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ON THE MOMENTS OF ELEMENTARY SYMMETRIC FUNCTIONS OF THE ROOTS OF TWO MATRICES AND APPROXIMATIONS TO A DISTRIBUTION

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1. Introduction and summary. Let $\mathbf{A}_1$ and $\mathbf{A}_2$ be two symmetric matrices of order p, $\mathbf{A}_1$, positive definite and having a Wishart distribution ([2], [23]) with f_1 degrees of freedom and $\mathbf{A}_2$, at least positive semi-definite and having a (pseudo) non-central (linear) Wishart distribution ([1], [3], [23], [24]) with f_2 degrees of freedom. Now let

$$\mathbf{A}_2 = \mathbf{CYY}'\mathbf{C}'$$

where $\mathbf{Y}$ is $p \times f_2$ and $\mathbf{C}$ is a lower triangular matrix such that

$$\mathbf{A}_1 + \mathbf{A}_2 = \mathbf{CC}'.$$

Now consider the $s(=$ minimum $(f_2, p))$ non-zero characteristic roots of the matrix $\mathbf{YY}'$. It can be shown that the density function of the characteristic roots of $\mathbf{Y}'\mathbf{Y}$ for $f_2 \leqq p$ can be obtained from that of the characteristic roots of $\mathbf{YY}'$ for $f_2 \geqq p$ if in the latter case the following changes are made: [23]

$$(1.1) \qquad (f_1, f_2, p) \rightarrow (f_1 + f_2 - p, p, f_2).$$

Now, in view of (1.1), we consider only the case $s = p$, based on the density function [12] of $L = \mathbf{YY}'$ for $f_2 \geqq p$.

In this paper, some results are obtained first regarding the ith elementary symmetric function (esf) of the characteristic roots of a non-singular matrix $\mathbf{P}$ ($\mathrm{tr}_i\,\mathbf{P}$) which are useful to compute the moments of $\mathrm{tr}_i\,\mathbf{L}$ and $\mathrm{tr}_i\,\{(\mathbf{I} - \mathbf{L})^{-1} - \mathbf{I}\}$. In particular, the first two moments of $\mathrm{tr}_2\,\mathbf{L}$ are obtained in the non-central linear case. These two moments of the above criteria in the central case have been obtained earlier by Pillai ([18], [19]). Further, from a study of the first four moments of $U^{(p)} = \mathrm{tr}\,\{(\mathbf{I} - \mathbf{L})^{-1} - \mathbf{I}\}$, [11], [14], two approximations to the distribution of $U^{(p)}$ were obtained in the general non-central case. The approximations are generalizations of those given by Khatri and Pillai [10] for the linear case. The accuracy comparisons of the approximations are also made.

2. Some results on ith esf of the roots of a matrix. In this section, we prove three lemmas which will be used to obtain the moments of $\mathrm{tr}_i\,\mathbf{L}$ and $\mathrm{tr}_i\,\{(\mathbf{I} - \mathbf{L})^{-1} - \mathbf{I}\}$ in the next section.

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LEMMA 1. *Let*

$$\mathbf{P} = \begin{pmatrix} x & \mathbf{a}' \\ \mathbf{a} & \mathbf{M} + \mathbf{a}\,\mathbf{a}'/x \end{pmatrix} \begin{matrix} 1 \\ p-1 \end{matrix}$$
$$\qquad\quad 1 \qquad p-1$$

be a non-singular matrix and let $\mathbf{M}$ *be equivalent to a diagonal matrix. Then, with* $\mathbf{M}^0 = \mathbf{I}_{p-1}$ *and* $\text{tr}_0\,\mathbf{M} = 1$,

$$\text{tr}_i\,\mathbf{P} = \text{tr}_i\,\mathbf{M} + x\,\text{tr}_{i-1}\,\mathbf{M} + x^{-1}\textstyle\sum_{j=0}^{i-1}(-1)^j(\mathbf{a}'\mathbf{M}^j\mathbf{a})(\text{tr}_{i-1-j}\,\mathbf{M}) \quad \textit{for} \quad i < p$$
$$= x\,|\mathbf{M}| \quad \textit{if} \quad i = p,$$

and

$$\textstyle\sum_{j=k}^{p-1+k}(-1)^j(\mathbf{a}'\mathbf{M}^j\mathbf{a})(\text{tr}_{p-1+k-j}\,\mathbf{M}) = 0 \quad \textit{for} \quad k = 0, 1, 2, \cdots.$$

