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COMMON TREATMENTS BETWEEN BLOCKS OF CERTAIN PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

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1. Summary and introduction. The study of common treatments between the blocks of statistical designs is not new. The block structure of BIB designs was most exhaustively studied by Connor [4], Shrikhande, Trehan, Chakrabarti [5] and others. Connor [3] extended this search to symmetrical Group Divisible (GD) designs restricting himself mainly to regular and semi-regular cases. Roy and Laha [6] were interested in partially balanced designs which are of the LB type. This paper deals with the study of treatments common to blocks of certain PBIB designs which are Kronecker product of certain known designs.

The credit for showing that the Kronecker product of two BIB designs has the association scheme of the rectangular type goes to Vartak [8]. In Section 3 of this paper we have obtained the matrices of characteristic vectors of the treatment characteristic matrix NN' of designs having rectangular association scheme. These are then used to set up bounds for the number of treatments common to any two blocks belonging to a group of blocks of the designs which are PBIB with three and two associate classes obtained by taking the Kronecker product of two BIB designs. Also the limits to the number of treatments common to some two blocks of a singular GD are determined.

2. Rectangular association scheme and the PBIB designs. Let us suppose that we have $v = v_1v_2$ treatments arranged in v_1 rows and v_2 columns. Corresponding to any treatment take as its first associates the remaining treatments in the same row as itself. Also take as its second associates the remaining treatments in the same column as itself. The remaining treatments are taken as its third associates. This then constitutes the rectangular association scheme. If treatments are now arranged in b blocks each of size $k < v$ such that (i) each treatment is replicated r times (ii) each block contains distinct treatments and (iii) each pair of ith associates occur together in λ_i blocks, we get a PBIB design N with rectangular association scheme. For any PBIB of this type, we have,

$$n_1 = v_2 - 1, \qquad n_2 = v_1 - 1, \qquad n_3 = n_1n_2, \tag{2.1}$$

and the association matrices are,

$$\begin{aligned} B_0 = I(v), \qquad B_1 &= I(v_1) \times [E(v_2, v_2) - I(v_2)], \\ B_2 &= [E(v_1, v_1) - I(v_1)] \times I(v_2), \\ \text{and} \quad B_3 &= [E(v_1, v_1) - I(v_1)] \times [E(v_2, v_2) - I(v_2)]; \end{aligned} \tag{2.2}$$

where $I(p)$ is the identity matrix of order p, $E(p, q)$ is a $p \times q$ matrix whose

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elements are all unity and $A \times B$ denotes the Kronecker product of the matrices A and B. Hence the matrices of parameters of the second kind are,

$$(2.3)\qquad (p_{jk}^1) = \begin{bmatrix} v_2 - 2 & 0 & 0 \\ 0 & 0 & v_1 - 1 \\ 0 & v_1 - 1 & (v_1 - 1)(v_2 - 1) \end{bmatrix},$$

$$(p_{jk}^2) = \begin{bmatrix} 0 & 0 & v_2 - 1 \\ 0 & v_1 - 1 & 0 \\ v_2 - 1 & 0 & (v_1 - 2)(v_2 - 1) \end{bmatrix},$$

$$(p_{jk}^3) = \begin{bmatrix} 0 & 1 & v_2 - 2 \\ 1 & 0 & v_1 - 2 \\ v_2 - 2 & v_1 - 2 & (v_1 - 2)(v_2 - 2) \end{bmatrix}.$$

From (2.2) it follows that if N is a PBIB with a rectangular type of association scheme,

$$(2.4)\qquad \begin{aligned} NN' &= \lambda_0 B_0 + \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3 B_3 \qquad (\lambda_0 = r) \\ &= I(v_1) \times A + [E(v_1, v_1) - I(v_1)] \times B \end{aligned}$$

where A and B are square matrices of order v_2 defined by,

$$(2.5)\qquad A = (r - \lambda_1)I(v_2) + \lambda_1 E(v_2, v_2), \qquad B = (\lambda_2 - \lambda_3)I(v_2) + \lambda_3 E(v_2, v_2).$$

Vartak [8] has shown that the roots of NN' are,

$$(2.6)\qquad \begin{aligned} &\theta_0 = rk, \qquad \theta_1 = \theta_3 + v_1(\lambda_2 - \lambda_3), \\ &\theta_2 = \theta_3 + v_2(\lambda_1 - \lambda_3) \quad \text{and} \quad \theta_3 = r - \lambda_1 - \lambda_2 + \lambda_3 \end{aligned}$$

with respective multiplicities 1, $v_2 - 1$, $v_1 - 1$ and $(v_1 - 1)(v_2 - 1)$.

