The Annals of Mathematical Statistics
1968, Vol. 39, No. 1, 282–285

ON A FURTHER ROBUSTNESS PROPERTY OF THE TEST AND ESTIMATOR BASED ON WILCOXON'S SIGNED RANK STATISTIC[1]

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1. Summary and introduction. The robust-efficiency of the test and estimator based on Wilcoxon's [7] signed rank statistic when the sample observations are drawn from different populations is studied here.

Let $X_1, \cdots, X_n$ be n independent random variables distributed according to continuous cumulative distribution functions (cdf) $F_1(x), \cdots, F_n(x)$, respectively. Let $\mathfrak{F}$ be the class of all continuous cdf's which are symmetric about their medians. If $F_1 = \cdots = F_n = F \varepsilon \mathfrak{F}$, the Wilcoxon's [7] signed rank statistic provides a robust test for and estimator of the median of $F(x)$, (cf. [2], [4], [6], [7]). The asymptotic relative efficiency (ARE) of this test and estimator has been studied by Hodges and Lehmann [1]. The present investigation is concerned with the study of the robust-efficiency of the same when $F_1, \cdots, F_n$ are not necessarily identical.

2. The main results. Let us define the sign-function $c(u)$ as 1, $\frac{1}{2}$ or 0 according as u is $>$, $=$ or $<$ 0, and let

$$S_i = c\ (X_i), \qquad R_i = \tfrac{1}{2} + \textstyle\sum_{j=1}^{n} c(|X_i| - |X_j|), \quad i = 1, \cdots, n. \tag{2.1}$$

By virtue of the assumed continuity of $F_1, \cdots, F_n$, ties among $|X_i|$, $i = 1, \cdots, n$, may be ignored, in probability. Wilcoxon's [7] signed rank statistic may then be defined as

$$W_n = [1/n(n+1)] \textstyle\sum_{i=1}^{n} S_i R_i - \tfrac{1}{4}. \tag{2.2}$$

The main theorem is concerned with the distribution of W_n when $F_1, \cdots, F_n$ are not necessarily identical. Let $\{H_n\}$ be a sequence of alternative hypotheses, specified by

$$H_n : F_i(x) = F_{in}(x) \text{ is symmetric about } \theta_n = n^{-\frac{1}{2}}\theta, \quad i = 1, \cdots, n; \tag{2.3}$$

where θ is real and finite, and $F_1, \cdots, F_n$ are all assumed to be absolutely continuous having continuous density functions $f_1(x), \cdots, f_n(x)$, respectively. It is also assumed that

$$\sup_i \textstyle\int_{-\infty}^{\infty} f_i^2(x)\, dx < \infty. \tag{2.4}$$

Instead of dealing with W_n, we shall deal with the allied U-statistic

$$U_n = \binom{n}{2}^{-1} \textstyle\sum_{1 \leq i < j \leq n} \phi(X_i, X_j); \qquad \phi(a, b) = c(a + b) - \tfrac{1}{2}. \tag{2.5}$$

Received 7 June 1966; revised 27 March 1967.

[1] Work supported by the Army Research Office, Durham, Grant DA-31-124-ARO-D-G432.

It is then easily verified that

(2.6) $U_n = 2[(n+1)(n-1)^{-1}W_n - (n-1)^{-1}(n^{-1}\sum_{i=1}^n S_i - \frac{1}{2})]$,

and hence,

(2.7) $|U_n - 2W_n| \leqq 2/(n-1)$, whatever be $F_1, \cdots, F_n$.

We also define

(2.8) $\bar{F}_n(x) = n^{-1}\sum_{i=1}^n F_{in}(x), \quad \bar{f}_n(x) = n^{-1}\sum_{i=1}^n f_i(x),$

$$\delta_n = 2\theta[\textstyle\int_{-\infty}^{\infty} \bar{f}_n^{\,2}(x)\,dx].$$

THEOREM 2.1. *Under* (2.3) *and* (2.4), $3^{\frac{1}{2}}(n^{\frac{1}{2}}U_n - \delta_n)$ *converges in law to a standard normal distribution, uniformly in* $F_1, \cdots, F_n$.