PROOF. Since $\mathbf{M}$ is equivalent to a diagonal matrix $\mathbf{D}$ (say), then there exists a matrix $\mathbf{Q}$ such that $\mathbf{M} = \mathbf{QDQ}^{-1}$. $\mathbf{D}$ is non-singular because $\mathbf{P}$ is non-singular. Hence there exists some $\theta(\leqq 1/\max_i|d_i|$, d_i $(i = 1, 2, \cdots, p-1)$ being the diagonal elements of $\mathbf{D}$) such that

$$\text{(2.1)} \qquad \textstyle\sum_{i=0}^{\infty}(-1)^i\mathbf{M}^i\theta^i = (\mathbf{I}_{p-1} + \theta\mathbf{M})^{-1}, \quad \text{a convergent series.}$$

Now, we have

$$|\mathbf{I}_p + \theta\mathbf{P}| = \begin{vmatrix} 1+\theta x & \theta\mathbf{a}' \\ \theta\mathbf{a} & \mathbf{I}_{p-1} + \theta\mathbf{M} + \theta\mathbf{a}\mathbf{a}'/x \end{vmatrix} = (1+\theta x)|\mathbf{I}_{p-1} + \theta\mathbf{M} + \theta x^{-1}(1+\theta x)^{-1}\mathbf{a}\mathbf{a}'|$$

and so

$$\text{(2.2)} \qquad |\mathbf{I}_p + \theta\mathbf{P}| = |\mathbf{I}_{p-1} + \theta\mathbf{M}|\{1 + \theta x + \theta x^{-1}\mathbf{a}'(\mathbf{I}_{p-1} + \theta\mathbf{M})^{-1}\mathbf{a}\}.$$

Moreover, we know that

$$\text{(2.3)} \qquad |\mathbf{I}_p + \theta\mathbf{A}| = \textstyle\sum_{i=0}^{p}\theta^i\,\text{tr}_i\,\mathbf{A} \quad \text{with} \quad \text{tr}_0\,\mathbf{A} = 1.$$

Using (2.1) and (2.3) in (2.2), we get

$$\text{(2.4)} \quad \textstyle\sum_{i=0}^{p}\theta^i\,\text{tr}_i\,\mathbf{P} = (\sum_{k=0}^{p-1}\theta^k\,\text{tr}_k\,\mathbf{M})\{1 + \theta x + x^{-1}\sum_{j=0}^{\infty}(-1)^j\theta^{j+1}(\mathbf{a}'\mathbf{M}^j\mathbf{a})\}$$

valid for $\theta \leqq 1/\max_i|d_i|$, d_i's being the ch. roots of $\mathbf{M}$.

Equating the coefficients of θ^i (for $i < p$), we get

$$\text{(2.5)} \quad \text{tr}_i\,\mathbf{P} = \text{tr}_i\,\mathbf{M} + x\,\text{tr}_{i-1}\,\mathbf{M} + x^{-1}\textstyle\sum_{j=0}^{i-1}(-1)^j(\mathbf{a}'\mathbf{M}^j\mathbf{a})(\text{tr}_{i-1-j}\,\mathbf{M}).$$

Now, directly, it is easy to see that

$$\text{(2.6)} \qquad \text{tr}_p\,\mathbf{P} = |\mathbf{P}| = x\,|\mathbf{M}| = xt_{p-1}\mathbf{M}$$

while the coefficient of θ^p in (2.5) is

$$\text{(2.7)} \qquad \text{tr}_p\,\mathbf{P} = xt_{p-1}\mathbf{M} + x^{-1}\textstyle\sum_{j=0}^{p-1}(-1)^j(\mathbf{a}'\mathbf{M}^j\mathbf{a})(\text{tr}_{p-1-j}\mathbf{M}).$$