In general a rectangular association scheme gives a three associate PBIB design. But, if $\lambda_1 = \lambda_3$ or $\lambda_2 = \lambda_3$, it easily follows from the definition of GD design [3], that the rectangular association scheme gives only a GD design. Also if $\lambda_1 = \lambda_2$, with the help of Lemma 4.1 of Vartak [7], we see from (2.3) that even if the association scheme is of the rectangular type the design obtained has only two associate classes, and in this case $\theta_1 = \theta_2$. If $\lambda_2 = \lambda_3$, we have $\theta_1 = \theta_3$; and $\theta_2 = \theta_3$ if $\lambda_1 = \lambda_3$. These special cases of two associate PBIB designs are implicit in the rectangular association scheme.

3. Characteristic vectors of NN'. We shall now obtain the characteristic vectors of NN' of (2.4) corresponding to its roots given in (2.6). In the rest of this paper characteristic vector means normalised characteristic vector.

THEOREM 3.1. *If $P: v \times (v_2 - 1)$, $M: v \times (v_1 - 1)$ and $Q: v \times (v_1 - 1)(v_2 - 1)$ are matrices of orthogonalised characteristic vectors corresponding to θ_1, θ_2 and θ_3,* *respectively, then*

$$(3.1) \quad PP' = (1/v_1)E(v_1, v_1) \times I(v_2) - (1/v)E(v, v),$$

$$MM' = (1/v_2)I(v_1) \times E(v_2, v_2) - (1/v)E(v, v), \quad \textit{and}$$

$$(3.2) \quad QQ' = I(v) - (1/v_2)I(v_1) \times E(v_2, v_2) - (1/v_1)E(v_1, v_1) \times I(v_2) + (1/v)E(v, v).$$

PROOF. Take any $v_2 - 1$ normalised and mutaully orthogonal vectors orthogonal to $E(1, v_2)$. Let the $v_2 \times (v_2 - 1)$ matrix of these vectors be represented by S_1. Then,

$$(3.3) \quad S_1S_1' = I(v_2) - (1/v_2)E(v_2, v_2).$$

Define,

$$(3.4) \quad P = (1/v_1^{\frac{1}{2}})E(v_1, 1) \times S_1.$$

The $v_2 - 1$ columns of P are orthogonal each having unit length, and

$$(3.5) \quad PP' = (1/v_1)E(v_1, v_1) \times I(v_2) - (1/v)E(v, v).$$

Let T be any typical column of S_1, where $T' = (t_1, \cdots, t_{v_2})$ and

$$(3.6) \quad t_1 + \cdots + t_{v_2} = 0.$$

The inner product of the first row of NN' and $Y = (1/v_1^{\frac{1}{2}})E(v_1, 1) \times T$, because of (3.6), is

$$(3.7) \quad [(r - \lambda_1) + (v_1 - 1)(\lambda_2 - \lambda_3)]t_1 = \theta_1 t_1.$$

In general, the inner product of the ith row of the v_2 rows in the first set and Y is $\theta_1 t_i$ $(i = 1, \cdots, v_2)$ and this is true for the remaining $v_1 - 1$ sets of v_2 rows in NN'. Thus,

$$(3.8) \quad NN'Y = \theta_1 Y.$$

Since Y is any typical column of P, (3.7) is true for each of its columns and we have,

$$(3.9) \quad NN'P = \theta_1 P.$$

Since θ_1 is of multiplicity $v_2 - 1$, the columns of P give a set of independent characteristic vectors corresponding to it.

