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ASSOCIATION MATRICES AND THE KRONECKER PRODUCT OF DESIGNS

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1. Summary and introduction. Vartak [4] has shown by enumeration that the Kronecker product of two PBIB designs with s and t associate classes is again a PBIB with at most $s + t + st$ associate classes. In this paper the same result is established more easily with the help of association matrices. In addition to this, it is shown that the association matrices of a PBIB which is the Kronecker product of two known designs, are the Kronecker product of those of the original designs and that the "augmented matrices of the parameters of the second kind" of the resulting design are the Kronecker product of the corresponding matrices of the given designs.

2. Some definitions. A PBIB design with m associate classes is defined as an arrangement of v treatments in b blocks each of size $k(<v)$ such that (i) each treatment is replicated r times; (ii) each block contains distinct treatments; (iii) corresponding to each treatment, others fall into m mutually exclusive classes known as associate classes, the ith class containing n_i treatments which (n_i) is independent of the treatment with which we start; (iv) the relation of association is symmetrical, i.e., if j is an ith associate of k, then k is an ith associate of j, and two treatments which are ith associates occur together in λ_i blocks; (v) if two treatments are ith associates, the number of treatments common to the jth associates of one and the kth associates of the other is $p^i_{jk} = p^i_{kj}$ and this is independent of the treatments with which we start.

Now we make the additional assumption that each treatment is its own zeroth associate and of no other treatment. Then $n_0 = 1$, $\lambda_0 = r$, $p^0_{ij} = p^0_{ji} = n_i$ if $j = i$ and zero otherwise, and $p^i_{j0} = p^i_{0j} = 1$ if $i = j$ and zero otherwise. Also we define the $v \times v$ matrices B_i $(i = 0, 1, \cdots, m)$ as

$$B_i = (b^k_{ji}) \tag{1}$$

where, $b^k_{ji} = 1$ if k and j are ith associates and zero otherwise. Clearly, $B_0 = I(v)$, where, $I(v)$ is the identity matrix of order v. B_i $(i = 0, \cdots, m)$ are called the association matrices of the association scheme corresponding to the PBIB defined above. These matrices are incidence matrices and are symmetric. They have unity in mutually exclusive positions on each row, B_i having n_i such elements in each of its row. Thus,

$$\sum_{i=0}^{m} B_i = E(v, v),$$

where, $E(v, v)$ is a $v \times v$ matrix with all its elements unity. Let

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$$P_i = (p^i_{jk}), \tag{2}$$

$j, k = 0, 1, \cdots, m$, for every $i = 0, 1, \cdots, m$. Then,

DEFINITION 2.1. $P_0, P_1, \cdots, P_m$ are called the "augmented matrices of the parameters of the second kind."

DEFINITION 2.2. The matrices obtained by deleting the first row and column of $P_1, \cdots, P_m$ are called the "usual matrices of parameters of the second kind."

3. Determination of a PBIB in terms of B_i and P_i. It is shown by Bose [1], Bose and Mesner [2] and Thompson [3] that

$$B_jB_k = p^0_{jk}B_0 + p^1_{jk}B_1 + \cdots + p^m_{jk}B_m. \tag{3}$$

Therefore the matrix

$$(B_jB_k) = P_0 \times B_0 + \cdots + P_m \times B_m, \qquad j, k = 0, 1, \cdots, m, \tag{4}$$

where, $A \times B$ denotes the Kronecker product of A and B. Also, if N is the incidence matrix of the PBIB design defined in Section 2, and N' its transpose

$$NN' = rB_0 + \lambda_1B_1 + \cdots + \lambda_mB_m. \tag{5}$$

The essence of the following theorem is due to Bose and Mesner [2].

THEOREM 1. *If B_i $(i = 0, 1, \cdots, m)$ are symmetric incidence matrices satisfying*

$$\text{(i) } B_0 = I(v), \qquad \text{(ii) } \textstyle\sum_{i=0}^m B_i = E(v, v), \qquad \text{(iii) } B_jB_k = \textstyle\sum_{i=0}^m p^i_{jk}B_i, \tag{6}$$

$(j, k = 0, \cdots, m)$ for some set of constants $p^i_{jk} = p^i_{kj}$ then, the incidence matrix N is the matrix of a PBIB *with v treatments if and only if N has equal column totals and*

$$NN' = rB_0 + \lambda_1B_1 + \cdots + \lambda_mB_m \tag{7}$$

for some m and some numbers $r, \lambda_1, \cdots, \lambda_m$.

Take B_i as the ith association matrix related to the design N and assume that it has n_i nonzero elements in each of its row. Identify the treatments corresponding to these elements as the ith associates of the treatment in the corresponding row of B_0. Then, since the (i, j)th element of NN' is the number of blocks in which the ith and jth treatments occur together, it follows from (7) that r is the number of replications of each treatment and λ_i is the number of blocks in which two treatments which are ith associates occur together. By a similar argument p^i_{kj} in (6) is the number of treatments common to the jth and kth associates of two treatments which are ith associates. That is, v, r, λ_i, n_i and p^i_{jk} have the same meaning as is attached to them in the definition of a PBIB.