Hence, (2.6) and (2.7) give

$$\sum_{j=0}^{p-1} (-1)^j(\mathbf{a}'\mathbf{M}^j\mathbf{a})(\mathrm{tr}_{p-1-j}\mathbf{M}) = 0, \tag{2.8}$$

while the coefficient of θ^{p+k} $(k \geqq 1)$ from (2.5) gives

$$\sum_{j=k}^{p-1+k} (-1)^j(\mathbf{a}'\mathbf{M}^j\mathbf{a})(\mathrm{tr}_{p-1+k-j}\mathbf{M}) = 0. \tag{2.9}$$

Thus, (2.5), (2.6), (2.8) and (2.9) establish the Lemma 1.

LEMMA 2. *Let*

$$L = \begin{pmatrix} l_{11} & \mathbf{l}' \\ \mathbf{l} & \mathbf{L}_{11} \end{pmatrix} \begin{matrix} 1 \\ p-1 \end{matrix} \qquad \begin{matrix} 1 & p-1 \end{matrix}$$

be a symmetric matrix of order p, $\mathbf{L}_{22} = \mathbf{L}_{11} - \mathbf{l}\mathbf{l}'/l_{11}$, $\mathbf{I}_{p-1} - \mathbf{L}_{22}$, *be positive definite and* $\mathbf{u} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-\frac{1}{2}}\mathbf{l}/\{l_{11}(1 - l_{11})\}^{\frac{1}{2}}$. *Then, with* $\mathbf{L}_{22}^0 = \mathbf{I}_{p-1}$ *and* $\mathrm{tr}_0\,\mathbf{L}_{22} = 1$,

$$\begin{aligned} \mathrm{tr}_i\,\mathbf{L} &= (\mathrm{tr}_i\,\mathbf{L}_{22} + \mathrm{tr}_{i-1}\,\mathbf{L}_{22}) \\ &\quad - (1 - l_{11})\{\mathrm{tr}_{i-1}\mathbf{L}_{22} - \textstyle\sum_{j=0}^{i-1} (-1)^j\mathbf{u}'(\mathbf{L}_{22}^j - \mathbf{L}_{22}^{j+1})\mathbf{u}(\mathrm{tr}_{i-1-j}\,\mathbf{L}_{22})\} \\ &\qquad \textit{for} \quad i < p \\ &= l_{11}\,|\mathbf{L}_{22}| \quad \textit{for} \quad i = p. \end{aligned} \tag{2.10}$$

Proof follows from Lemma 1 by noting

$$x = l_{11}, \qquad \mathbf{a} = \mathbf{l} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{\frac{1}{2}}\mathbf{u}\{l_{11}(1 - l_{11})\}^{\frac{1}{2}} \quad \text{and} \quad \mathbf{M} = \mathbf{L}_{22}.$$

LEMMA 3. *Let* $\mathbf{L}$, $\mathbf{L}_{22}$ *and* $\mathbf{u}$ *be defined as in Lemma 2. Let* $\mathbf{U} = (\mathbf{I}_p - \mathbf{L})^{-1} - \mathbf{I}_p$ *and* $\mathbf{M} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-1} - \mathbf{I}_{p-1}$. *Then*

$$\begin{aligned} \mathrm{tr}_i\,\mathbf{U} &= l_{11}\{(1 - l_{11})(1 - \mathbf{u}'\mathbf{u})\}^{-1}\,\mathrm{tr}_{i-1}\,\mathbf{M} + \mathrm{tr}_i\,\mathbf{M} \\ &\quad + (1 - \mathbf{u}'\mathbf{u})^{-1} \textstyle\sum_{j=0}^{i-1} (-1)^j\mathbf{u}'(\mathbf{M}^j + \mathbf{M}^{j+1})\mathbf{u}(\mathrm{tr}_{i-1-j}\,\mathbf{M}) \\ &\qquad \textit{for} \quad i < p \\ &= l_{11}\{(1 - l_{11})(1 - \mathbf{u}'\mathbf{u})\}^{-1}\,|\mathbf{M}| \quad \textit{for} \quad i = p. \end{aligned} \tag{2.11}$$