Next consider any set of $v_1 - 1$ mutually orthogonal and normalised vectors orthogonal to $E(1, v_1)$. We shall denote the $v_1 \times (v_1 - 1)$ matrix of these vectors by S_2. Then,

$$(3.10) \quad S_2S_2' = I(v_1) - (1/v_1)E(v_1, v_1).$$

Define,

$$(3.11) \quad M = (1/v_2^{\frac{1}{2}})S_2 \times E(v_2, 1),$$

so that

$$(3.12)\qquad MM' = (1/v_2)I(v_1) \times E(v_2, v_2) - (1/v)E(v, v).$$

If $L' = (u_1, \cdots, u_{v_1})$ is the transpose of any arbitrary column of S_2, $X = (1/v_2^{\frac{1}{2}})L \times E(v_2, 1)$ is of unit length and the inner product of the first row of NN' and X is,

$$(3.13)\qquad [(r - \lambda_2) + (v_2 - 1)(\lambda_1 - \lambda_3)]u_1 = \theta_2 u_1,$$

for, $u_1 + \cdots + u_{v_1} = 0$. The relation (3.13) is true for the first v_2 rows of NN'. In general we see that the inner product of X and the ith set of v_2 rows of NN' is $\theta_2 u_i$ $(i = 1, \cdots, v_1)$. Thus,

$$(3.14)\qquad NN'X = \theta_2 X.$$

Since X is any arbitrary column of M, we have,

$$(3.15)\qquad NN'M = \theta_2 M.$$

As the multiplicity of θ_2 is $v_1 - 1$, the columns of M form a set of characteristic vectors corresponding to θ_2.

We know that the characteristic vector corresponding to θ_0 is $(1/v^{\frac{1}{2}})E(v, 1)$. Hence if Q is the $v \times (v_1 - 1)(v_2 - 1)$ matrix whose columns constitute a set of orthogonal characteriseic vectors corresponding to θ_3, and

$$(3.16)\qquad R = [(1/v^{\frac{1}{2}})E(v, 1)MPQ],$$

$$(3.17)\qquad RR' = I(v) = (1/v)E(v, v) + MM' + PP' + QQ'.$$

Thus from (3.17), using (3.5) and (3.12),

$$(3.18)\qquad QQ' = I(v) - (1/v_1)E(v_1, v_1) \times I(v_2) - (1/v_2)I(v_1) \times E(v_2, v_2) + (1/v)E(v, v).$$

4. Relation between rectangular association scheme and the Kronecker product of designs. Let

$$(4.1)\qquad N_1 : b_1, v_1, r_1, k_1, \lambda_1', \qquad N_2 : b_2, v_2, r_2, k_2, \lambda_2'$$

be two BIB designs. Vartak [7] has shown that the Kronecker product $N_1 \times N_2 = N$ is in general a PBIB with three associate classes. He has also shown [8] that the association scheme of this design is of the rectangular type. The first kind of parameters of N are,

$$(4.2)\qquad b = b_1 b_2, \quad v = v_1 v_2, \quad r = r_1 r_2, \quad k = k_1 k_2, \quad n_1 = v_2 - 1$$
$$n_2 = v_1 - 1, \quad n_3 = n_1 n_2, \quad \lambda_1 = r_1 \lambda_2', \quad \lambda_2 = r_2 \lambda_1', \quad \lambda_3 = \lambda_1' \lambda_2'.$$

Vartak [7] has further proved that a necessary and sufficient condition for the Kronecker product to be a PBIB with two associate classes is that $v_1 = v_2$ and $k_1 = k_2$. This case is then equivalent to taking $\lambda_1 = \lambda_2$ and $v_1 = v_2$ in a three associate PBIB having rectangular type of association scheme. The parameters in this particular case are,

$$(4.3)\quad v = v_1^2, \quad b = b_1 b_2, \quad k = k_1^2, \quad n_1 = 2(v_1 - 1),$$
$$n_2 = (v_1 - 1)^2, \quad \lambda_1 = r_1\lambda_2' \text{ and } \lambda_2 = \lambda_1'\lambda_2'.$$

The case in which $\lambda_2 = \lambda_3$, when N is a Kronecker product, is obtained if $N_2 = E(v_2, 1)$ and then it gives a singular GD.

The roots of NN' expressed in terms of the parameters of the component BIBD are,

$$(4.4)\quad \theta_0 = rk, \quad \theta_1 = \theta_3 + v_1\lambda_1'(r_2 - \lambda_2'), \quad \theta_2 = \theta_3 + v_2\lambda_2'(r_1 - \lambda_1') \text{ and}$$
$$\theta_3 = (r_1 - \lambda_1')(r_2 - \lambda_2').$$

When N is a singular GD, $r_2 - \lambda_2' = 0$ and therefore θ_3 and θ_1 are zero and,

$$(4.5)\qquad \theta_2 = v_2(r_1 - \lambda_1').$$

From (4.4) it is evident that N is a three associate PBIB if and only if all roots are distinct.

