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SOME RULES FOR A COMBINATORIAL METHOD FOR MULTIPLE PRODUCTS OF GENERALIZED k-STATISTICS

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1. Introduction. Dwyer and Tracy [2] gave some rules which are useful in obtaining formulae for products of two generalized k-statistics in terms of linear combinations of such statistics. These rules included generalizations of certain rules of Fisher [3] and Kendall [4], Wishart [8], and Tukey [7]. All the rules in [2] can be generalized to give rules applicable when forming products of more than two generalized k-statistics. The rules concern determination of pattern functions associated with various patterns. This paper indicates a generalization of the four such rules in [2] and establishes four additional rules.

2. General notation and background material. A π-part partition of a positive integer p, denoted by P as in [2], may be represented by

$$P = p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \tag{2.1}$$

with the convention $p_1 > p_2 > \cdots > p_s > 0$, where the order of P is the number of parts $\pi = \sum_{i=1}^{s} \pi_i$ and where $\sum_{i=1}^{s} p_i \pi_i = p$. It may also be written in the form

$$P = p_1 p_2 \cdots p_\pi \tag{2.1$'$}$$

with the convention $p_1 \geqq p_2 \geqq \cdots \geqq p_\pi > 0$. Here p itself may be considered as a 1-part partition of p. We call p_i a proper part of p if $p_i < p$.

The augmented symmetric function of the sample values $x_1, x_2, \cdots, x_n$ is given by [4], p. 276,

$$\sum x_i^{p_1} x_j^{p_1} \cdots x_q^{p_2} x_r^{p_2} \cdots x_u^{p_s} x_v^{p_s} \cdots = [p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}] = [p_1 p_2 \cdots p_\pi] = [P] \tag{2.2}$$

and the average augmented symmetric function (sample), which Tukey [7], p. 38, calls the symmetric mean or bracket $\langle p_1 p_1 \cdots p_\pi \rangle$, may be written as

$$m_P' = \langle P \rangle = [P]/n^{(\pi)} \tag{2.3}$$

and hence [4], p. 276, [1], p. 42,

$$E(m_P') = \mu_P' = \mu'_{p_1} \mu'_{p_2} \cdots \mu'_{p_\pi} \tag{2.4}$$

where μ''s are the moments about the origin.

The partition coefficient $C(P)$, as defined in [2], is the number of ways that the distinct units of p may be combined into sets of indistinguishable parcels de-

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scribed by the partition. Thus

$$C(P) = p!/(p_1!)^{\pi_1} \cdots (p_s!)^{\pi_s}(\pi_1! \cdots \pi_s!). \tag{2.5}$$

For example, 4 students of a class can be grouped into sets of indistinguishable sections of two students each in just 3 ways.

In this notation, the formula for the cumulant κ_p in terms of moments of order p and less [3], p. 201, [4], p. 279, can be written in the form

$$\kappa_p = \sum (-1)^{\pi-1}(\pi - 1)!C(P)\mu P' \tag{2.6}$$

where $\sum$ denotes summation over all partitions of p of order π, $\pi = 1, 2, \cdots, p$.

The k-statistic k_p is defined [3], p. 203, as the sample symmetric function such that $E(k_p) = \kappa_p$. In virtue of (2.4) and (2.6), we get the formula, see [4], p. 279,

$$\begin{aligned} k_p &= \sum (-1)^{\pi-1}(\pi - 1)!C(P)m_P' \\ &= \sum (-1)^{\pi-1}(\pi - 1)!C(P)[P]/n^{(\pi)} \end{aligned} \tag{2.7}$$

where $\sum$ denotes summation over all partitions of p.

The generalized k-statistic $k_P = k_{p_1 \cdots p_\pi}$ is defined [1] as the sample symmetric function such that $E(k_P) = \kappa_{p_1} \cdots \kappa_{p_\pi}$. Tukey [7], p. 52, calculated k_P by a symbolic multiplication ($\circ$) of brackets in which products of brackets are replaced by brackets enclosing the product factors. The notation of (2.7) can be adapted to generalized k-statistics with the use of subscripts [2]. Thus $P = p_1 \cdots p_i \cdots p_\pi$ is a specified partition. A partition of p_i, of order π_i, is indicated by $P_i = P_{i1}^{\pi_{i1}} \cdots p_{i\alpha}^{\pi_{i\alpha}} \cdots p_{it}^{\pi_{it}}$ where $p_i = \sum_{\alpha=1}^{t} p_{i\alpha}\pi_{i\alpha}$ and

$$\pi_i = \sum_{\alpha=1}^{t} \pi_{i\alpha}. \tag{2.8}$$

This use of π_i is different from the one in (2.1) where π_i indicates the multiplicity of p_i in the specified P. Since we are concerned hereafter with a specified P in a given k_P, we interpret π_i below in the sense of (2.8).

The partition coefficient of P_i is

$$C(P_i) = p_i!/(p_{i1}!)^{\pi_{i1}} \cdots (p_{it}!)^{\pi_{it}}(\pi_{i1}! \cdots \pi_{it}!). \tag{2.9}$$

A partition obtained by partitioning one or more of the p_i is indicated by P_I and the generalized k-statistic k_P may be expressed as

$$\begin{aligned} k_P = k_{p_1p_2\cdots p_i \cdots p_\pi} &= k_{p_1} \circ k_{p_2} \circ \cdots \circ k_{p_i} \circ \cdots \circ k_{p_\pi} \qquad [7] \\ &= \sum (-1)^{\Sigma(\pi_i-1)} \prod (\pi_i - 1)! \prod C(P_i)[P_I]/n^{(\Sigma \pi_i)} \end{aligned} \tag{2.10}$$

where the summation extends over all P_I.