Proof follows from lemma 1 by noting (see [8])

$$\mathbf{U} = (\mathbf{I}_p - \mathbf{L})^{-1} - \mathbf{I}_p = \begin{pmatrix} x & \mathbf{a}' \\ \mathbf{a} & \mathbf{M} + \mathbf{a}\mathbf{a}'/x \end{pmatrix},$$

where $x = l_{11}/\{(1 - l_{11})(1 - \mathbf{u}'\mathbf{u})\}$, $\mathbf{a} = l_{11}^{\frac{1}{2}}(\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-\frac{1}{2}}\mathbf{u}/\{(1 - l_{11})^{\frac{1}{2}}(1 - \mathbf{u}'\mathbf{u})\}$ and $\mathbf{M} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-1} - \mathbf{I}_{p-1}$. Note that $\mathbf{M}^j + \mathbf{M}^{j+1} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-1}\mathbf{M}^j = \mathbf{M}^j(\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-1} = (\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-\frac{1}{2}}\mathbf{M}^j(\mathbf{I}_{p-1} - \mathbf{L}_{22})^{-\frac{1}{2}}$.

3. Moments of $\mathrm{tr}_i\,\mathbf{L}$. First note that the distributions of l_{11}, $\mathbf{u}$ and $\mathbf{L}_{22}$ in Lemma 2 are available in [8], [9] except that the non-centrality parameter will be denoted here by λ in place of $2\lambda^2$ given there. Now let $\mathbf{L}_0$ be the $\mathbf{L}$ matrix when $\lambda = 0$ and let $l_{11,0}$ be the top left corner element of $\mathbf{L}_0$. Then

$$x_1 = E(1 - l_{11,0}) - E(1 - l_{11}) = f_1\delta(\nu), \tag{3.1}$$

(3.2) $$x_2 = E(1 - l_{11,0})^2 - E(1 - l_{11})^2 = \tfrac{1}{2}f_1(f_1 + 2)\Delta_1,$$

(3.3) $$x_3 = E(1 - l_{11,0})^3 - E(1 - l_{11})^3 = \tfrac{1}{8}f_1(f_1 + 2)(f_1 + 4)\Delta_2,$$

and

(3.4) $$x_4 = E(1 - l_{11,0})^4 - E(1 - l_{11})^4 = (1/48)f_1(f_1 + 2)(f_1 + 4)(f_1 + 6)\Delta_3$$

where $\nu = f_1 + f_2$,

$$\begin{aligned}\delta(\nu) &= \tfrac{1}{2}\lambda\nu^{-1}\exp(-\tfrac{1}{2}\lambda)\textstyle\sum_{i=0}^{\infty}(\tfrac{1}{2}\lambda)^i(i!)^{-1}(\tfrac{1}{2}\nu + i + 1)^{-1}\\ &= \tfrac{1}{2}\lambda\nu^{-1}\textstyle\int_0^1(1 - y)^{\frac{1}{2}\nu}\exp(-\tfrac{1}{2}\lambda y)\,dy\end{aligned}$$

(3.5) $$\begin{aligned}&= \lambda\nu^{-1}\textstyle\sum_{i=0}^{\infty}(-\lambda)^i/(\nu + 2)(\nu + 4)\cdots(\nu + 2i + 2)\\ &\qquad \text{if}\quad \lambda < \nu + 2\\ &= \nu^{-1}[\textstyle\sum_{i=0}^{\frac{1}{2}\nu}\binom{\frac{1}{2}\nu}{i}(-1)^i(i!)(\tfrac{1}{2}\lambda)^{-i}\\ &\qquad - (-1)^{\frac{1}{2}\nu}(\tfrac{1}{2}\nu!)\exp(-\lambda/2)(\tfrac{1}{2}\lambda)^{-\frac{1}{2}\nu}] \quad \text{if}\quad \tfrac{1}{2}\nu \quad \text{is an integer,}\end{aligned}$$