5. Common treatments between blocks of $N = N_1 \times N_2$. It is shown by Connor [4] that if y_i is the number of treatments common to some two blocks of the design N_i of (4.1),

$$(5.1)\qquad x_i \leqq y_i \leqq z_i,$$

where

$$(5.2)\qquad x_i = \max(0, 2k_i - v_i, k_i + \lambda_i' - r_i)$$

and,

$$(5.3)\qquad z_i = \min[k_i, r_i - \lambda_i' - k_i + (1/r_i)(2k_i\lambda_i')] \qquad (i = 1, 2).$$

Hence it is easy to see that if 'x' is the number of treatments common to some two blocks of N,

$$(5.4)\qquad x_1x_2 \leqq x \leqq k_1z_2.$$

We shall now divide the blocks of N into b_1 groups each consisting of v_2 blocks and obtained by taking the Kronecker product of any one block of N_1 and the matrix N_2. By interchange, if necessary, of the blocks of N_1, any group of blocks can be made the first in N, and by interchange of blocks of N_2, the first two blocks of N may be made to correspond to any two blocks of N_2.

It is easy to show that, if Z is a characteristic vector corresponding to a non-zero root θ of NN', $(1/\theta^{\frac{1}{2}})N'Z$ is a characteristic vector corresponding to the same root of $N'N$.

THEOREM 5.1. *If N_1 and N_2 are defined by* (4.1) *and y denotes the number of treatments common to any two blocks within a group of blocks of a three associate* PBIB *design $N = N_1 \times N_2$, then y always satisfies the relation,*

$$(5.5)\qquad \max(0, a_1) \leqq y \leqq \min(k_1z_2, f_1),$$

where

$$(5.6)\quad a_1 = (k - \theta_3) - (k_1^2/v_1\theta_1)(\theta_1 - \theta_3)(k - y_2),$$

$$(5.7)\quad f_1 = (\theta_3 - k) + (2\theta_3/b\theta_2)(rk - \theta_3) + (1/\theta_1)(\theta_1 - \theta_3)(k_1^2/v_1)(k + y_2) - (2/v)k^2 + (2/v_2\theta_2)k_1k_2^2(\theta_1 - \theta_3),$$

and y_2, z_2 *are defined by* (5.1) *and* (5.3) *respectively.*

PROOF. Let,

$$(5.8)\quad B = (1/b)(rk - \theta_3)E(b, b) + (1/\theta_1)(\theta_1 - \theta_3)N'PP'N + (1/\theta_2)(\theta_2 - \theta_3)N'MM'N.$$

Using (3.5) and (3.12),

$$(5.9)\quad N'PP'N = (1/v_1)k_1^2E(b_1, b_1) \times N_2'N_2 - (1/v)k^2E(b, b)$$

and,

$$(5.10)\quad N'MM'N = (1/v_2)k_2^2N_1'N_1 \times E(b_2, b_2) - (1/v)k^2E(b, b).$$

The non-zero roots of $NN' - B$ are all θ_3 and hence if Y is a real column vector with v elements,

$$(5.11)\quad 0 \leqq Y'(N'N - B)Y \leqq \theta_3 Y'Y.$$

Taking successively

$$(5.12)\quad Y' = (1/2^{\frac{1}{2}}, -1/2^{\frac{1}{2}}, 0, \cdots, 0) \quad \text{and} \quad Y' = (1/2^{\frac{1}{2}}, 1/2^{\frac{1}{2}}, 0, \cdots, 0)$$

in (5.11) after using (5.9) and (5.10) in the expression for B, we get $y \geqq a_1$ and $y \leqq f_1$. Combining this result with (5.4) we get (5.5).

In general a_1 and f_1 have two values. In such cases a_1 must be given its least and f_1 its highest values in (5.5).