In analogous notation,

$$\begin{aligned} k_Q &= k_{q_1q_2\cdots q_j \cdots q_\chi} \\ &= \sum (-1)^{\Sigma(\chi_j-1)} \prod (\chi_j - 1)! \prod C(Q_j)[Q_J]/n^{(\Sigma \chi_j)}. \end{aligned} \tag{2.11}$$

Then

$$(2.12)\quad k_{p_1\cdots p_\pi k_{q_1}\cdots q_\chi} = k_P k_Q = \sum \prod \{(-1)^{\pi_i - 1}(-1)^{\chi_j - 1} \cdot (\pi_i - 1)!(\chi_j - 1)!\, C(P_i)C(Q_j)\}[P_I Q_J]/n^{(\rho)}$$

where $P_I Q_J$ denotes an array (bipartition [2]) formed by pairing partitions P_I and Q_J, $[P_I Q_J]$ is the corresponding augmented symmetric function, and ρ, the order of $P_I Q_J$, is the number of rows in the array. For example, for the product $k_2 k_{11}$, Dwyer and Tracy [2], p. 1175, consider the bipartitions $P_I Q_J$:

	(a)	(b)	(c)	(d)	(e)
	21	20	11	11	10
(2.13)	01	01	11	10	10
		01		01	01
					01

$$(2.14)\quad k_P k_Q \cdots = \sum \prod \{(-1)^{\pi_i - 1}(-1)^{\chi_j - 1} \cdots (\pi_i - 1)!(\chi_j - 1)! \cdots C(P_i)C(Q_j) \cdots\}[P_I Q_J \cdots]/n^{(\rho)},$$

where ρ is the order of $P_I Q_J \cdots$, later called a multipartition.

3. Generalization of the method for products of two generalized k-statistics. Following [2], which is essentially a modification of Wishart's method [8], we expand the right-hand side of (2.14) in terms of generalized k-statistics. The procedure is to take expected values of symmetric means $[P_I Q_J \cdots]/n^{(\rho)}$ as $\mu'_{P_I Q_J \cdots}$, change these to population cumulants using a multipartite notation [4], [6], p. 1175, and then obtain the formula for $k_P k_Q \cdots$ as a linear function of generalized k-statistics by taking estimates. These steps correspond to those used by Tukey [7] in his algebraic method.

The transformation from the μ''s to the cumulants by the multipartite notation essentially implies a modification of Wishart's method [8] and generalization of the method in [2]. Only those arrays of $PQ \cdots$ are considered which represent $P_I Q_J \cdots$. These arrays, consisting of more than two columns, are called multipartitions, extending the idea of bipartitions in [2]. The multipartitions $P_I Q_J \cdots$ represented in (2.14) are called admissible multipartitions of $PQ \cdots$.

We illustrate this by considering the product $k_2 k_{11} k_{11}$. In a modification of Wishart's [8] example in which he obtains $k_2 k_2$ by a combinatorial method and then manipulates it algebraically to obtain the product of generalized k-statistics $k_2 k_{11}$, Dwyer and Tracy [2], p. 1175, obtain the product $k_2 k_{11}$ directly by a combinatorial method. The bipartitions appropriate to the required product, i.e. the admissible bipartitions of PQ, are listed in (2.13).

Since we are now interested in $k_2 k_{11} k_{11}$, we need to consider multipartitions obtained by appending a third column to the bipartitions (2.13) by suffixing the partition $\begin{smallmatrix}1\\1\end{smallmatrix}$ in all possible ways. The admissible multipartitions so obtained are listed in Table 1. The numbering scheme denotes correspondence to the bi-

TABLE 1

Multipartitions related to the product $k_2k_{11}k_{11}$

Number	Admissible Multiparti-tion	Multipartition Function	Combinatorial Coefficient	Contribution to Required Product
a_1	211 011	$1/n^2(n-1)$	4	$4k_{42}/n^2(n-1)$
a_2	211 010 001	$1/n^2$	4	$4k_{411}/n^2$
a_3	210 011 001	$1/n^2$	4	$4k_{321}/n^2$
a_4	210 010 001 001	$1/n$	2	$2k_{3111}/n$
b_1	201 011 010	$1/n^2$	4	$4k_{321}/n^2$
b_2	200 011 011	$1/n(n-1)$	2	$2k_{222}/n(n-1)$
b_3	201 010 010 001	$1/n$	2	$2k_{3111}/n$
b_4	200 011 010 001	$1/n$	4	$4k_{2211}/n$
b_5	200 010 010 001 001	1	1	k_{21111}
c_1	111 111	$-1/n^2(n-1)^2$	4	$-4k_{33}/n^2(n-1)^2$
c_2	111 110 001	$-1/n^2(n-1)$	8	$-8k_{321}/n^2(n-1)$

TABLE 1—(*Continued*)