(3.6) $$\Delta_1 = \delta(\nu) - \delta(\nu + 2), \qquad \Delta_2 = \delta(\nu) - 2\delta(\nu + 2) + \delta(\nu + 4) \quad\text{and}$$
$$\Delta_3 = \delta(\nu) - 3\delta(\nu + 2) + 3\delta(\nu + 4) - \delta(\nu + 6).$$

The results (3.1) $\cdots$ (3.4) are obtained by using the partial fractions for $[\nu(\nu + 2)(\nu + 4)\cdots]^{-1}\cdots$.

Moreover, let

(3.7) $$\beta_{1(i)} = \text{tr}_{i-1}\mathbf{L}_{22} - \textstyle\sum_{j=0}^{i-1}(-1)^j\mathbf{u}'(\mathbf{L}_{22}^j - \mathbf{L}_{22}^{j+1})\mathbf{u}(\text{tr}_{i-1-j}\mathbf{L}_{22}),$$

and

(3.8) $$\alpha_{1(i)} = \text{tr}_i\mathbf{L}_{22} + \text{tr}_{i-1}\mathbf{L}_{22}.$$

Then

(3.9) $$E(\text{tr}_i\mathbf{L}) = E(\text{tr}_i\mathbf{L}_0) + x_1E\beta_{1(i)},$$

(3.10) $$E(\text{tr}_i\mathbf{L})^2 = E(\text{tr}_i\mathbf{L}_0)^2 - x_2E\beta_{1(i)}^2 + 2x_1E\alpha_{1(i)}\beta_{1(i)},$$

(3.11) $$E(\text{tr}_i\mathbf{L})^3 = E(\text{tr}_i\mathbf{L}_0)^3 + x_3E\beta_{1(i)}^3 - 3x_2E\beta_{1(i)}\alpha_{1(i)} + 3x_1E\beta_{1(i)}\alpha_{1(i)}^2$$

and

(3.12) $$E(\text{tr}_i\mathbf{L})^4 = E(\text{tr}_i\mathbf{L}_0)^4 - x_4E\beta_{1(i)}^4 + 4x_3E\beta_{1(i)}\alpha_{1(i)} - 6x_2E\beta_{1(i)}^2\alpha_{1(i)}^2 + 4x_1E\beta_{1(i)}\alpha_{1(i)}^3.$$

Now consider $i = 2$. We have

(3.13) $$\begin{aligned}E\beta_{1(2)} &= E\{(a + 2)\,\text{tr}\,\mathbf{L}_{22} + 2\,\text{tr}_2\mathbf{L}_{22}\}/f_1\\ &= \{(p - 1)(f_2 - 1)(\nu - p)\}/\{(\nu - 1)(\nu - 2)\},\end{aligned}$$

(3.14) $$\begin{aligned}E\beta_{1(2)}\alpha_{1(2)} = \{&(a + 4)E\,\text{tr}\,\mathbf{L}_{22}\,\text{tr}_2\mathbf{L}_{22} + (a + 2)E(\text{tr}\,\mathbf{L}_{22})^2\\ &+ 2E(\text{tr}_2\mathbf{L}_{22})^2\}/f_1\end{aligned}$$

and

$$E\beta_{1(2)}^2 = \{(a+2)(a+4)E(\mathrm{tr}\,\mathbf{L}_{22})^2 + 8E(\mathrm{tr}_2\,\mathbf{L}_{22})^2 + 4(a+3)E(\mathrm{tr}\,\mathbf{L}_{22}\,\mathrm{tr}_2\,\mathbf{L}_{22}) - 4E(\mathrm{tr}\,\mathbf{L}_{22}\,\mathrm{tr}_3\,\mathbf{L}_{22}) - 8E\,\mathrm{tr}_4\,\mathbf{L}_{22} - 12E\,\mathrm{tr}_3\,\mathbf{L}_{22} - 4E\,\mathrm{tr}_2\,\mathbf{L}_{22}\}/\{f_1(f_1+2)\}, \tag{3.15}$$

