The Annals of Mathematical Statistics
1968, Vol. 39, No. 2, 643–656

RELATIONSHIP OF GENERALIZED POLYKAYS TO UNRESTRICTED SUMS FOR BALANCED COMPLETE FINITE POPULATIONS[1]

BY EDWARD J. CARNEY[2]

Iowa State University

1. Introduction and summary. The polykays of Tukey [8] and the bipolykays of Hooke [7] were generalized by Dayhoff [4] for arbitrary balanced complete finite population structures. The expected mean squares in the analysis of variance of such structures may be expressed as linear functions of variance components corresponding to the factors classifying the population. Since the variance components serve to measure the relative influence of the factors it is often desired to estimate these quantities. Unbiased estimates may be obtained by substituting observed mean squares for population mean squares and solving the resulting linear equations for the variance components. Alternative expressions for the expected mean squares involve linear functions of quantities called cap sigmas (Wilk [10], Zyskind [11], White [9]), and the variance component estimates may be expressed as linear functions of sample cap sigmas. Dayhoff [3] shows that cap sigmas are generalized polykays of degree two and that the variances and covariances of variance components are linear functions of generalized polykays of degree four.

Since generalized polykays have the property of inheritance on the average (i.e., averages of sample generalized polykays over all random samples are the same generalized polykays of populations responses), the variances and covariances of the unbiased variance component estimates may be estimated unbiasedly by linear functions of generalized polykays of degree four. The generalized polykays are in turn linear functions of generalized symmetric means (Hooke [6], Dayhoff [4]) which may be computed directly from the observations. However, such computations are very difficult to carry out by hand because the formulas may involve thousands of distinct generalized polykays and generalized symmetric means. Furthermore, direct computation of generalized symmetric means for moderately large numbers of levels of the factors is not only impossible to carry out by hand but even far too costly using the most advanced digital computers, because a single generalized symmetric mean may require hundreds of billions of operations in its evaluation [2].

In order to make the computations of the variance-covariance matrix for estimated variance components economically feasible several requirements must be met. First the variance-covariance formulas in terms of the generalized sym-

Received 21 August 1967.

[1] This work was prepared in connection with work on Contract AF33(615)1737, Aerospace Research Laboratories, United States Air Force, Wright-Patterson Air Force Base, Ohio.

[2] Now at University of Rhode Island.

metric functions must be generated by computer algorithms. Secondly, a way of computing the generalized symmetric means must be developed which significantly reduces the number of additions and multiplications. Thirdly, the above algorithms must be made general enough for application to the many possible balanced complete response structures which may be encountered as the relationship of nesting among the factors varies.

In meeting the above requirements it is necessary to determine a logical system of relationships which allows the development of a few relatively simple algorithms to perform the various tasks on the computer, and which may be applied generally to the many different structures possible. The present paper results from an attempt to find such logical relationships among the various quantities which may be used in programs for digital computation of the variance-covariance matrix of estimated variance components. Study of the patterns of subscript restrictions which specify the generalized symmetric means leads to the development of algebraic relationships between the generalized polykays and the generalized symmetric means, which may be formulated in terms of a lattice of ordered partitions. Similar relationships exist between the numerators of the generalized symmetric means and quantities called unrestricted sums. These latter quantities may be computed much more efficiently than the generalized symmetric means themselves. (Hooke [6] gives an example of such a computation for a two factor structure using quantities similar to the unrestricted sums.) The various relationships, in addition to their intrinsic theoretical interest provide the necessary logical basis for the development of digital computer algorithms for performance of the algebraic and numerical computations for estimation of the variances and covariances of the variance component estimates.

2. Notation. Four types of quantities will be used in the following sections. These will be denoted by various types of brackets enclosing the "ordered partitions" or sets of ordered partitions as indicated below:

Quantity	*Simple*	*Generalized*
Symmetric mean	$\langle\alpha\rangle$	$\langle\alpha^1/\alpha^2/\cdots/\alpha^f\rangle$
Polykay	(α)	$(\alpha^1/\alpha^2/\cdots/\alpha^f)$
Symmetric sum	$\lvert\alpha\rvert$	$\lvert\alpha^1/\alpha^2/\cdots/\alpha^f\rvert$
Unrestricted sum	$[\alpha]$	$[\alpha^1/\alpha^2/\cdots/\alpha^f]$

The symbols α, α^i, within the various types of brackets are the "ordered partitions" and may be considered as lists of symbols, $\alpha = \alpha_1\alpha_2 \cdots \alpha_m$, while the quantities $\alpha^1/\alpha^2/\cdots/\alpha^f$ will denote a set of f such lists, and may be considered as $f \times m$ matrices, each row representing the subscript restrictions for a factor. The abbreviation gsm will frequently be used for "generalized symmetric mean."

3. The lattice of ordered partitions.

DEFINITION 3.1. An ordered partition of weight m is defined to be a set of $m(m-1)/2$ consistent statements of equality or inequality for m positions.

Any list of m symbols, if it can be determined which are equal and which unequal, is a representation of an ordered partition, and will be called an ordered partition. Ordinarily an ordered partition will be specified by a list of m symbols, but the equality-inequality of the symbols determine the ordered partition, not the particular symbols themselves.

DEFINITION 3.2. Let $\alpha = \alpha_1\alpha_2 \cdots \alpha_m$ and $\beta = \beta_1\beta_2 \cdots \beta_m$ be ordered partitions of weight m. Then α is said to be an ordered subpartition of β ($\alpha \leqq \beta$) if and only if $\alpha_i = \alpha_j$ implies $\beta_i = \beta_j$ for all pairs (i, j) $i = 1, 2, \cdots, m; j = 1, 2, \cdots, m$.

It is clear that if two ordered partitions, α and β, of equal weight, are such that $\alpha \leqq \beta$ and $\beta \leqq \alpha$, then they are the same ordered partition.