Number	Admissible Multipartition	Multipartition Function	Combinatorial Coefficient	Contribution to Required Product
c_3	110 110 001 001	$-1/n(n-1)$	2	$-2k_{2211}/n(n-1)$
d_1	111 101 010	$-1/n^2(n-1)$	8	$-8k_{321}/n^2(n-1)$
d_2	111 100 011	0	8	0
d_3	110 101 011	$1/n^2(n-1)^2$	8	$8k_{222}/n^2(n-1)^2$
d_4	111 100 010 001	0	8	0
d_5	110 101 010 001	0	8	0
d_6	110 100 011 001	0	8	0
d_7	110 100 010 001 001	0	4	0
e_1	101 101 010 010	$-1/n(n-1)$	2	$-2k_{2211}/n(n-1)$
e_2	101 100 011 010	0	8	0

TABLE 1—(*Concluded*)

Number	Admissible Multipartition	Multipartition Function	Combinatorial Coefficient	Contribution to Required Product
e_3	100 100 011 011	0	2	0
e_4	101 100 010 010 001	0	4	0
e_5	100 100 011 010 001	0	4	0
e_6	100 100 010 010 001 001	0	1	0

partitions (2.13). It may be noted that multipartitions like

222,	220, 002	220, 001 001	200, 010 012	100 110 012

are inadmissible since 2 in the second or third column is not a partition of 11.

We form the combined multipartitions of a given multipartition in Table 1 by adding rows, calling the admissible results c-multipartitions. Thus multipartitions a_1 and c_1 have no c-multipartitions, whereas others do, since rows may be added as long as the resulting entry in the second or third column is not 2.

In modification of Wishart's method [8], the n-coefficient, the non-combinatorial factor of the coefficient associated with a multipartition, is obtained from (2.14) as

$$\prod \{(-1)^{\pi_i-1}(-1)^{\chi_j-1}\cdots(\pi_i-1)!(\chi_j-1)!\cdots\}n^{(\rho)}/n^{(\Sigma\pi_i)}n^{(\Sigma\chi_j)}\cdots. \tag{3.1}$$

The sum of the n-coefficients for the multipartition and all its c-multipartitions is called its multipartition (or partition) function. Thus the multipartition function of a_1 is $(1/n)(1/n^{(2)})(1/n^{(2)})n^{(2)} = 1/n^2(n-1)$, since there is no c-multipartition. For a_2, there is a c-multipartition $\begin{smallmatrix}211\\011\end{smallmatrix}$ hence the multipartition function of a_2 is

$$(1/n)(1/n^{(2)})(1/n^{(2)})n^{(3)} + (1/n)(1/n^{(2)})(1/n^{(2)})n^{(2)} = 1/n^2.$$

One has to be careful with multipartitions like c_3, which has 4 c-multipartitions $\begin{matrix}111\\110\\001\end{matrix}$ and 2 c-multipartitions $\begin{matrix}111\\111\end{matrix}$ Its partition function is thus $-(1/n^{(2)})^3 n^{(4)} - 4(1/n^{(2)})^3 n^{(3)} - 2(1/n^{(2)})^3 n^{(2)} = -1/n(n-1)$.

The combinatorial coefficient, the number of ways the multipartition can be formed, is indicated in the fourth column in Table 1. Such can be better understood by considering elements 2 in the first column as composed of distinct units e_1, e_2, the $\begin{matrix}1\\1\end{matrix}$ in the second column as identifying distinct units e_3, e_4, and the $\begin{matrix}1\\1\end{matrix}$ in the third column as identifying distinct units e_5, e_6. Thus the multipartition a_3, $\begin{matrix}210\\011\\001\end{matrix}$ with combinatorial coefficient 4, represents the 4 equivalent multipartitions in which either one of e_3, e_4 in column 2 is matched with either of e_5, e_6 in column 3 (in the second row), while the 2 in column 1 always represents $e_1 + e_2$.

The resulting generalized k-statistics have subscripts indicated by the row sums, hence the required product is the sum of the individual contributions of each multipartition, shown in the last column of Table 1, yielding the formula

$$\begin{aligned} k_2k_{11}k_{11} = {} & 4k_{42}/n^2(n-1) - 4k_{33}/n^2(n-1)^2 + 4k_{411}/n^2 \\ & + 8(n-3)k_{321}/n^2(n-1) + 2(n^2-n+4)k_{222}/n^2(n-1)^2 \\ & + 4k_{3111}/n + 4(n-2)k_{2211}/n(n-1) + k_{21111}. \end{aligned}$$

This agrees with the result of Wishart [8], p. 7, who did not have a direct pure combinatorial approach for products of generalized k-statistics but only for products of type $k_ak_bk_c \cdots$, which were solved simultaneously to obtain the desired product of generalized k-statistics.

4. Definitions associated with partition patterns. A partition pattern is defined as an algebraic multipartition in which the relative positions of the partitions of the P_I, Q_J, $\cdots$ are fixed though rows or columns may be interchanged. Thus

$$\begin{matrix} p_{11} & q_1 & 0 \\ p_{12} & 0 & r_{11} \\ p_{21} & q_{21} & r_{12} \\ p_{22} & q_{22} & r_2 \end{matrix} \tag{4.1}$$

is a pattern and equivalent patterns may be obtained by interchanging rows or columns. The multipartition d_5 in Table 1, $\begin{matrix}110\\101\\010\\001\end{matrix}$ and the multipartitions $d_5' =$

$\begin{matrix}110\\011\\100\\001\end{matrix}, d_5'' = \begin{matrix}110\\100\\011\\001\end{matrix}$ belong to equivalent patterns since d_5' is obtained by interchanging columns 1 and 2 of d_5, whereas d_5'' is obtained from d_5' by interchanging second and third rows. (The form d_5'' is required later to apply Rule 7(b) to the multipartition d_5.)