PROOF. We essentially follow the line of proof of Theorem 8.1 of Hoeffding ([3], pp. 310–313), and let

(2.9) $\psi_{1(i)j}(x_i) = E\{\phi(x_i, X_j)\} - E\{\phi(X_i, X_j)\}, \qquad i \neq j = 1, \cdots, n;$

(2.10) $\psi_{1(i)}(X_i) = (n-1)^{-1}\sum_{j=1\neq i}^n \psi_{1(i)j}(X_i), \qquad i = 1, \cdots, n;$

$$\bar{\psi}_n = n^{-1}\textstyle\sum_{i=1}^n \psi_{1(i)}(X_i).$$

Then, it follows from the boundedness of $\phi(X_i, X_j)$'s that

(2.11) $n^{\frac{1}{2}}\{[U_n - E(U_n)] - 2\bar{\psi}_n\} = R_n; \quad E(R_n^{\,2}) = O(n^{-1}),$

uniformly in $F_1, \cdots, F_n$. Also, from (2.3), (2.4) and (2.5), we obtain

(2.12) $|n^{\frac{1}{2}}E\{U_n \mid H_n\} - \delta_n| = o(1)$, uniformly in $F_1, \cdots, F_n$.

So, it suffices to show that $(12n)^{\frac{1}{2}}\bar{\psi}_n$ converges in law to a standard normal distribution (under $\{H_n\}$). Using (2.5), (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned}\psi_{1(i)}(X_i) = (n-1)^{-1}\{n[\{\tfrac{1}{2} - \bar{F}_n(-X_i)\} - \\ \textstyle\int_{-\infty}^{\infty}\{\tfrac{1}{2} - \bar{F}_n(-x)\}\,dF_{in}(x)] - [\{\tfrac{1}{2} - F_{in}(-X_i)\} \\ - \textstyle\int_{-\infty}^{\infty}\{\tfrac{1}{2} - F_{in}(-x)\}\,dF_{in}(x)]\}, \qquad i = 1, \cdots, n.\end{aligned} \tag{2.13}$$

Also, under H_n in (2.3)

(2.14) $[\frac{1}{2} - F_{in}(-x)] = [F_{in}(x + 2n^{-\frac{1}{2}}\theta) - \frac{1}{2}], \qquad i = 1, \cdots, n.$

Hence, from (2.13) and (2.14), we obtain, on using (2.4) and (2.8), that

(2.15) $V\{n^{\frac{1}{2}}\bar{\psi}_n \mid H_n\} = \frac{1}{12} + o(1)$, uniformly in $F_1, \cdots, F_n$.

Upon noting that $\{\psi_{1(i)}(X_i), i = 1, \cdots, n\}$ are independent and bounded valued random variables, the rest of the proof follows from the Berry-Esseen theorem (cf. [5], p. 288). Hence the theorem.

REMARK. As is pointed out by the referee, a proof of Theorem 2.1 for the particular case of symmetry about 0 (i.e., $\theta = 0$ in (2.3),) is contained on page

12 of Chapter 3 of the preliminary version of J. W. Pratt's book on nonparametric statistics. The same result also follows directly from the sign-invariance permutation arguments of Sen and Puri [6a], which does not require the identity of $F_1, \cdots, F_n$.

If the cdf $F_i(x)$ has a finite variance σ_i^2 ($i = 1, \cdots, n$), and we define $\bar{\sigma}_n^2 = n^{-1} \sum_{i=1}^n \sigma_i^2$, it is easy to show that according to the usual definition of ARE (cf. [1]), the ARE of the Wilcoxon's signed rank test with respect to Student's t-test is

$$e_{W_n/t_n} = 12\bar{\sigma}_n^2[\textstyle\int_{-\infty}^{\infty} \bar{f}_n^2(x)\, dx]^2. \tag{2.16}$$

Since, $\bar{f}_n(x)$ is also a continuous cdf with the variance $\bar{\sigma}_n^2$, it follows that the lower bound to the ARE of Wilcoxon's test with respect to t-test (viz. 0.864), considered by Hodges and Lehmann [1], is also valid when $F_1, \cdots F_n$ are not necessarily identical. For a class of cdf's, (2.16) provides sharper bounds to e_{W_n/t_n}. Let $\mathfrak{F}_0(\subset \mathfrak{F})$ be a class of cdf's, such that if F_1, F_2 both belong to $\mathfrak{F}_0$, then

$$\begin{aligned}\textstyle\int_{-\infty}^{\infty} f_1(x)\, dF_2(x) &\geqq (\textstyle\int_{-\infty}^{\infty} f_1(x)\, dF_1(x))(\textstyle\int_{-\infty}^{\infty} f_2(x)\, dF_2(x)) \\ &\quad \cdot\{\tfrac{1}{2}((\textstyle\int_{-\infty}^{\infty} f_1(x)\, dF_1(x))^2 + (\textstyle\int_{-\infty}^{\infty} f_2(x)\, dF_2(x))^2)\}^{-\frac{1}{2}} > 0.\end{aligned} \tag{2.17}$$