THEOREM 3.1. *The set L_m of ordered partitions of weight m, with the subpartition partial ordering, is a lattice.*

PROOF. From the definition above it follows that the subpartition relationship is antisymmetric and transitive so that L_m is a partially ordered set. Let α and β be two ordered partitions of weight m and define $\gamma = \alpha \wedge \beta$ to be the ordered partition $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \cdots, (\alpha_m, \beta_m)$ formed by the ordered pairs (α_i, β_i), $i = 1, 2, \cdots, m$. $\alpha_i \neq \alpha_j$ implies $\gamma_i \neq \gamma_j$ and $\beta_i \neq \beta_j$ implies $\gamma_i \neq \gamma_j$ so that $\gamma \leqq \alpha$, $\gamma \leqq \beta$ and hence γ is a lower bound of α and β. Let $\delta \leqq \alpha$ and $\delta \leqq \beta$. Then $\delta_i = \delta_j$ implies $\alpha_i = \alpha_j$ and $\beta_i = \beta_j$, so that $\gamma_i = \gamma_j$ and hence γ is a greatest lower bound.

Now consider the ordered partition $\lambda = \alpha \vee \beta$ constructed according to the following rules:

1. $\lambda_1 = 0$. If $\alpha_i = \alpha_1$, or if $\beta_i = \beta_1$ let $\lambda_i = 0$.

2. Let R_0 be the set of all integers $i = 1, 2, \cdots, m$ such that $\lambda_i = 0$. If $i \varepsilon R_0$ and $\alpha_i = \alpha_j$, or $\beta_i = \beta_j$ let $\lambda_j = 0$. Continue until there exists no j such that $\alpha_j = \alpha_i$ or $\beta_j = \beta_i$ with $i \varepsilon R_0$. If $R_0 = \{1, 2, \cdots, m\}$ then λ is the 1 part ordered partition $00 \cdots 0$.

3. If $R_0 \neq \{1, 2, \cdots, m\}$ let i be the first integer not in R_0 and set $\lambda_i = 1$. Construct successively sets $R_1, R_2, \cdots$, setting $\lambda_i = k$, if $i \varepsilon R_k$, considering for the set R_k those positive integers less than or equal to m which are not in the set $R_0 \cup \cdots \cup R_{k-1}$, until all λ_i, $i = 1, 2, \cdots, m$, are determined.

Let λ be so constructed for ordered partitions α and β of weight m. If $\alpha_i = \alpha_j$ or $\beta_i = \beta_j$ then $\lambda_i = \lambda_j$ by construction so that λ is an upper bound for α and β. Let i and j be integers such that $\lambda_i = \lambda_j$. Then there is some set R_k of integers $i = k_1$, $k_2 \cdots, k_s = j$ such that there exists a set of pairs of equal elements of α or β, say $\epsilon_{k_1} = \epsilon_{k_2}$, $\epsilon'_{k_2} = \epsilon'_{k_3}$, $\cdots$, $\epsilon^{(s-2)}_{k_{s-1}} = \epsilon^{(s-2)}_{k_s}$, where each pair $\epsilon^{(u)}_{k_{u+1}} = \epsilon^{(u)}_{k_{u+2}}$, is the pair $\alpha_{k_{u+1}} = \alpha_{k_{u+2}}$ or the pair $\beta_{k_{u+1}} = \beta_{k_{u+2}}$. If δ is any ordered partition such that $\alpha \leqq \delta$ and $\beta \leqq \delta$ then $\delta_{k_1} = \delta_{k_2}$, $\delta_{k_2} = \delta_{k_3}$, $\cdots$, $\delta_{k_{s-1}} = \delta_{k_s}$. Thus $\delta_i = \delta_j$; and hence $\lambda_i = \lambda_j$ implies $\delta_i = \delta_j$ so that $\lambda \leqq \delta$, and λ is the least upper bound of α and β.

It follows that L_m is a lattice, since the set of ordered partitions is partially ordered and each pair of elements has the greatest lower bound $\alpha \wedge \beta$ and the least upper bound $\alpha \vee \beta$.

4. Symmetric means and polykays. An ordered partition $\alpha = \alpha_1\alpha_2 \cdots \alpha_m$ determines a partition of the integer m with the sets of identical symbols as parts. For example, 001232 determines the four part partition 2, 2, 1, 1, of 6. Let $\phi(\alpha)$ denote the number of parts of the partition determined by α. Thus $\phi(\alpha)$ may be defined as the number of distinct symbols α_i in any representation of α.

Consider a set of real numbers $\{y_i; i = 1, 2, \cdots, n\}$ and the set of all n^m m-tuples (with repetition) of these numbers. Let $a = i_1i_2 \cdots i_m$ denote the list of subscripts of an m-tuple $y_{i_1}, y_{i_2}, \cdots, y_{i_m}$, and let π_a denote the product $\prod_{k=1}^m y_{i_k}$ of the numbers having the subscripts a. There are $n(n-1) \cdots (n - \phi(\alpha) + 1) = (n)_{\phi(\alpha)}$ m-tuples such that the list of subscripts a form the ordered partition α.

DEFINITION 4.1. Let α be an ordered partition of weight m. The mth degree symmetric sum $|\alpha|$ is defined by

$$|\alpha| = \textstyle\sum_{a=\alpha} \pi_a ,$$

where the summation is over all sets of subscripts a such that a (when viewed as an ordered partition) is the ordered partition α.

DEFINITION 4.2. The mth degree symmetric mean $\langle\alpha\rangle$ is the symmetric sum $|\alpha|$ divided by the number of terms in this sum: $\langle\alpha\rangle = \sum_{a=\alpha} \pi_a/(n)_{\phi(\alpha)}$.