Any pattern resulting from admissible addition of rows of a pattern is a c-pattern. The pattern (4.1) has the c-pattern

$$\begin{matrix} p_1 & q_1 & r_{11} \\ p_{21} & q_{21} & r_{12} \\ p_{22} & q_{22} & r_2 \end{matrix} \qquad (4.2)$$

only, since only parts of the partition of same p_i, q_j or r_k can be combined. Thus we can add the first two rows, combining p_{11} and p_{12}, but not the last two, since we can not combine r_{12} and r_2.

For the algebraic multipartition called a pattern, the procedure in Section 3 gives a pattern function by adding the n-coefficient of the pattern and all its c-patterns. However, the process is extensive if it is applied directly to all the patterns emerging from a required product of generalized k-statistics. Work is simplified considerably by observing certain rules which these functions obey, from examples such as the one considered in Section 3. Before discussing these, we need some more definitions.

An extended pattern, as in [2], p. 1177, consists of an initial pattern plus additional rows consisting of single non-zero entries p_i or q_j or $\cdots$, Examples are multipartitions a_2, a_3, a_4 in Table 1, where the initial patterns are 211, $\begin{matrix}210\\011\end{matrix}$, 210 respectively.

An augmented pattern consists of an initial pattern plus additional columns whose non-zero entries are such that they are carried all the way through in all c-patterns and do not impose any further restrictions in the addition of rows. Examples are multipartitions b_2, c_1 in Table 1 with initial patterns $\begin{matrix}20\\01\\01\end{matrix}$, $\begin{matrix}11\\11\end{matrix}$ respectively.

A pattern is said to be composed of blocks if all its non-zero entries fall into two or more blocks, each confined to separate rows and columns, e.g. multipartition a_4 in Table 1. The concept is extended to two blocks connected by a row or column; these are called row-bordered and column-bordered blocks respectively. The connecting row or column is termed solid if it does not contain any zero entries.

5. Rules for computing pattern functions

RULE 1. *General Rule.* The pattern function is obtained by adding the n-coefficient (3.1) for the pattern and for each of its c-patterns.

This rule, which is essentially an operational restatement of the definition of a pattern function, can be used in Section 3 to construct the results of Table 1, though additional rules, discussed below, are easier to apply when appropriate.

COROLLARY. For products of k's single subscripts, all rows may be added and the partition parts of p, q, $\cdots$ may be replaced by $\times$'s to get the general rule of Fisher [3], p. 221, [4], p. 283, and Wishart [8], p. 4

(5.1) *n-coefficient*

$$= (-1)^{\pi-1}(-1)^{\chi-1} \cdots (\pi - 1)!(\chi - 1)! \cdots n^{(\rho)}/n^{(\pi)}n^{(\chi)} \cdots .$$

RULE 2. *Pattern Rule.* The multipartition functions of all multipartitions having equivalent patterns are identical and equal to the pattern function.

This is a generalization of Fisher's pattern rule [4], p. 283.

The pattern function as defined is independent of the relative positions of the rows and/or columns and hence is the same for all equivalent patterns.

For example, in Table 1, partitions a_3 and b_1 have equivalent patterns (seen by interchanging columns 2 and 3) and so they have the same pattern function $1/n^2$. Partitions a_4 and b_3 also have equivalent patterns (seen by interchanging columns 2 and 3 and then rows 2 and 4) with function $1/n$.

It should be noted that two multipartitions may look identical, but may not have the same pattern, depending upon the product being considered. For example, the multipartition

$$\begin{matrix} 110 \\ 101 \\ 011 \end{matrix} \quad (5.2)$$

for the product k_{11}^3 does not have the same pattern as multipartition $d_3 = \begin{matrix} 110 \\ 101 \\ 011 \end{matrix}$ for the product $k_2k_{11}^2$. In the first column of d_3, 11 is a partition of 2, whereas in (5.2), it is not. Thus d_3 admits a c-multipartition $\begin{matrix} 211 \\ 011 \end{matrix}$ whereas (5.2) does not. Written in general terms, (5.2) and d_3 have the patterns

$$\begin{matrix} p_1 & q_1 & 0, \\ p_2 & 0 & r_1 \\ 0 & q_2 & r_2 \end{matrix} \qquad \begin{matrix} p_{11} & q_1 & 0 \\ p_{12} & 0 & r_1 \\ 0 & q_2 & r_2 \end{matrix}$$

respectively. The pattern function of (5.2) is $(n - 2)/n^2(n - 1)^2$ which differs from $-(n - 2)/n^2(n - 1)^2 + 1/n^2(n - 1) = 1/n^2(n - 1)^2$ for d_3.

RULE 3. *Rule of proper parts.* The pattern function is 0 for any pattern which has at least one row whose single non-zero entry is a proper part.

By the pattern rule, the multipartition functions for all multipartitions having equivalent patterns are the same, no matter what the values of p_i, q_j, $\cdots$. Consider the case where all p_i, q_j, $\cdots$ are greater than 1 and let the proper part appearing alone in a row be 1. Then each k-statistic arising from the pattern has a unit subscript and thus depends on the choice of origin. Since the product expansion for the pattern with all other entries greater than 1 yields only seminvariant k-statistics (without unit subscripts), it is independent of the choice of origin. Hence the pattern function must be zero.