It may be noted that for $F_1, \cdots, F_n$ being all normal or all double exponential distributions, differing only in scale parameters, (2.17) is satisfied; $\mathfrak{F}_0$ may also hold some other distributions (viz., convex exponential cdf's with strictly positive density everywhere). Let then

$$e_i = 12\sigma_i^2(\textstyle\int_{-\infty}^{\infty} f_i^2(x)\, dx)^2; \qquad g_{in} = \sigma_i^2/(\sum_{j=1}^n \sigma_j^2), \qquad i = 1, \cdots, n. \tag{2.18}$$

THEOREM 2.2. *If $F_1, \cdots, F_n$ all belong to $\mathfrak{F}_0$,*

$$e_{W_n/t_n} \geqq [\textstyle\sum_{i=1}^n g_{in}/e_i]^{-1}, \tag{2.19}$$

where the equality sign holds only when $\int_{-\infty}^{\infty} f_i(x)\, dF_j(x) =$ constant, for all i, $j = 1, \cdots, n$.

PROOF. Using the elementary inequalities between the arithmetic and harmonic means and between the sum of powers, we have

$$\begin{aligned}(\textstyle\int_{-\infty}^{\infty} \bar{f}_n^2(x)\, dx)^2 &= (n^{-2} \textstyle\sum_{i=1}^n \sum_{j=1}^n \int_{-\infty}^{\infty} f_i(x) f_j(x)\, dx)^2 \\ &\geqq (n^{-2} \textstyle\sum_{i=1}^n \sum_{j=1}^n [1/\int_{-\infty}^{\infty} f_i(x) f_j(x)\, dx\,)^{-2} \\ &\geqq (n^{-2} \textstyle\sum_{i=1}^n \sum_{j=1}^n [1/\int_{-\infty}^{\infty} f_i(x) f_j(x)\, dx]^2)^{-1},\end{aligned} \tag{2.20}$$

where the equality sign holds only when $\int_{-\infty}^{\infty} f_i(x)\, dF_j(x) =$ constant, for all $i, j = 1, \cdots, n$. Since, $F_1, \cdots F_n$ all belong to $\mathfrak{F}_0$, on using (2.17), (2.20) reduces to

$$\begin{aligned}(\textstyle\int_{-\infty}^{\infty} \bar{f}_n^2(x)\, dx)^2 &\geqq \{(2n^2)^{-1} \textstyle\sum_{i=1}^n \sum_{j=1}^n (\int_{-\infty}^{\infty} f_i^2(x)\, dx)^{-2} \\ &\qquad + (\textstyle\int_{-\infty}^{\infty} f_j^2(x)\, dx)^{-2})\}^{-1} \\ &= (n^{-1} \textstyle\sum_{i=1}^n [\int_{-\infty}^{\infty} f_i^2(x)\, dx]^{-2})^{-1} \\ &= (12 n^{-1} \textstyle\sum_{i=1}^n \sigma_i^2/e_i)^{-1}.\end{aligned} \tag{2.21}$$

The rest of the proof follows from (2.16), (2.18), (2.19) and (2.21). Hence, the theorem.

If the cdf's $F_1, \cdots, F_n$ differ only in scale parameters $\sigma_1, \cdots, \sigma_n$, it is easy to show that $e_1 = \cdots = e_n = e$, and hence, under the condition of Theorem 2.2 we obtain

$$e_{W_n/t_n} \geqq e, \tag{2.22}$$

where the equality sign holds only when $\sigma_1 = \cdots = \sigma_n$. In particular, if F_i is normal with variance σ_i^2, $i = 1, \cdots, n$, it is easily seen that

$$e_{W_n/t_n} = 3\pi^{-1}\bar{\sigma}_n^2(n^{-2} \textstyle\sum_{i=1}^n \sum_{j=1}^n (2/(\sigma_i^2 + \sigma_j^2))^{\frac{1}{2}})^2 \geqq 3/\pi \tag{2.23}$$

where the equality sign holds only when $\sigma_1 = \cdots = \sigma_n$. Further, by varying $\sigma_1, \cdots, \sigma_n$ arbitrarily, (2.23) can be made arbitrarily large. This clearly indicates the robust-efficiency of the signed rank test when the variables are not identically distributed.

Since, the same efficiency results also apply to the (point as well as confidence interval) estimator of the median obtained by using the signed rank statistic, (cf. [2], [4], [6]), the conclusions of Theorems 2.1 and 2.2 are true for the estimation case too.

3. Acknowledgment. Thanks are due to the referee for his useful comments on the paper.

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