DEFINITION 4.3. Let α be an ordered partition of weight m. The mth degree

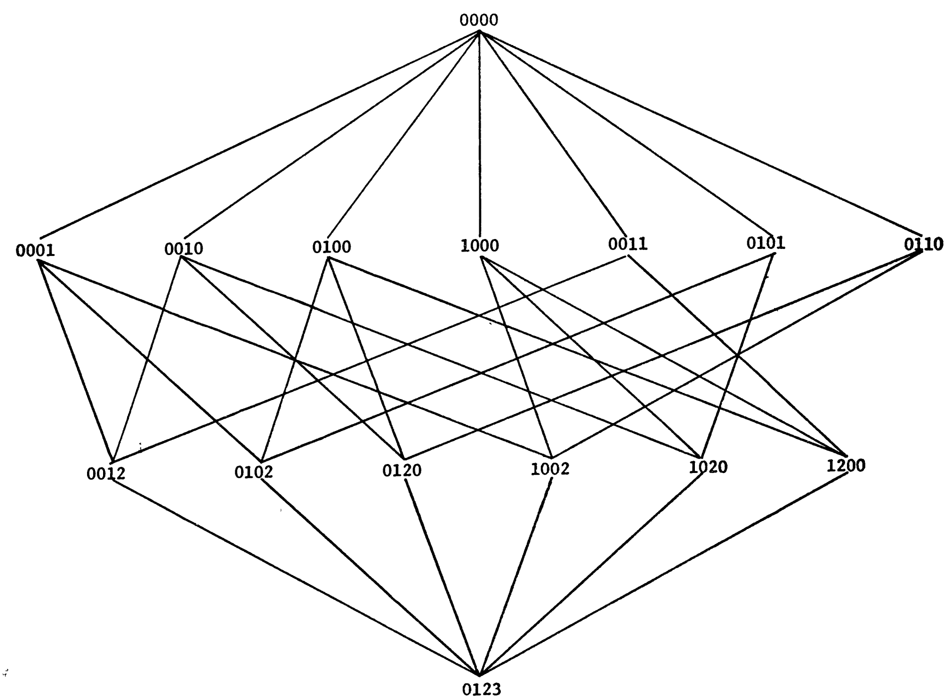

FIG. 1. The lattice of ordered partitions of weight four

polykay (α) is defined implicitly by the formula

$$\langle\alpha\rangle = \sum_{\beta \leqq \alpha} (\beta).$$

DEFINITION 4.4. Let the elements of the lattice L_m of ordered partitions of weight m be $\alpha^1, \alpha^2, \cdots, \alpha^r$ in some order which does not violate the partial ordering, say $\alpha^i \leqq \alpha^j$ implies $i < j$. The matrix Λ with elements

$$\lambda_{ij} = 1, \quad \text{if} \quad \alpha^j \leqq \alpha^i,$$

$$\lambda_{ij} = 0, \quad \text{otherwise},$$

will be called the incidence matrix of the lattice.

The definition of polykays may now be written as $\langle\boldsymbol{\alpha}\rangle = \Lambda(\boldsymbol{\alpha})$ where $\langle\boldsymbol{\alpha}\rangle$ and $(\boldsymbol{\alpha})$ denote the vector of all symmetric means and all polykays of degree m.

Because of the ordering of the elements α^i the matrix Λ is upper triangular with ones on the diagonal. Let $T = \Lambda - I$. The strictly triangular matrix T may be partitioned into m^2 blocks according to $\phi(\alpha^i)$. If $\phi(\alpha^i) = \phi(\alpha^j)$ we must have $\alpha^i = \alpha^j$ or α^i and α^j not comparable. Therefore T has diagonal blocks which are all zero. From this it follows that T is nilpotent of index m, which leads to the following expression for the inverse of Λ.

THEOREM 4.1. *Let* Λ *be the matrix of Definition* 4.4. *Then if* $T = \Lambda - I$, $T^0 \equiv I$

$$\Lambda^{-1} = \sum_{j=0}^{m-1} (-1)^j T^j.$$

PROOF. $\Lambda(\sum_{j=0}^{m-1} (-1)^j T^j) = (I + T) \sum_{j=0}^{m-1} (-1)^j T^j = \sum_{j=0}^{m-1} (-1)^j T^j + \sum_{j=0}^{m-1} (-1)^j T^{j+1} = T^0 + T^m = T^0 = I.$

As a result of the theorem we may write

$$(\boldsymbol{\alpha}) = \Lambda^{-1}\langle\alpha\rangle = (\sum_{j=0}^{m-1} (-1)^j T^j)\langle\boldsymbol{\alpha}\rangle,$$

where $(\boldsymbol{\alpha})$, $\langle\boldsymbol{\alpha}\rangle$ denote vectors formed by the sets of polykays and symmetric means, respectively.

We note that the form of the inverse implies that if $\lambda_{ij} = 0$ then $[\Lambda^{-1}]_{ij} = 0$, which is useful later on. We further note that the definition of polykays here is equivalent to that of Hooke [7], the difference in the present development being the utilization of the algebraic properties of the ordered partitions.

5. Unrestricted sums.

DEFINITION 5.1. Let $\alpha \varepsilon L_m$. The mth degree unrestricted sum $[\alpha]$ is defined by;

$$[\alpha] = \sum_{\alpha \leqq a} \pi_a.$$

Because of Definition 4.1 and 5.1 we have the theorem:

THEOREM 5.1.

$$[\alpha] = \Lambda' \, |\boldsymbol{\alpha}|.$$

This theorem allows the polykays to be computed in terms of the unrestricted sums. To make this explicit let N denote the diagonal matrix $n_{ii} = (n)_{\phi(\alpha^i)}$.

