; condition (11) is satisfied if, for example, the x_k are identically distributed with distribution function F where $F(-x) = o(1 - F(x))$, $x \to \infty$, and $1 - F(x) \sim x^{-\alpha} L(x)$, $1 < \alpha < 2$, L slowly varying.

Theorems 1 and 2 may be combined to show that if the variables x_k are identically distributed and symmetric, then Et_c is finite or infinite according as $c < 1$ or $c \geqq 1$ if and only if $Ex_k^2 = 1$.

REFERENCES

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