where $a = f_1 - p$, $\mathrm{tr}\,\mathbf{L}_{22} = \mathrm{tr}_1\,\mathbf{L}_{22}$, $E\,\mathrm{tr}_i\,\mathbf{L}_{22} = \binom{p-1}{i}\prod_{j=1}^{i}\{(f_2-j)/(\nu-j)\}$, $E\,\mathrm{tr}\,\mathbf{L}_{22}\,\mathrm{tr}_i\,\mathbf{L}_{22} = (E\,\mathrm{tr}_i\mathbf{L}_{22})\{(p-1)(f_2-1)(\nu-i+1)+2i(a+1)\}/\{(\nu+1)(\nu-i-1)\}$, and $E(\mathrm{tr}_2\,\mathbf{L}_{22})^2$ can be obtained from $E(\mathrm{tr}_2\,\mathbf{L}_0)^2$ by changing p to $p-1$ and f_2 to $f_2 - 1$. Note that $E(\mathrm{tr}_2\,\mathbf{L}_0)^2$ is available in Pillai ([18], [19]). Using the results (3.13) to (3.15) in (3.9) and (3.10) we get the first two moments of $\mathrm{tr}_2\,\mathbf{L}$.

4. Approximations to the distribution of $U^{(p)}$. The moments of $U^{(p)}$ (a constant times Hotelling's T_0^2) have been studied by Pillai in the central case [14], [15], [16], [17], [22] and in the non-central linear case by Khatri and Pillai [8], [9], [10] who obtained the first four moments of $U^{(p)}$. Further, more recently, Khatri and Pillai extended this study to the most general case [11], i.e., to the case of number of population roots λ_i $(i = 1, 2, \cdots, r)$, $r \leqq p$. Constantine [4] has derived independently the first four moments of Hotelling's T_0^2 statistic in terms of generalized Laguerre polynomials and has computed the first two moments in the central case for illustration.

Pillai [20] has given an approximation to the distribution of $U^{(2)}$ in the linear case for $f_1 > f_2$. This has been generalized to the case of $U^{(p)}$ by Khatri and Pillai [10] in the linear case for $f_1 > (p-1)f_2$. The following is a further generalization of the latter to the most general case in the light of the first four general non-central moments.

$$g(U^{(p)}) = (U^{(p)})^{p_1-1}/\{(1+U^{(p)}/k)^{p_1+q_1+1}k^{p_1}\beta(p_1, q_1+1)\}, \quad 0 < U^{(p)} < \infty, \tag{4.1}$$

where

$$p_1 = 2q_1/\{q_1(h-1) - 2h\},$$

$$q_1 = 2\{c^2(a-3)h - (c+d)^2(a-1)\}/\{c^2(a-3)(h+1) - 2(c+d)^2(a-1)\},$$

$$k = c\{q_1(h-1) - 2h\}/\{2(a-1)\},$$

$$h = (c+1.99d)^3(a-1)/\{(c+d)^2(a-5)c\},$$

$$c = pf_2 + \sum \lambda_i \quad \text{and} \quad d = \{f_1 + (1-p)f_2 - 1\}/a.$$

Further, as a generalization of Patnaik's non-central F approximation, [13], a second approximation for the distribution of $U^{(p)}$ was suggested by Khatri and Pillai in the linear case [10]. That second approximation is further generalized as below:

$$(4.2)\quad g_1(U^{(p)}) = (U^{(p)})^{\frac{1}{2}\nu_1-1}/\{(1+U^{(p)}/k_1)^{\frac{1}{2}(\nu_1+\nu_2)}k_1^{\frac{1}{2}\nu_1}\beta(\tfrac{1}{2}\nu_1,\tfrac{1}{2}\nu_2)\},\quad 0<U^{(p)}<\infty,$$

where

$$\nu_1 = [2\{\mu_1'(U^{(p)})\}^2(a-1)]/[(a-3)\mu_2'(U^{(p)}) - (a-1)\{\mu_1'(U^{(p)})\}^2],$$

$$\nu_2 = a+1 \quad \text{and} \quad k_1 = \{\mu_1'(U^{(p)})\}(a-1)/\nu_1.$$

It may be pointed out that approximation (4.2) has been obtained by equating the first two respective moments of the approximate and exact distributions, while (4.1) has been suggested using the first three exact moments but equating only the first approximate and exact moments.