TABLE 1

The symmetric matrix $(\Lambda' N \Lambda)^{-1}$

$$
(n)_4^{-1}\left[\begin{array}{ccccccccccccccc}
n^3+n^2 & -n^2-n & -n^2-n & -n^2-n & -n^2-n & -n^2+n & -n^2+n & -n^2+n & 2n & 2n & 2n & 2n & 2n & 2n & -6 \\
 & (n-1)^2 & n+1 & n+1 & n+1 & n-1 & n-1 & n-1 & -n+1 & -n+1 & -2 & -n+1 & -2 & -2 & 2 \\
 & & (n-1)^2 & n+1 & n+1 & n-1 & n-1 & n-1 & -n+1 & -2 & -n+1 & -2 & -n+1 & -2 & 2 \\
 & & & (n-1)^2 & n+1 & n-1 & n-1 & n-1 & -2 & -n+1 & -n+1 & -2 & -2 & -n+1 & 2 \\
 & & & & (n-1)^2 & n-1 & n-1 & n-1 & -2 & -2 & -2 & -n+1 & -n+1 & -n+1 & 2 \\
 & & & & & n^2-3n+1 & 1 & 1 & -n+2 & -1 & -1 & -1 & -1 & -n+2 & 1 \\
 & & & & & & n^2-3n+1 & 1 & -1 & -n+2 & -1 & -1 & -n+1 & -1 & 1 \\
 & & & & & & & n^2-3n+1 & -1 & -1 & -n+2 & -n+2 & -1 & -1 & 1 \\
 & & & & & & & & n-2 & 1 & 1 & 1 & 1 & 1 & -1 \\
 & & & & & & & & & n-2 & 1 & 1 & 1 & 1 & -1 \\
 & & & & & & & & & & n-2 & 1 & 1 & 1 & -1 \\
 & & & & & & & & & & & n-2 & 1 & 1 & -1 \\
 & & & & & & & & & & & & n-2 & 1 & -1 \\
 & & & & & & & & & & & & & n-2 & -1 \\
 & & & & & & & & & & & & & & 1
\end{array}\right]
$$

Then we have

$$(\boldsymbol{\alpha}) = \Lambda^{-1}\langle\boldsymbol{\alpha}\rangle = \Lambda^{-1}N^{-1}|\boldsymbol{\alpha}| = \Lambda^{-1}N^{-1}(\Lambda^{-1})'[\boldsymbol{\alpha}] = (\Lambda'N\Lambda)^{-1}[\boldsymbol{\alpha}].$$

For the fourth degree case the symmetric matrix $\Lambda^{-1}N^{-1}(\Lambda^{-1})'$ is shown in Table 1. This gives formulas such as

$$\begin{aligned}(0012) = {} & (n(n-1)(n-2)(n-3))^{-1}\{2n[0000] - (n-1)([0001] + [0010]) \\ & - 2([0100] - [1000]) - (n-2)[0011] - ([0101] + [0110]) \\ & + (n-2)[0012] + ([0102] + [0120] + [1002] + [1020] \\ & + [1200]) - [0123]\}.\end{aligned}$$

Regarding (0012) as the simple polykay (2, 1, 1) (i.e., considering only one subscript), this is

$$\begin{aligned}(2, 1, 1) = {} & (n(n-1)(n-2)(n-3))^{-1}\{2n \textstyle\sum y^4 - 2(n+1)\sum y^3 \sum y \\ & + (n+3)(\textstyle\sum y^2)(\sum y)^2 - (\sum y)^4\}.\end{aligned}$$

6. Crossed structures. Consider an f-factor crossed structure. A generalized symmetric mean for such a structure will be denoted by $\langle\alpha^1/\alpha^2/\cdots/\alpha^f\rangle$, where α^i is an ordered partition for the ith factor subscripts.

DEFINITION 6.1. Let the symbolic multiplication $\otimes$ satisfy the following conditions:

(a) $a\langle\alpha\rangle \otimes b\langle\beta\rangle = ab\langle\alpha/\beta\rangle$, where a and b are real numbers.

(b) $\langle\alpha/\beta\rangle \otimes \langle\gamma\rangle = \langle\alpha\rangle \otimes \langle\beta/\gamma\rangle = \langle\alpha/\beta/\gamma\rangle$.

(c.1) $\langle\alpha\rangle \otimes (\langle\beta\rangle + \langle\gamma\rangle) = \langle\alpha/\beta\rangle + \langle\alpha/\gamma\rangle$.

(c.2) $(\langle\alpha\rangle + \langle\beta\rangle) \otimes \langle\gamma\rangle = \langle\alpha/\gamma\rangle + \langle\beta/\gamma\rangle$.

DEFINITION 6.2. The generalized polykay $(\alpha^1/\alpha^2/\cdots/\alpha^f)$ for an f-factor crossed structure is given by the formula

$$(\alpha^1/\alpha^2/\cdots/\alpha^f) = (\alpha^1) \otimes (\alpha^2) \otimes \cdots \otimes (\alpha^f).$$

Consider the f-fold cartesian product $L = L_m \times L_m \times \cdots \times L_m$. The elements of L will be written as $\alpha^{i_1}/\alpha^{i_2}/\cdots/\alpha^{i_f}$, where the α^{i_j} are elements of L.

DEFINITION 6.3. Let α and β be elements of L. Then α will be said to be a subpartition of β if and only if $\alpha^{i_j} \leqq \beta^{i_j}$, $j = 1, 2, \cdots, f$. Using this definition the elements of L are partially ordered, and the following theorem is immediate.