A more formal proof is of interest. Consider a pattern with a proper part $p_{i,r+1}$ of p_i appearing as the only non-zero entry in row **R**. Let the other entries in column **C** containing $p_{i,r+1}$ consist of the remaining r parts $p_{i1}, \cdots, p_{ir}$ of p_i, s zero and t other non-zero entries. If A denotes the product of $\prod_u (-1)^{\tau_u - 1} \cdot (\tau_u - 1)!/n^{(\Sigma_u \tau_u)}$, τ_u being the order of partition of t_u in a column, for all columns except C, the n-coefficient for the pattern is

$$(5.3) \qquad (-1)^r r! \, n^{(r+s+t+1)} A/n^{(r+t+1)} = (-1)^r r! (n - r - t - 1)^{(s)} A.$$

The c-patterns obtained by adding row **R** to other rows are of two types:

(1) s c-patterns where one of the s zeros in column **C** is replaced by $p_{i,r+1}$, resulting by adding row **R** to one of the s rows having a zero entry in column **C**. The n-coefficient for each such c-pattern being $(-1)^r r! \, n^{(r+s+t)} A/n^{(r+t+1)}$, the contribution of s such c-patterns to the pattern function is

$$(5.4) \qquad (-1)^r r! \, s(n - r - t - 1)^{(s-1)} A.$$

(2) r c-patterns resulting from the addition of row **R** to a row having a part of p_i in column **C**, the n-coefficient for each being $(-1)^{r-1}(r - 1)! \, n^{(r+s+t)} A/n^{(r+t)}$. The contribution of these r c-patterns to the pattern function is

$$(5.5) \qquad (-1)^{r-1} r! \, (n - r - t)^{(s)} A.$$

The total contribution to the pattern function, being the sum of (5.3), (5.4), (5.5), is 0. The total contribution is similarly zero for each c-pattern involving additions of rows other than **R** among themselves. Hence the pattern function for the pattern is 0.

This rule, when applied to multipartitions in Table 1, gives 0 as the partition function for multipartitions d_2, d_4, d_6, d_7 and e_2 through e_6. It thus eliminates 9 out of 25 multipartitions from further consideration. The only other multipartition which can yet be eliminated is d_5 (as a consequence of Rule 7(b) below).

COROLLARY. Turkey [7], p. 45, when writing products of two generalized k-statistics as linear functions of the same, gave a rule that the unit weight of the linear function can not exceed the unit weight of the product, where unit weight of a polynomial in k-statistics is defined as the highest number of unit parts appearing in any term of the polynomial.

This rule can be generalized for multiple products and follows as a corollary of Rule 3 since any unit subscript over and above the subscripts of the original set must result in a row of a partition having a unit proper part as its only non-zero element. Such a generalization of Tukey's rule, however, is not as effective for multiple products as Rule 3. The only multipartition it eliminates in Table 1 is e_6.

RULE 4. *Rule for extended patterns.* The pattern function of an extended pattern equals that of the initial pattern.

Consider an initial pattern extended by a row **R** in which the only non-zero element is a p_i. Let the column **C** containing this entry have s zero and r other non-zero entries. If A indicates the product of signs and factorials for all columns and $1/n^{(\tau)}$ for all columns except **C**, τ being the number of non-zero entries in a column, the sum of the n-coefficients for the extended pattern and its s c-patterns resulting from adding row **R** to the initial ones is

$$An^{(r+s+1)}/n^{(r+1)} + sAn^{(r+s)}/n^{(r+1)} = An^{(r+s)}/n^{(r)}$$

which equals the n-coefficient for the initial pattern.

This equality holds for each c-pattern of the initial pattern, obtained by adding together any of the initial rows. Hence the pattern functions of the extended and the initial patterns are the same. The argument holds when more rows of this type are added.

It may be observed in Table 1 that $a_2 = \begin{matrix} 211 \\ 010 \\ 001 \end{matrix}$ being an extended pattern of 211 in $k_2k_1^2$, has the same pattern function $1/n^2$.

COROLLARY. *A pattern composed of rows each of which has a single non-zero entry which is not a proper part has pattern function unity.* Thus b_5 in Table 1 have partition function 1. In the products k_2c_{11} [2], p. 1176, the partition function of $\begin{matrix} 20 \\ 01 \\ 01 \end{matrix}$ is also 1.

RULE 5. *Rule for augmented patterns.* The pattern function of an augmented pattern is the product of the pattern function of the initial pattern and the coefficients $\prod_u (-1)^{\tau_u - 1}(\tau_u - 1)!/n^{(\Sigma_u \tau_u)}$ for the additional columns, τ_u being the order of partition of t_u in a column.

A column of this type leads to multiplication of the n-coefficient for the initial pattern and each of its c-patterns by $\prod_u (-1)^{\tau_u - 1}(\tau_\iota - 1)!/n^{(\Sigma_u \tau_u)}$, and all the c-patterns of the augmented pattern are just c-patterns of the initial pattern augmented by this column. Hence the rule. Application of the rule, using the first two columns to form the initial pattern, gives 1. $(1/n^{(2)})$ as the partition function for b_2 and $(-1/n^{(2)})(1/n^{(2)})$ for c_1 in Table 1.