4. The main results.

THEOREM 2.

$$N_1; b_1, v_1, r_1, k_1, \lambda_i', n_i' \qquad (i = 1, \cdots, s)$$

$$N_2; b_2, v_2, r_2, k_2, \lambda_j'', n_j'' \qquad (j = 1, \cdots, t)$$

are two PBIB *designs with s and t associate classes with* (a) $P_i' = (q_{hk}^i)$ *and* $P_j'' = (g_{um}^f)$ $(i = 0, \cdots, s; j = 0, \cdots, t)$ *as their respective augmented matrices of parameters of the second kind and* (b) B_i' *and* B_j'' *as their respective association matrices* $(i = 0, \cdots, s; j = 0, \cdots, t)$. *Then the Kronecker product* $N_1 \times N_2 = N$ *is a* PBIB *with at most $s + t + st$ associate classes with*

(i) *parameters of the first kind as*

$$(8) \quad b = b_1b_2, \quad v = v_1v_2, \quad r = r_1r_2, \quad k = k_1k_2, \quad \lambda_j = r_1\lambda_j'', \; \lambda_{t+i} = r_2\lambda_i', \\ \lambda_{t+s+ij} = \lambda_i'\lambda_j'', \quad n_j = n_j'', \quad n_{t+i} = n_i', \quad n_{t+s+ij} = n_i'n_j'' \\ (i = 1, \cdots, s; j = 1, \cdots, t);$$

(ii) *augmented matrices of parameters of the second kind as*

$$(9) \quad P_0 = P_0' \times P_0'', \quad P_j = P_0' \times P_j'', \quad P_{t+i} = P_i' \times P_0'', \quad P_{t+s+ij} = P_i' \times P_j'', \\ (i = 1, \cdots, s; j = 1, \cdots, t)$$

(iii) *the association matrices as*

$$(10) \quad B_0 = B_0' \times B_0'', B_j = B_0' \times B_j'', B_{t+i} = B_i' \times B_0'', B_{t+s+ij} = B_i' \times B_j'' \\ (i = 1, \cdots, s; j = 1, \cdots, t);$$

and the design has all the $s + t + st$ classes distinct if $\lambda_i'\lambda_j'' \neq \lambda_u'\lambda_m''$ *except when* $i = u$ *and* $j = m$ $(i,u = 0, \cdots, s; j, m = 0, \cdots, t)$.

Proof. It is easy to show that N is an incidence matrix which has $v_1v_2 = v$ rows and $b_1b_2 = b$ columns each column having unity in $k_1k_2 = k$ places. Since, $NN' = N_1N_1' \times N_2N_2'$ and $N_1N_1' = r_1B_0' + \lambda_1'B_1' + \cdots + \lambda_s'B_s'$, $N_2N_2' = r_2B_0'' + \lambda_1''B_1'' + \cdots + \lambda_t''B_t''$,

$$(11) \quad NN' = r_1r_2B_0' \times B_0'' + r_1 \textstyle\sum_{j=1}^t \lambda_j''B_0' \times B_j'' + r_2 \sum_{i=1}^s \lambda_i'B_i' \times B_0'' \\ + \textstyle\sum_{i=1}^s \sum_{j=1}^t \lambda_i'\lambda_j''B_i' \times B_j''.$$

$B_i' \times B_j''$ is symmetric, as B_i' and B_j'' are symmetric. The Kronecker product $B_i' \times B_j''$ is obtained by replacing every unity of B_i' by B_j'' and every zero by a zero matrix of order $v_2 \times v_2$. Hence $B_i' \times B_j''$ is a $v \times v$ matrix having on each of its row $n_i'n_j''$ unit elements and the rest zero. From the properties of B_i' and B_j'' it follows that the matrices $B_i' \times B_j''$ $(i = 1, \cdots, s; j = 1, \cdots, t)$ have nonzero elements in mutually exclusive positions in the corresponding rows. Also $B_0' \times B_0'' = I(v)$. Thus defining B_i as in (10) the sum of elements of each row of the sum of $B_0, \cdots, B_{t+s+st}$ is

$$(12) \quad 1 + \textstyle\sum_{j=1}^t n_j'' + \sum_{i=1}^s n_i' + \sum_{i=1}^s \sum_{j=1}^t n_i'n_j'' = v$$

for, $\sum_{j=1}^t n_j'' = v_2 - 1$ and $\sum_{i=1}^s n_i' = v_1 - 1$. This shows that

$$(13) \quad \textstyle\sum_{i=0}^{s+t+st} B_i = E(v, v).$$

Again if $B_a = B_h' \times B_j''$ and $B_c = B_k' \times B_m''$,

$$(14) \quad B_aB_c = (\textstyle\sum_{i=0}^s q_{hk}^i B_i')(\sum_{f=0}^t g_{jm}^f B_f'') = \sum_{i=0}^{s+t+st} p_{ac}^i B_i$$