THEOREM 6.1. *The elements of L form a lattice with the greatest lower bound and least upper bound of any two elements* α, β *given by* $\alpha \wedge \beta = \alpha^{i_1} \wedge \beta^{i_1}/\alpha^{i_2} \wedge \beta^{i_2}/\cdots/\alpha^{i_f} \wedge \beta^{i_f}$ *and* $\alpha \vee \beta = \alpha^{i_1} \vee \beta^{i_1}/\alpha^{i_2} \vee \beta^{i_2}/\cdots/\alpha^{i_f} \vee \beta^{i_f}$, *respectively.*

The symbolic multiplication can be extended to apply to the numerators of the generalized symmetric means and to the corresponding unrestricted sums. Consider, for example, a crossed structure having n_k levels, $k = 1, 2, \cdots, f$. We may write the formula for the vector, P_k, of polykays in terms of the vector, U_k, of unrestricted sums, for the kth factor (ignoring all the others) as

$$P_k = \Lambda^{-1}N_k^{-1}(\Lambda^{-1})'U_k = F_kU_k, \quad \text{say}.$$

The formulas for the generalized polykays may now be expressed as

$$\begin{aligned} P &= P_1 \otimes P_2 \otimes \cdots \otimes P_f \\ &= (F_1U_1) \otimes (F_2U_2) \otimes \cdots \otimes (F_fU_f) \\ &= (F_1 \otimes F_2 \otimes \cdots \otimes F_f)(U_1 \otimes U_2 \otimes \cdots \otimes U_f) \\ &= FU, \quad \text{say.} \end{aligned}$$

The operator should be interpreted to be the Kronecker product of matrices, with multiplication of symbolic elements taken to be the symbolic multiplication of Definition 6.1, while multiplication of numeric elements is taken to be ordinary multiplication of real numbers. Thus F is the usual Kronecker product of the matrices $F_1, F_2, \cdots, F_f$, while U is the Kronecker symbolic product of the identical symbolic vectors $U_1, U_2, \cdots, U_k$. The situation is simply illustrated by the second degree polykays for a two-factor crossed structure.

We have, for a single factor,

$$\begin{pmatrix} (00) \\ (01) \end{pmatrix} = \Lambda^{-1} N_k^{-1} (\Lambda')^{-1} \begin{pmatrix} [00] \\ [01] \end{pmatrix}.$$

Here Λ is the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ and N_k is diag $(n_k, n_k(n_k - 1))$, giving

$$F_k = \Lambda^{-1} N_k^{-1} (\Lambda^{-1})' = (n_k(n_k - 1))^{-1} \begin{bmatrix} n_k & -1 \\ -1 & 1 \end{bmatrix}.$$

The two factor generalized polykays are then

$$\begin{pmatrix} (00/00) \\ (00/01) \\ (01/00) \\ (01/01) \end{pmatrix} = \begin{pmatrix} (00) \\ (01) \end{pmatrix} \otimes \begin{pmatrix} (00) \\ (01) \end{pmatrix} = (F_1 \otimes F_2) \begin{pmatrix} [00] \\ [01] \end{pmatrix} \otimes \begin{pmatrix} [00] \\ [01] \end{pmatrix}$$

$$= (n_1 n_2 (n_1 - 1)(n_2 - 1))^{-1} \begin{bmatrix} n_1 n_2 & -n_1 & -n_2 & 1 \\ -n_1 & n_1 & 1 & -1 \\ -n_2 & 1 & n_2 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{pmatrix} [00/00] \\ [00/01] \\ [01/00] \\ [01/01] \end{pmatrix}$$

i.e.,

$$\begin{aligned} (00/00) &= (n_1n_2(n_1 - 1)(n_2 - 1))^{-1}\{n_1n_2 \textstyle\sum_i \sum_j y_{ij}^2 - n_1 \sum_i (\sum_j y_{ij})^2 \\ &\quad - n_2 \textstyle\sum_j (\sum_i y_{ij})^2 + (\sum_i \sum_j y_{ij})^2\}, \\ (00/01) &= (n_1n_2(n_1 - 1)(n_2 - 1))^{-1}\{-n_1 \textstyle\sum_i \sum_j y_{ij}^2 + n_1 \sum_i (\sum_j y_{ij})^2 \\ &\quad + \textstyle\sum_j (\sum_i y_{ij})^2 - (\sum_i \sum_j y_{ij})^2\}, \\ (01/00) &= (n_1n_2(n_1 - 1)(n_2 - 1))^{-1}\{-n_2 \textstyle\sum_i \sum_j y_{ij}^2 + \sum_i (\sum_j y_{ij})^2 \\ &\quad + n_2 \textstyle\sum_j (\sum_i y_{ij})^2 - (\sum_i \sum_j y_{ij})^2\}, \end{aligned}$$

$$(01/01) = (n_1n_2(n_1 - 1)(n_2 - 1))^{-1}\{\sum_i \sum_j y_{ij}^2 - \sum_i (\sum_j y_{ij})^2 - \sum_j (\sum_i y_{ij})^2 + (\sum_i \sum_j y_{ij})^2\}.$$

7. Nested structures. The levels of a nested factor in a balanced complete structure are different within every different combination of the levels of nesting factors. If the subscripts of a nested factor, say the ith, are numbered sequentially $1, 2, \cdots, n_i$, then the order of the subscripts of the nested factor within any combination of levels of nested factors is arbitrary. Any particular assignment of subscripts to nested factors results in an artificial crossed structure. The generalized symmetric means for the nested structure are defined in terms of the crossed structure as follows [4].

DEFINITION 7.1. Let $\langle\alpha^1/\alpha^2/\cdots/\alpha^f\rangle^*$ denote a generalized symmetric mean for the artificial crossed structure. Suppose that the ith factor is nested in the factors numbered $j_1, j_2, \cdots, j_s$. Then the generalized symmetric mean formed by the ordered partition $\beta^1/\beta^2/\cdots/\beta^f$, $\beta^i = \alpha^{j_1} \wedge \alpha^{j_2} \wedge \cdots \wedge \alpha^{j_s}$, is generalized symmetric mean for the nested structure.

DEFINITION 7.2. For a given balanced complete structure an f-fold ordered partition $\alpha^1/\alpha^2/\cdots/\alpha^f$ will be said to be admissible if and only if the ordered partition for each nested factor is a subpartition of the greatest lower bound of the ordered partitions for all nesting factors.