5. Accuracy comparisons. For $p = 2$, Pillai and Jayachandran [21] have

TABLE 1

Values of $G(U^{(2)})$, $G_1(U^{(2)})$ and $F(U^{(2)})$

f_1	f_2	λ_1	λ_2	$U^{(2)}$	$G(U^{(2)})$	$G_1(U^{(2)})$	$F(U^{(2)})$
13	3	1	1	1.45081	.895	.891	.888
23	7	1	1	1.31973	.914	.911	.910
13	5	1	2	2.17706	.892	.889	.885
23	3	1.5	1.5	0.68072	.844	.833	.829
13	5	1	3	2.17706	.868	.863	.858
33	5	2	2	0.65171	.830	.823	.819

obtained the cdf of $U^{(2)}$ which is also given in [10]. Denoting this cdf by $F(U^{(2)})$, the cdf from (4.1) by $G(U^{(2)})$ and from (4.2) by $G_1(U^{(2)})$, some numerical comparisons may be made on the accuracy of the approximations from Table 1.

The values of $U^{(2)}$ in Table 1 are taken from [21]. As in [10], for $p > 2$, the method of comparison assumes the exact cdf to be a Pearson type with the first four moments the same as those of the exact. Thus using the "Table of percentage points of Pearson curves for given $\beta_1^{\frac{1}{2}}$ and β_2, expressed in standard measure" [7], upper 5 per cent points are obtained for selected values for f_1, f_2, and λ_i $(i = 1, \cdots, p)$, and similar upper percentage points are obtained for approximations (4.1) and (4.2). These are presented in Table 2. In Table 2, for $p = 3, f_1 = 84$ and $f_2 = 14$, the 95 per cent point from Pearson type approximation (given under the exact column) is 0.858. From Ito's asymptotic formula, [5], [6], the probability corresponding to 0.858 is 0.957. But for $p = 3, f_1 = 64$ and $f_2 = 14$, corresponding to 1.278, the probability from Ito's formula is 0.965. Since the values of f_1 are not too large, these results are to be expected. In fact, for the power tabulations [6] Ito has taken values of $f_1 = 100$ or above. It may be pointed out that for computing from Ito's formulae, Patnaik's approximation [13] to the non-central chi-square has been used, as did Ito.

TABLE 2

Upper 5 per cent points using the exact moment quotients and the approximations (4.1) *and* (4.2)

p	f_1	f_2	λ_1	λ_2	λ_3	λ_4	λ_5	Percentage Points		
								Eqn. (4.1)	Eqn. (4.2)	Exact
2	23	3	0	25				2.768	2.937	2.931
3	24	4	1	2	3			1.655	1.693	1.685
3	84	14	1	2	3			0.853	0.861	0.858
3	24	4	2	3	6			2.065	2.109	2.123
3	64	14	2	3	6			1.266	1.279	1.278
5	56	6	1	1	1	1	1	1.058	1.068	1.061
5	56	6	1	1	2	2	2	1.141	1.155	1.146

Tables 1 and 2 show that approximation (4.1) becomes closer to the exact as p increases. However, approximation (4.2) still maintains its accuracy noted for $p = 1$ (Patnaik's), [13], even for larger values of p considered in the tables above. Further, it should be pointed out that the condition $f_1 > (p - 1)f_2$ applies for both approximations. The findings about the approximations in the general case discussed above are similar to those obtained for less general cases discussed earlier [10], [20].

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