COROLLARY. *The pattern function of an augmented pattern having r columns, each of which has a single non-zero entry, is $1/n^r$ times the pattern function of the initial pattern.*

In Table 1, a_2, which may be obtained by augmenting the first column to $\begin{matrix} 11 \\ 10 \\ 01 \end{matrix}$, having partition function $1/n$ in k_{11}^2, has partition function $(1/n)(1/n) = 1/n^2$.

RULE 6. *Blocks rule.* The pattern function of a pattern composed of blocks is the product of the pattern functions of these blocks.

This is a generalization of Wishart's rule [8], p. 4, for products of single-subscript k-statistics. Fisher [3], p. 221, and Kendall [4], p. 283, ignore such patterns since these coefficients are 0 in the cumulant formulae which they feature.

Consider first the case of two blocks **A** and **B** having a and b rows and α and β columns respectively. Suppose $a \geqq b$ without loss of generality. Let the α columns of **A** have $a_1, \cdots, a_\alpha$ non-zero entries and the β columns of **B** have $b_1, \cdots, b_\beta$ non-zero entries. At this stage, we do not consider any addition within the blocks. Let A and B denote the products of signs and factorials for the blocks **A** and **B** respectively. There are $a^{(r)}b^{(r)}/r!$ c-patterns obtained by adding r rows of **B** to r rows of **A**, $(r = 1, 2, \cdots, b)$, each having n-coefficient $AB\, n^{(a+b-r)}/(\prod_i n^{(a_i)} \cdot \prod_j n^{(b_j)})$. The contribution to the pattern function from the pattern and its c-patterns, not treating addition within blocks, is thus

$$\begin{aligned} &ABn^{(a+b)}/(\textstyle\prod_i n^{(a_i)} \prod_j n^{(b_j)}) \\ &\qquad + \textstyle\sum_{r=1}^{b} (a^{(r)}b^{(r)}/r!)(ABn^{(a+b-r)}/\prod_i n^{(a_i)} \prod_j n^{(b_j)}) \\ &\qquad = (AB/\textstyle\prod_i n^{(a_i)} \prod_j n^{(b_j)}) \sum_{r=0}^{b} a^{(r)}b^{(r)}n^{(a+b-r)}/r!. \end{aligned} \tag{5.6}$$

The summation in (5.6) can be written as

$$n^{(a)} \textstyle\sum_{r=0}^{b} \binom{b}{r} a^{(r)}(n-a)^{(b-r)} = n^{(a)}n^{(b)} \tag{5.7}$$

by Vandermonde's theorem in the form [5], p. 9,

$$n^{(b)} = \textstyle\sum_{r=0}^{b} \binom{b}{r}(n-m)^{(r)}m^{(b-r)}. \tag{5.8}$$

Hence, this contribution (5.6) is

$$(An^{(a)}/\textstyle\prod_i n^{(a_i)})(Bn^{(b)}/\prod_j n^{(b_j)}) \tag{5.9}$$

which is the product of the n-coefficients for blocks **A** and **B**.

Let now the c-patterns obtained by considering addition within blocks be denoted by $\begin{matrix}\mathbf{A}_u & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_v\end{matrix}$, where **0** denotes a block of zeros. We consider

$$\begin{matrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{matrix} = \begin{matrix}\mathbf{A}_0 & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_0\end{matrix} \text{ included in the set } \left\{\begin{matrix}\mathbf{A}_u & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_v\end{matrix}\right\}.$$

Then, the pattern function of $\begin{matrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{matrix}$, by definition, is

$$\begin{aligned} &\textstyle\sum_{u,v} \text{Contribution of } \begin{matrix}\mathbf{A}_u & \mathbf{0}\\ \mathbf{0} & \mathbf{B}_v\end{matrix} \text{ to pattern function} \\ &\qquad = \textstyle\sum_{u,v} (n\text{-coefficient of } \mathbf{A}_u)\ (n\text{-coefficient of } \mathbf{B}_v) \text{ by (5.9)} \\ &\qquad = (\textstyle\sum_u n\text{-coefficient of } \mathbf{A}_u)\ (\sum_v n\text{-coefficient of } \mathbf{B}_v) \\ &\qquad = (\text{pattern function of } \mathbf{A})\ (\text{pattern function of } \mathbf{B}). \end{aligned} \tag{5.10}$$

If now there are three blocks **A**, **B**, **C**, we can treat $\begin{matrix}\mathbf{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{B}\end{matrix}$ as one block and **C** as the second. Repeating this, the rule is proved for any number of blocks.

Examples in Table 1 are multipartitions a_4, b_2, b_3, b_4, c_3, e_1, e_6. The blocks in b_3, b_4, e_1 become obvious after rearranging rows and columns. For example, in b_3, one can interchange columns 2 and 3 and then rows 2 and 4.

RULE 7. *Rule for column-bordered blocks.* The pattern function of a pattern $\begin{matrix}\mathbf{A} & \mathbf{C}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_2 & \mathbf{B}\end{matrix}$, consisting of two blocks **A** and **B** connected by a column **C**, whose parts associated with the two blocks are denoted by $\mathbf{C}_1$ and $\mathbf{C}_2$, is given by the following rule:

(a) When there is no partition with parts in both $\mathbf{C}_1$ and $\mathbf{C}_2$, the pattern function is the product of the pattern functions for the blocks $\mathbf{AC}_1$ and $\mathbf{C}_2$ **B**.

(b) When there is at least one partition which has parts in both $\mathbf{C}_1$ and $\mathbf{C}_2$, the pattern function is zero.