Consider, as an example, a structure with factors A and B crossed and C nested in A and B. Then, letting the number of levels of the factors also be denoted by A, B and C, the gsm

$$\langle 0011/0012/0012\rangle = \sum\nolimits^{\neq} y_{ijk}^2 y_{i'j'k'} y_{i'j''k''}/A(A-1)B(B-1)C^3,$$

where $\sum^{\neq}$ denotes the sum of all terms such that the unequally primed subscripts remain unequal, is an admissible one for this structure because $0012 \leqq 0011 \wedge 0012 = 0012$. On the other hand the ordered partition 0001/0010/0011 is not admissible because 0011 is not a subpartition of $0001 \wedge 0010 = 0012$.

One may consider forming all possible artificial crossed structures from a given nested structure by permuting the subscripts of nested factors within each combination of nesting factors in all possible ways. The operator notation E_l will denote the process of averaging a function defined on such a crossed structure over all of these artificial crossed structures. A definition of the gms's for nested structures which is equivalent to 7.1 [4] is:

DEFINITION 7.3. $\langle\alpha\rangle$ is a gsm for a balanced complete structure if and only if there exists a gsm $\langle\beta\rangle^*$ for an artificial crossed structure such that $E_l\langle\beta\rangle^* = \langle\alpha\rangle$.

It follows that if α is an admissible ordered partition for a balanced complete structure then $E_l\langle\alpha\rangle^* = \langle\alpha\rangle$.

The generalized symmetric sums $|\alpha|$ for an arbitrary balanced complete structure are simply the numerators of the gsm's $\langle\alpha\rangle$. The expressions for the denominators of the gsm's for nested structures are slightly more complex than for crossed structures. Let α be a simple ordered partition of weight m. Then α determines a partition of the integer m. Let $\phi(\alpha)$ denote the number of parts of this partition. Then for a crossed structure the number of terms in the gsm $\langle \alpha^1/\alpha^2/\cdots/\alpha^f \rangle$ is

$$(n_1)_{\phi(\alpha^1)}(n_2)_{\phi(\alpha^2)} \cdots (n_f)_{\phi(\alpha^f)}$$
$$= n_1(n_1 - 1) \cdots (n_1 - \phi(\alpha^1) + 1)n_2(n_2 - 1) \cdots (n_2 - \phi(\alpha^2) + 1)$$
$$\cdots n_f(n_f - 1) \cdots (n_f - \phi(\alpha^f) + 1).$$

For a nested structure gsm let γ^i denote the glb of the ordered partitions for factors which nest the ith factor. Each part, $\gamma^i_{(j)}$, say, of the partition determined by the ordered partition γ^i determines an ordered partition, $\alpha^i_{(j)}$, say, formed by the positions of α^i which correspond to the jth set of equal positions of γ^i. Then the gsm $\langle \alpha^1/\alpha^2/\cdots/\alpha^f \rangle$ has the number of terms $\prod_{i=1} s_i$, where

$$s_i = \prod_{j=1}^{\phi(\gamma^i)} (n_i)_{\phi(\alpha^i_{(j)})} .$$

The generalized polykays and unrestricted sums for nested structures are defined in terms of the expectation, E_l, over all artificial cross-labeled structures. Let h, h^* denote functions defined for the nested and artificial crossed structures respectively. In particular let (α), $(\alpha)^*$ and $[\alpha]$, $[\alpha]^*$ denote generalized polykays and unrestricted sums. Then we define, for an admissible ordered partition α.

Definition 7.4.

$$(\alpha) = E_l(\alpha)^*.$$

Definition 7.5.

$$[\alpha] = E_l[\alpha]^*.$$

The following theorems complement the above definitions.

Theorem 7.1. *Let α^i/α^j be an ordered partition for a two factor structure in which the first factor nests the second. Then*

$$E_l(\alpha^i/\alpha^j)^* = (\alpha^i/\alpha^j), \quad \text{if} \quad \alpha^j \leqq \alpha^i$$
$$= 0, \quad \text{otherwise.}$$

Proof. Since the condition for α^i/α^j to be admissible is $\alpha^j \leqq \alpha^i$ the first assertion of the theorem is merely a restatement of Definition 7.4. Suppose $\alpha^j \nleqq \alpha^i$. Then

$$(7.1.1) \quad E_l(\alpha^i/\alpha^j)^* = E_l\{(\alpha^i) \otimes (\alpha^j)\}^* = E_l\{(\textstyle\sum_r \lambda^{ir}\langle\alpha^r\rangle) \otimes (\alpha^j)\}^*,$$

where λ^{ir} is the element in the ith row and rth column of Λ^{-1}. A particular term of this expression may be written

$$E_l\{\langle\alpha^r\rangle \otimes (\alpha^j)\}^* = E_l\{\langle\alpha^r\rangle \otimes (\textstyle\sum_s \lambda^{js}\langle\alpha^s\rangle)\}^*$$
$$= \textstyle\sum_s \lambda^{js} E_l\langle\alpha^r/\alpha^s\rangle^*$$
$$= \textstyle\sum_s \lambda^{js}\langle\alpha^r/\alpha^r \wedge \alpha^s\rangle$$
$$= \textstyle\sum_s \lambda^{js}\langle\alpha^r/\alpha^r \wedge \alpha^s\rangle^*$$
$$= E_l\{\langle\alpha^r\rangle \otimes (\textstyle\sum_s \lambda^{js}\langle\alpha^r \wedge \alpha^s\rangle)\}^*$$
$$= E_l\{\langle\alpha^r\rangle \otimes (\textstyle\sum_s \lambda^{js} \sum_{\alpha^t \leqq \alpha^r \wedge \alpha^s} (\alpha^t))\}^*.$$