EXAMPLE 5.1. Let,

$$N_1 = \begin{bmatrix} 101010 \\ 100101 \\ 011001 \\ 010110 \end{bmatrix} \quad \text{and} \quad N_2 = \begin{bmatrix} 110 \\ 011 \\ 101 \end{bmatrix}$$

be two BIB designs with, $b_1 = 6$, $v_1 = 4$, $r_1 = 3$, $k_1 = 2$, $\lambda_1' = 1$ and $b_2 = v_2 = 3$, $r_2 = k_2 = 2$, $\lambda_2' = 1$. The parameters of N include, $b = 18$, $v = 12$, $r = 6$, $k = 4$, $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = 1$. From (4.4) we get, $\theta_1 = 6$, $\theta_2 = 8$, and $\theta_3 = 2$. From (5.2) and (5.3) we have, $x_1 = 0$, $z_1 = 1$; $x_2 = z_2 = 1$. Then (5.4) gives that the number of treatments x common to some two blocks of N satisfies,

$$(5.13)\quad 0 \leqq x \leqq 2.$$

Also,

$$a_1 = 2 - \tfrac{1}{3} \quad \text{and} \quad f_1 = 4 + (11/18).$$

Hence from (5.5) the number of treatments common to some two intragroup blocks is two, and this is exact. It is easy to see that the limits given in (5.13) are the actual ones.

THEOREM 5.2. *If N_1 and N_2 are as defined in* (4.1) *and if $N = N_1 \times N_2$ is a* PBIB *with two associate classes, then the number of treatments y common to some two blocks within a group of blocks of this design satisfies the relation,*

$$\max(0, a_2) \leqq y \leqq \min(k_1 z_2, f_2) \tag{5.14}$$

where

$$a_2 = (k - \theta_3) - (k_1^2/v_1\theta_1)(\theta_1 - \theta_3)(k_2 - y_2) \tag{5.15}$$

$$f_2 = (\theta_3 - k) + (2\theta_3/b\theta_1)(rk - \theta_3) + (k/v_1\theta_1)(\theta_1 - \theta_3)(3k_1 + y_2 - 2k/v_1) \tag{5.16}$$

and y_2, z_2 *are defined by* (5.1) *and* (5.3) *respectively.*

Since the Kronecker product $N_1 \times N_2$ is a two associate PBIB if and only if $v_1 = v_2$ and $k_1 = k_2$ and if these conditions are satisfied, $\theta_2 = \theta_1$. Making these changes in the expressions for a_1 and f_1 we get (5.15) and (5.16). Combining these with (5.4) and (5.5), (5.14) follows.

EXAMPLE 5.2. Let N_1 be the same as in the Example 5.1 and $N_2 = N_1$. Then, from (4.3) we have, $b = 36$, $v = 16$, $r = 9$, $k = 4$, $\lambda_1 = 3$ and $\lambda_2 = 1$. Therefore, from (4.4) we get, $\theta_3 = 4$ and $\theta_1 = \theta_2 = 12$. Hence (5.4) gives that, if x is the number of treatments common to some two blocks of N

$$0 \leqq x \leqq 2, \tag{5.17}$$

and these limits are exact.
Also, $a_2 = -\frac{4}{3}$ or $-\frac{2}{3}$ and $f_2 = 2 + \frac{25}{27}$ or $3 + (\frac{1}{3})$. Hence from (5.14), $0 \leqq y \leqq 2$, and these limits also are exact.

THEOREM 5.3. *If x is the number of treatments common to some two blocks of a singular* GD *design with parameters, $b = b_1$, $v = v_1v_2$, $r = r_1$, $k = k_1v_2$, $\lambda_1 = r$ and $\lambda_2 = \lambda_1'$, x satisfies the relation*

$$k - \theta_2 \leqq x \leqq (\theta_2 - k) + (2/b)(rk - \theta_2), \tag{5.18}$$

where $\theta_2 = v_2(r - \lambda_2)$.

PROOF. Let N_1 be as in (4.1) and take $N_2 = E(v_2, 1)$. Then the only non-zero root other than rk of $N = N_1 \times N_2$ is $\theta_2 = v_2(r - \lambda_2)$. Now define,

$$B = (1/b)(rk - \theta_2)E(b, b). \tag{5.19}$$

Then the only non-zero root of $NN' - B$ is θ_2. Taking Y as in (5.12) and proceeding as before we get (5.18).

This paper offers an efficient tool for the discussion of common treatments between blocks of certain designs which are Kronecker product of known designs. However the method is not without difficulties. All roots of NN' and a matrix of orthogonal characteristic vectors corresponding to each root (except for one) are to be determined exactly. This may offer some resistance to the easy extension of this method.

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