We consider the case of a solid connecting column only. The rule holds when **C** is not solid, for patterns required for multiple products at least through weight 12.

(a) Consider the pattern $\begin{matrix}\mathbf{A} & \mathbf{C}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_2 & \mathbf{B}\end{matrix}$ by itself first, without its c-patterns. If no partition has parts in both $\mathbf{C}_1$ and $\mathbf{C}_2$, the pattern functions of the two blocks **A** $\mathbf{C}_1$ and $\mathbf{C}_2$**B** are independent of each other. Let the number of rows in **A** $\mathbf{C}_1$, $\mathbf{C}_2$ **B** be a, b respectively. If the factor due to **A** in the n-coefficient of **A** $\mathbf{C}_1$ is denoted by A and the signs and factorials for $\mathbf{C}_1$ by C_1, and similarly for $\mathbf{C}_2$ **B**, the n-coefficient is

$$AB\ C_1C_2\, n^{(a+b)}/n^{(a+b)} = A\ B\ C_1C_2,$$

which is the product of the n-coefficients $A\ C_1\, n^{(a)}/n^{(a)} = A\ C_1$ of **A** $\mathbf{C}_1$ and $B\ C_2$ of $\mathbf{C}_2$**B**.

Let us now consider addition within the blocks, denoting a typical c-pattern by $\begin{matrix}\mathbf{A}_u & \mathbf{C}_{1u} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{2v} & \mathbf{B}_v\end{matrix}$. Treating

$$\begin{matrix}\mathbf{A} & \mathbf{C}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_2 & \mathbf{B}\end{matrix} = \begin{matrix}\mathbf{A}_0 & \mathbf{C}_{10} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{20} & \mathbf{B}_0\end{matrix}$$

as one of these, the pattern function is

$$\sum\nolimits_{u,v} \text{Contribution of } \begin{matrix}\mathbf{A}_u & \mathbf{C}_{1u} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{2v} & \mathbf{B}_v\end{matrix} \text{ to pattern function}$$

$$= \sum\nolimits_{u,v} (n\text{-coefficient of } \mathbf{A}_u\ C_{1u})\ (n\text{-coefficient of } \mathbf{C}_{2v}\ \mathbf{B}_v)$$

$$= (\sum\nolimits_u n\text{-coefficient of } \mathbf{A}_u\ \mathbf{C}_{1u})\ (\sum\nolimits_v n\text{-coefficient of } \mathbf{C}_{2v}\ \mathbf{B}_v)$$

$$= (\text{pattern function of } \mathbf{A}\ \mathbf{C}_1)\ (\text{pattern function of } \mathbf{C}_2\ \mathbf{B}).$$

(b) Let now a particular partition in column **C** have c_1 parts in $\mathbf{C}_1$ and c_2 parts in $\mathbf{C}_2$. If the α columns of **A** have $a_1, \cdots, a_\alpha$ non-zero entries and the β columns of **B** have $b_1, \cdots, b_\beta$ non-zero entries, the n-coefficient of the pattern is

$$(5.11) \qquad (-1)^{c_1+c_2-1}(c_1 + c_2 - 1)!\,(ABC_1C_2/\textstyle\prod_i n^{(a_i)} \prod_j n^{(b_j)})$$

where A, B are the products of signs and factorials for blocks $\mathbf{A}$, $\mathbf{B}$ and C_1, C_2 for entries in $\mathbf{C}_1$, $\mathbf{C}_2$, which are not parts of the particular partition.

For the particular partition having c_1 parts in $\mathbf{C}_1$ and c_2 parts in $\mathbf{C}_2$, suppose $c_1 \geqq c_2$ without loss of generality. Although there may be other such partitions, consider c-patterns obtained by adding rows in one block containing parts of this partition to similar rows in the other block, without adding rows in the same block. A typical c-pattern then consists of r rows of $\mathbf{B}$ added to r rows of $\mathbf{A}$, $1 \leqq r \leqq c_2$, there being $c_1^{(r)}c_2^{(r)}/r!$ of them, and each one has an n-coefficient

$$(5.12)\qquad (-1)^{c_1+c_2-r-1}(c_1 + c_2 - r - 1)!\,(ABC_1C_2/\textstyle\prod_i n^{(a_i)} \prod_j n^{(b_j)}).$$

Then the contribution to the pattern function from the pattern and such c-patterns is (5.11) added to $c_1^{(r)}c_2^{(r)}/r!$ times (5.12) for each r, which is

$$\begin{aligned}&\textstyle\sum_{r=0}^{c_2} (-1)^{c_1+c_2-r-1}(c_1 + c_2 - r - 1)!\,(c_1^{(r)}c_2^{(r)}/r!)\\ &\qquad\cdot(ABC_1C_2/\textstyle\prod_i n^{(a_i)} \prod_j n^{(b_j)})\\ (5.13)\qquad &\qquad= (-1)^{c_1+c_2-1}(c_1 - 1)!\,(ABC_1C_2/\textstyle\prod_i n^{(a_i)} \prod_j n^{(b_j)})\\ &\qquad\cdot\textstyle\sum_{r=0}^{c_2} (-1)^{-r}(c_1^{(r)}c_2^{(r)}/r!)c_1^{[c_2-r]}.\end{aligned}$$

Since $x^{[z]} = (-1)^z(-x)^{(z)}$, the summation in (5.13) can be written as

$$(5.14)\qquad (-1)^{c_2} \textstyle\sum_{r=0}^{c_2} \binom{c_2}{r} c_1^{(r)}(-c_1)^{(c_2-r)}$$

which is 0 by Vandermonde's theorem (5.8).