Now $\alpha^t \leqq \alpha^r \wedge \alpha^s$ implies $\alpha^t \leqq \alpha^r$ and $\alpha^t \leqq \alpha^s$ and conversely so that this may be written;

$$E_l\{\langle\alpha^r\rangle \otimes (\alpha^j)\}^* = E_l\{\langle\alpha^r\rangle \otimes \textstyle\sum_s \lambda^{js} \sum_t \lambda_{rt}\lambda_{st}(\alpha^t)\}^*$$
$$= E_l\{\langle\alpha^r\rangle \otimes (\textstyle\sum_t \lambda_{rt}(\alpha^t) \sum_s \lambda^{js}\lambda_{st})\}^*$$
$$= E_l\{\langle\alpha^r\rangle \otimes \textstyle\sum_t \lambda_{rt}(\alpha^t)\delta_{jt}\}^*$$
$$= E_l\{\langle\alpha^r\rangle \otimes \lambda_{rj}(\alpha^j)\}^*$$
$$= \lambda_{rj}E_l\{\langle\alpha^r\rangle \otimes (\alpha^j)\}^*.$$

Since
$$\lambda_{rj} = 1, \quad \alpha^j \leqq \alpha^r,$$
$$= 0, \quad \text{otherwise},$$

we have
$$E_l\{\langle\alpha^r\rangle \otimes (\alpha^j)\} = 0, \quad \text{if} \quad \alpha^j \nleqq \alpha^r.$$

Substituting in (7.1.1) gives
$$E_l(\alpha^i/\alpha^j)^* = E_l\{\textstyle\sum_r \lambda^{ir}\lambda_{rj}\langle\alpha^r\rangle \otimes (\alpha^j)\}^*.$$

Now, examination of the form of the inverse given by Theorem 4.1 shows that $\lambda^{ir} \neq 0$ implies $\lambda_{ir} = 1$. Also $\lambda_{ir} = 1$ and $\lambda_{rj} = 1$, imply $\alpha^j \leqq \alpha^r \leqq \alpha^i$ so that then $\lambda_{ij} = 1$. Thus either $\lambda^{ir}\lambda_{rj} = \lambda^{ir}\lambda_{ij}$ or $\lambda^{ir}\lambda_{rj} = 0$, and we may write

$$E_l(\alpha^i/\alpha^j)^* = E_l\{\textstyle\sum_r \lambda^{ir}\lambda_{ij}\langle\alpha^r\rangle \otimes (\alpha^j)\}^* = \lambda_{ij}E_l\{\textstyle\sum_r \lambda^{ir}\langle\alpha^r\rangle \otimes (\alpha^j)\}^*$$

from which the second assertion of the theorem follows.

COROLLARY 7.1. *If the ordered partition $\theta = \theta^1/\theta^2/\cdots/\theta^f$ is not admissible for a balanced complete response structure, then, for that structure $E_l(\theta)^* = 0$.*

The proof of the corollary is obtained by considering a single nested factor and one of its nesters for which the conditions of the theorem apply.

The theorem and corollary show that the set of admissible ordered partitions serve to specify a set of generalized polykays for an arbitrary balanced complete structure. The computation of the generalized polykays for such structures is aided by the theorems which follow:

THEOREM 7.2. *Let $\alpha = \alpha^1/\alpha^2/\cdots/\alpha^f$ denote an admissible ordered partition for a given balanced complete structure. Then $[\alpha] = [\alpha]^*$.*

PROOF. By definition $[\alpha] = E_l[\alpha]^*$. Let $Y^{(1)}, Y^{(2)}, \cdots, Y^{(M)}$ denote all the different crossed structures which may be obtained from the given structure by randomly cross labeling the nested factors within each combination of nesting factors. Let $[\alpha]^{(k)}$ denote the unrestricted sum determined by α for the structure $Y^{(k)}$. Then

$$[\alpha] = E_l[\alpha]^* = M^{-1} \textstyle\sum_{k=1}^{M} [\alpha]^{(k)}.$$

Consider any two of the crossed structures, say $Y^{(k)}$ and $Y^{(k')}$. $Y^{(k')}$ can be obtained from $Y^{(k)}$ by permuting the labels (subscripts) of some one or more of the nested factors. Let $\pi_\beta^{(k)}$ denote the product of m elements of $Y^{(k)}$ with the subscripts forming the $f \times m$ matrix β. We may write

$$[\alpha]^{(k)} = \sum_{\beta \geqq \alpha} \pi_\beta^{(k)},$$
$$[\alpha]^{(k')} = \sum_{\beta \geqq \alpha} \pi_\beta^{(k')},$$

where the summation is over all sets of subscripts β which form an ordered partition of which α is a subpartition (using the partial ordering of Theorem 6.1).

Consider any term of $[\alpha]^{(k)}$, say $\pi_\beta^{(k)}$. Suppose that the permutation $Y^{(k)} \to Y^{(k')}$ changes $\pi_\beta^{(k)}$ to $\pi_\gamma^{(k')}$. Let α^i be the (simple) ordered partition for the ith factor and let the m positions of α^i be $\alpha_1^i, \alpha_2^i, \cdots, \alpha_m^i$. If the ith factor is nested in no other factor it follows that $\gamma^i = \beta^i$, since the subscripts of nonnested factors will not be changed by the permutation. On the other hand suppose the ith factor to be nested in those factors numbered $i_1, i_2, \cdots, i_s$. Then $\alpha^i = \alpha^{i_1} \wedge \cdots \wedge \alpha^{i_s}$ since α is admissible. Suppose $\alpha_j^i = \alpha_{j'}^i$. Then $\beta_j^i = \beta_{j'}^i$, since $\beta \geqq \alpha$. But since $\beta_j^i = \beta_{j'}^i$ is a subscript within a single combination of all nesting factors it follows that $\gamma_j^i = \gamma_{i'}^j$, and therefore that $\gamma \geqq \alpha$. Thus

$$[\alpha]^* = [\alpha]^{(1)} = [\alpha]^{(2)} = \cdots = [\alpha]^{(M)} = [\alpha],$$

as claimed.