where, $p^i_{ac} = p^i_{ca}$ as q^i_{hk} and g^f_{jm} are symmetric. Thus defining λ_i as in (8) N, B_i and λ_i $(i = 0, \cdots, s + t + st)$ satisfy all requirements of the Theorem 1. N is, therefore, a PBIB with at most $s + t + st$ associate classes satisfying (i) and (iii) of Theorem 2 with p^i_{ac} as the number of treatments common to the ath and cth classes of two treatments which are ith associates. To prove the condition (ii) of the theorem we observe that the matrix

$$(15)\quad (B_aB_c) = (P_0'B_0' + \cdots + P_s'B_s') \times (P_0''B_0'' + \cdots + P_t''B_t'') = \textstyle\sum_{i=0}^{s+t+st} P_iB$$

where P_i and B_i are defined in (9) and (10). This shows that P_i is the augmented matrix of parameters of the second kind corresponding to the ith class. Here we have assumed that the treatments corresponding to the zonzero elements in any row of B_i are the ith associates of the treatment which corresponds to the treatment in the same row of B_0. This completes the proof of our theorem.

The relations (9) and (10) give a logical and at the same time less cumbersome method of determining the different associate classes and the second kind of parameters.

If $\lambda_i = \lambda_j$ in any PBIB, then the corresponding classes may be combined with the help of the lemma given by Vartak [4].

5. Illustrations. (i) Let $N_1 : b_1, v_1, r_1, k_1, \lambda_1'$; $N_2 : b_2, v_2, r_2, k_2, \lambda_2''$ be BIB designs. Then by our notation,

$$B_0' = I(v_1),\quad B_1' = E(v_1, v_1) - I(v_1);\quad B_0'' = I(v_2),\quad B_1'' = E(v_2, v_2) - I(v_2),$$

$$P_0' = \begin{Bmatrix} 1 & 0 \\ 0 & v_1 - 1 \end{Bmatrix},\quad P_1' = \begin{Bmatrix} 1 & 1 \\ 1 & v_1 - 2 \end{Bmatrix},$$

$$P_0'' = \begin{Bmatrix} 1 & 0 \\ 0 & v_2 - 1 \end{Bmatrix},\quad P_1'' = \begin{Bmatrix} 1 & 1 \\ 1 & v_2 - 2 \end{Bmatrix}.$$

TABLE 1

Treatments	I associates $B_0' \times B_1' + B_1' \times B_0'$									II associates $B_1' \times B_1'$								
	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
1	0	1	1	1	0	0	1	0	0	0	0	0	0	1	1	0	1	1
2	1	0	1	0	1	0	0	1	0	0	0	0	1	0	1	1	0	1
3	1	1	0	0	0	1	0	0	1	0	0	0	1	1	0	1	1	0
4	1	0	0	0	1	1	1	0	0	0	1	1	0	0	0	0	1	1
5	0	1	0	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1
6	0	0	1	1	1	0	0	0	1	1	1	0	0	0	0	1	1	0
7	1	0	0	1	0	0	0	1	1	0	1	1	0	1	1	0	0	0
8	0	1	0	0	1	0	1	0	1	1	0	1	1	0	1	0	0	0
9	0	0	1	0	0	1	1	1	0	1	1	0	1	1	0	0	0	0

In general, $N_1 \times N_2$ is a PBIB with three associate classes. The first kind of parameters and the association matrices are obtained by strictly following (8) and (10). The usual matrices of parameters of the second kind are obtained from P_i by suppressing the first row and column. They are, respectively,

$$\begin{Bmatrix} v_2 - 2 & 0 & 0 \\ 0 & 0 & v_1 - 1 \\ 0 & v_1 - 1 & (v_1 - 1)(v_2 - 2) \end{Bmatrix}, \quad \begin{Bmatrix} 0 & 0 & v_2 - 1 \\ 0 & v_1 - 1 & 0 \\ v_2 - 1 & 0 & (v_1 - 2)(v_2 - 1) \end{Bmatrix}$$

$$\begin{Bmatrix} 0 & 1 & v_2 - 2 \\ 1 & 0 & v_1 - 2 \\ v_2 - 2 & v_1 - 2 & (v_1 - 2)(v_2 - 2) \end{Bmatrix}$$

and these are the same as those obtained by Vartak [4].

(ii) Let $N_1 : b_1 = v_1 = 3, r_1 = k_1 = 2, \lambda_1' = 1$ and N_2 be identical BIB designs. Then,

$$B_0' = B_0'' = I(3), \qquad B_1' = B_1'' = \begin{Bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{Bmatrix}.$$

N is a PBIB with only two associate classes as the classes corresponding to $\lambda_1 = \lambda_2$ can be combined. We shall designate the class corresponding to this value of λ as the first associates. The treatments and the different associate classes of N are given in Table 1.

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