If we now consider addition of other rows within blocks or from two blocks, restricted earlier, resulting in a typical c-pattern $\begin{matrix}\mathbf{A}_u & \mathbf{C}_{1u} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{2v} & \mathbf{B}_v\end{matrix}$ and consider

$$\begin{matrix}\mathbf{A} & \mathbf{C}_1 & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_2 & \mathbf{B}\end{matrix} = \begin{matrix}\mathbf{A}_0 & \mathbf{C}_{10} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{20} & \mathbf{B}_0\end{matrix}$$

as one of these, the pattern function is

$\sum_{u,v}$ Contribution of $\begin{matrix}\mathbf{A}_u & \mathbf{C}_{1u} & \mathbf{0}\\ \mathbf{0} & \mathbf{C}_{2v} & \mathbf{B}_v\end{matrix}$ to pattern function

$= \sum_{u,v} (0) = 0.$

A case where Rule 7(a) applies is multipartition a_3 in Table 1. Its pattern function is seen to be the product of $1/n$ for 21 and $1/n$ for $\begin{matrix}11\\01\end{matrix}$. Rule 7(b) applies to multipartition d_5 which is first rearranged by interchanging columns 1 and 2, and rows 2 and 3. Since the connecting second column has parts of 2 in both blocks, the pattern function is 0.

RULE 8. *Rule for row-bordered blocks.* The pattern function of a pattern which consists of λ blocks with a row connecting each two consecutive blocks is $1/n^{\lambda-1}$ times the product of the pattern functions of the blocks.

We prove the rule by induction. The rule is trivially true for $\lambda = 1$. Consider $\lambda = 2$. Let there be a, b rows ($a \geqq b$) and α, β columns respectively in the blocks $\mathbf{A}$, $\mathbf{B}$. The connecting row is made up of the last row of block $\mathbf{A}$ and the first row of block **B**. We do not treat additions within blocks at this stage. In particular, therefore, the connecting row is not involved in c-patterns. Thus, in a typical c-pattern, r of the $(b - 1)$ rows of **B** can be added to r of the $(a - 1)$ rows of **A**, $(r \leqq b - 1)$. There are $(a - 1)^{(r)}(b - 1)^{(r)}/r!$ such c-patterns, the n-coefficient of each being $ABn^{(a+b-r-1)}/\prod_i n^{(a_i)} \prod_j n^{(b_j)}$, where A, B are the products of signs and factorials for the two blocks. The sum of the n-coefficients is thus

$$\begin{aligned}(AB/\textstyle\prod n^{(a_i)} \prod n^{(b_j)}) \sum_{r=0}^{b-1} ((a - 1)^{(r)}(b - 1)^{(r)}/r!)n^{(a+b-r-1)} \\ = (ABn^{(a)}/\textstyle\prod n^{(a_i)} \prod n^{(b_j)}) \\ \cdot \textstyle\sum_{r=0}^{b-1} ((a - 1)^{(r)}(b - 1)^{(r)}/r!)(n - a)^{(b-r-1)} \\ = ABn^{(a)}(n - 1)^{(b-1)}/\textstyle\prod n^{(a_i)} \prod n^{(b_j)}\end{aligned}$$

by Vandermonde's theorem (5.8)

$$= (1/n)(An^{(a)}/\textstyle\prod n^{(a_i)})(Bn^{(b)}/\prod n^{(b_j)}). \tag{5.15}$$

Now, considering additions within blocks, we obtain c-patterns for which (5.15) holds. Following an argument similar to that for (5.10), the pattern function is seen to be $1/n$ times the product of the pattern functions for the two blocks. Depending upon the nature of the entries in the connecting row, some other rows may be added to it, but since it remains the connecting row, the proof holds.

Now, let the pattern consist of λ row-bordered blocks $\mathbf{A}_\nu$, $\nu = 1, 2, \cdots, \lambda$. Let **A** denote the part of the pattern consisting of the blocks $\mathbf{A}_1, \cdots, \mathbf{A}_{\lambda-1}$ and let **B** consist of the blocks $\mathbf{A}_{\lambda-1}$, $\mathbf{A}_\lambda$ (together with the corresponding **0** blocks). If the rule is true for $\lambda - 1$, the pattern function of **A** is $(1/n^{\lambda-2}) \prod_{\nu=1}^{\lambda-1}$ (Pattern function of $\mathbf{A}_\nu$).

If for $\mathbf{A}_{\lambda-1}$, we substitute **B** whose pattern function is $1/n$ times the product of the pattern functions of $\mathbf{A}_{\lambda-1}$ and $\mathbf{A}_\lambda$, we see that the pattern function of the given pattern is $(1/n^{\lambda-1}) \prod_{\nu=1}^{\lambda}$ (Pattern function of $\mathbf{A}_\nu$).

Thus the rule is proved by induction.

6. Conclusion.

These rules prove useful in determining the pattern functions of particular patterns encountered in obtaining multiple products of generalized k-statistics by the combinatorial method. A list of these pattern functions, in generalization of all those of Fisher [3] for single-subscript k-statistics, is presented in [6]. These are used, in turn, to obtain specific formulae for multiple products of generalized k-statistics, which are being prepared for publication.

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