This theorem means that an unrestricted sum for an admissible ordered partition of a balanced complete structure is the same function of the observations whatever the structure. Which ordered partitions will be admissible depends, of course, upon the given structure.

The computation of generalized polykays for nested structures is facilitated by the following theorem which is essentially a rule for eliminating terms in the expansion of generalized symmetric sums before performing the symbolic multiplication. The proof, which depends upon several lemmas, may be found in [2].

THEOREM 7.3. *Let $\alpha^{i_1}/\alpha^{i_2}/\cdots/\alpha^{i_f}$ be an admissible f-fold ordered partition for a balanced complete response structure. Then*

$$\alpha^{i_1}/\alpha^{i_2}/\cdots/\alpha^{i_f}| = \sum_{r_1} \sum_{r_2} \cdots \sum_{r_f} \lambda^{r_1 i_1} \lambda^{r_2 i_2} \cdots \lambda^{r_f i_f} \lambda_{q_1 r_1} \lambda_{q_2 r_2} \cdots \lambda_{q_f r_f} \cdot [\alpha^{r_1}/\alpha^{r_2}/\cdots/\alpha^{r_f}],$$

where α^{q_k} is the glb *among $\alpha^{i_1}, \alpha^{i_2}, \cdots, \alpha^{i_f}$ which correspond to factors nesting the* kth *factor.*

Application of the theorem may be illustrated by computation of the second degree generalized symmetric sums for a two factor structure in which the first factor nests the second. The admissible generalized symmetric sums are |00/00|, |00/01|, |01/01| (01/00 is inadmissible because of Definition 7.2). For a crossed structure we would have the formulas

$$|00/00| = [00] \otimes [00] = [00/00]$$
$$|00/01| = [00] \otimes ([01] - [00]) = [00/01] - [00/00]$$
$$|01/01| = ([01] - [00]) \otimes ([01] - [00])$$
$$= [01/01] - [01/00] - [00/01] + [00/00].$$

Applying the theorem the first two formulas remain the same but the third formula becomes

$$|01/01| = ([01] - [00]) \otimes [01] = [01/01] - [00/01].$$

8. Unique generalized symmetric functions. The set of admissible ordered partitions of a given weight for a given f-factor balanced complete structure determines a set of generalized polykays, a set of generalized symmetric sums, a set of generalized symmetric means, and a set of unrestricted sums. However, because multiplication is commutative these sets of symmetric functions are redundant. In order to obtain a unique set of symmetric functions, the set of f-factor ordered partitions must be reduced to the set of f-dimensional partitions. The number of such partitions of fourth degree, and hence the number of generalized polykays, symmetric sums, symmetric means and unrestricted sums is $\frac{1}{24}(15^f + 9.7^f + 14.3^f)$ [5], [1]. While the formulas obtained from the ordered partitions are valid, in actual computation it is extremely desirable to reduce the set of formulas to those for the set of f-dimensional partitions, and to transform the terms to representatives of this same set and collect coefficients.

9. Utility of results for computation. The computation of generalized polykays of degree four using the generalized symmetric means directly is not in general feasible for crossed structures having more than 2 factors. Using algorithms based upon the relationships above it is possible to handle four-factor crossed structures and five factor nested structures economically [2]. The gain in computing capability appears more significant when it is noted that the number of 4th degree generalized polykays for a four-factor structure is

$$\frac{1}{24}(15^4 + 9\cdot 7^4 + 14\cdot 3^4) = 3057,$$

compared to 33 4th degree generalized polykays for two crossed factors. On the other hand, computation of the 519,153 4th degree generalized polykays for a six factor crossed structure does not seem feasible with the most advanced digital computer. With respect to nested structures the situation is somewhat improved not only because the number of generalized polykays is reduced, but also because those which require the largest number of arithmetic operations (these are unrestricted sums whose symbols contain disjoint 2-2 partitions) are often among those which are inadmissible because of the nesting [2].

10. Acknowledgment. The research reported here was performed as part of the Ph.D. thesis of the author at Iowa State University under the direction of Professor Oscar Kempthorne, whose encouragement and many helpful suggestions are gratefully acknowledged. The author also wishes to thank Dr. J. H. Bennett for pointing out to him the means of counting the generalized symmetric functions for crossed structures.

REFERENCES

[1] BENNETT, J. H. (1956). Partitions in more than one dimension. *J. Roy. Statist. Soc.* **18** 104–112.

[2] CARNEY, E. J. (1967). Computation of variances and covariances of variance component estimates. Ph.D. thesis, Library, Iowa State Univ.

[3] DAYHOFF, EUGENE (1964). On the equivalence of polykays of the second degree and Σ's. *Ann. Math. Statist.* **35** 1663–1672.

[4] DAYHOFF, EUGENE (166). Generalized polykays, an extension of simple polykays and bipolykays. *Ann Math. Statist.* **37** 226–241.

[5] FISHER, R. A. (1950) A class of enumerations of importance in genetics. *Proc. Roy. Soc. of London* **136** 509–520.

[6] HOOKE, ROBERT (1954). The estimation of polykays in the analysis of variance. Princeton Univ. *Statist. Res. Group. Mem. Rep.* **56.**

[7] HOOKE, ROBERT (1956). Symmetric functions of a two-way array. *Ann. Math. Statist.* **27** 55–79.

[8] TUKEY, JOHN W. (1956). Keeping moment like sampling computations simple. *Ann. Math. Statist.* **27** 37–54.

[9] WHITE, R. F. (1963). Randomization analysis of the general experiment. Ph.D. thesis, Library, Iowa State Univ.

[10] WILK, M. B. (1955). Linear models and randomized experiments. Ph.D. thesis, Library, Iowa State Univ.

[11] ZYSKIND, GEORGE (1958). On structure, relation, Σ, and expectation of mean squares. *Sankya, Series* **24 115–148.**