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NOTES

A CLARIFICATION CONCERNING CERTAIN EQUIVALENCE CLASSES OF GAUSSIAN PROCESSES ON AN INTERVAL[1]

By J. Feldman

The Hebrew University of Jerusalem

Two Gaussian processes will be called *equivalent* if the measures they induce on path space are mutually absolutely continuous. In [1], we considered the set of Gaussian processes obtained from stationary processes on the real line by restriction to a finite interval. A complete (although slightly awkward) description was given of the equivalence class, within this set, of the restriction of a process with rational spectral density. More recently, in [3], [4], [5], this theorem was generalized by Rozanov. However, the statements and proofs given were inexact, and there was one serious omission in the proposed necessary and sufficient conditions. It is the purpose of this note to give a correct and reasonably concise statement of Rozanov's more general theorem, together with an example to show the necessity of the condition which he omitted. An example to the same effect has also been obtained by B. Eisenberg. We refrain from including any proofs; the interested reader can presumably construct his own by use of [1], [3], [4], [5]. The details have been written out in [2].

Theorem. Let $d\mu(\lambda) = f(\lambda)\, d\lambda$, where f is a function on the real line, and, for a certain fixed n, and positive constant c,

$$c^{-1}|f(\lambda)| \leqq (1 + \lambda^2)^{-n} \leqq c\,|f(\lambda)|, \qquad -\infty < \lambda < \infty.$$

Let $\rho(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\mu(\lambda)$. Let ν be any other finite nonnegative measure on the real line, and let

$$\sigma(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\nu(\lambda).$$

Then the following conditions are necessary and sufficient that the two Gaussian processes with mean zero and parameter sets $[-T, T]$, and whose covariances $E\{X_s X_t\}$ are respectively $\rho(s - t)$ and $\sigma(s - t)$, be equivalent.

(a) The function $\tau = \rho - \sigma$ has, on the open interval $(-2T, 2T)$, $2n - 1$ derivatives. The $(2n - 1)$st derivative is an absolutely continuous function. Thus, $(1 - d^2/dt^2)\tau$ is an a.e. defined function, call it Φ. Furthermore,

$$\int_{-T}^{T}\int_{-T}^{T} |\Phi(s - t)|^2\, ds\, dt < \infty.$$

(b) If F is an entire function of exponential type $\leqq T$, $\int |F|^2\, d\mu < \infty$, and $F = 0$ ν – a.e., then F is zero.

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Notice that if the support of ν is not a discrete set on the line, then condition (b) is automatically fulfilled. However, the following example shows that the condition cannot be dispensed with in general.

EXAMPLE. Let $T = 1$, $n = 1$, $d\mu(\lambda) = (2/\pi)(1 + \lambda^2)^{-1}$. Then $\rho(t) = e^{-|t|}$. Let $\sigma(t) = (t-1)^2$ for $0 \leqq t \leqq 2$, and repeat it periodically. A calculation shows that

$$\sigma(t) = \sum_{n=-\infty}^{\infty} \sigma_n e^{i\pi n t}, \quad \text{where} \quad \sigma_n \geqq 0 \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |\sigma_n| < \infty .$$

Thus $\sigma(t) = \int_{-\infty}^{\infty} e^{it\lambda}\, d\nu(\lambda)$, where ν is an atomic measure with mass σ_n at the point n. Now: $\sigma - \rho$ has two continuous derivatives in $(-2T, 2T)$, and the second derivative is bounded. Thus condition (a) of the theorem is satisfied. However condition (b) fails, as may be seen by considering the function $F(\lambda) = \sin \lambda$. And it is easy to see that the two processes are *not* equivalent: $X_{-1} = X_1$ almost everywhere for the σ-process, but not for the ρ-process.

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