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MONOTONE CONVERGENCE OF BINOMIAL PROBABILITIES AND A GENERALIZATION OF RAMANUJAN'S EQUATION

BY KUMAR JOGDEO[1] AND S. M. SAMUELS[2]

Courant Institute of Mathematical Sciences, N.Y. and Purdue University

1. Introduction and summary. Let the following expressions denote the binomial and Poisson probabilities,

$$(1.1)\qquad B(k; n, p) = \sum_{j=0}^{k} b(j; n, p) = \sum_{j=0}^{k} \binom{n}{j} p^j (1-p)^{n-j},$$

$$(1.2)\qquad P(k; \lambda) = \sum_{j=0}^{k} p(k; \lambda) = \sum_{j=0}^{k} e^{-\lambda}\lambda^k/k!.$$

Section 2 contains two basic theorems which generalize results of Anderson and Samuels [1] and Jogdeo [7]. These two theorems serve as lemmas for the more detailed results of Sections 3 and 4.

Section 3 is devoted to a study of the median number of successes in Poisson trials (i.e. independent trials where the success probability may vary from trial to trial). The study utilizes a method first introduced by Tchebychev [12], generalized by Hoeffding [6], and used by Darroch [5] and Samuels [10]. The results correspond to those for the modal number of successes obtained by Darroch.

Ramanujan (see [8]) considered the following equation, where n is a positive integer:

$$(1.3)\qquad \tfrac{1}{2} = P(n-1; n) + y_n p(n; n),$$

and correctly conjectured that $\frac{1}{3} < y_n < \frac{1}{2}$. In Section 4 we show that for the corresponding binomial equation,

$$(1.4)\qquad \tfrac{1}{2} = B(k-1; n, k/n) + z_{k,n} b(k; n, k/n),$$

$\frac{1}{3} < z_{k,n} < \frac{2}{3}$ and, for each k and for $n \geqq 2k$, $z_{k,n}$ decreases to y_k as $n \to \infty$.

2. Basic theorems.

THEOREM 2.1. *Let $\{m_n\}$, $\{m_n p_n\}$ and $\{m_n(1-p_n)\}$, $n = 1, 2, \cdots$, be nondecreasing sequences of positive integers with m_n's strictly increasing and the p_n's between 0 and 1. Then the sequence $\{B(m_n p_n + r; m_n, p_n)\}$ is strictly increasing (decreasing) if r is a negative (non-negative) integer.*

THEOREM 2.2. *Consider $n + 1$ independent trials with success probabilities $p_0 \geqq p_1 \geqq \cdots \geqq p_n$, $p_1 > p_n$, and $p_1 + p_2 + \cdots + p_n = k$ (positive integer).*

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Then

$$P(\leqq k \text{ successes}) < B(k-1; n, k/n) + (1-p_0)b(k; n, k/n).$$

PROOF OF THEOREM 2.1. $B(m_n p_n + r; m_n, p_n)$ may be interpreted as the probability of $\leqq (m_{n+1}p_{n+1} + r)$ successes in m_{n+1} trials in which m_n of the success probabilities are equal to p_n, $m_{n+1}p_{n+1} - m_n p_n$ of them are equal to 1 and the remaining $m_{n+1}(1 - p_{n+1}) - m_n(1 - p_n)$ are equal to 0. To complete the proof we need only to apply the theorem of Hoeffding [6] (see [10], Lemma 1) which states (in part) that if the mean number of successes in n independent trial is λ, then the probability of $\leqq k$ successes is $\geqq$ ($\leqq$) $B(k; n, \lambda/n)$ if $k \geqq \lambda$ ($\leqq \lambda - 1$), with equality holding only if all n success probabilities are equal to λ/n.

PROOF of THEOREM 2.2. Fix p_0 and consider the class of all choices of $p_1, \cdots, p_n$ with $1 \geqq p_0 \geqq p_1 \geqq \cdots \geqq p_n \geqq 0$ and $p_1 + p_2 + \cdots + p_n = k$. We need to show that, within this class, the probability of $\leqq k$ successes in the $n + 1$ trials attains its unique maximum when $p_1 = p_2 = \cdots = p_n = k/n$. Since by compactness and continuity the maximum probability is attained, it suffices to prove that it is not attained when $p_1 > p_n$. Now

$$P[\leqq k \text{ successes}] = p_1 p_n[f^*(k) - f^*(k-1)] - (p_1 + p_n)f^*(k) + A,$$

where $f^*(k)$ is the probability of k successes in the other $n - 1$ trials and A does not depend on p_1 and p_n. By a corollary to Theorem 1 of [10], the hypotheses of the present theorem are sufficient to guarantee that $f^*(k) > f^*(k - 1)$. Hence the probability of $\leqq k$ successes can be increased by replacing p_1 and p_n by $(p_1 + p_n)/2$, which completes the proof.

Note that if $(n - 1)p_0 \geqq k$, we may take $p_1 = p_2 = \cdots = p_{n-1} = k/(n - 1)$, $p_n = 0$ and conclude that

$$B(k-1; n-1, k/(n-1)) + (1-p_0)b(k; n-1, k/(n-1)) < B(k-1; n, k/n) + (1-p_0)b(k; n, k/n). \tag{2.1}$$

3. Median number of successes. The modal number of successes in Poisson trials has been studied in [5] and [10], and the following is known:

(a) If the mean number of successes is an integer k, then the mode is also k (and is unique). If the mean is between two integers k and $k + 1$ then the mode is k or $k + 1$.

(b) The set of all possible mean numbers of successes in n independent trials which uniquely determine the mode is

$$A_n = \bigcup_{k=0}^{n-1} (I_k \cup J_{k+1})$$

where I_k is the interval $[k, k + 1/(k + 2))$, and J_k is $(k - 1/(n - k + 2), k]$.

In this section we obtain analogous results for the median.

THEOREM 3.1. *Let t be a fixed positive integer >1. Then as $n \to \infty$, $B(n - 1; nt, 1/t)$ increases to $\frac{1}{2}$ while $B(n; nt, 1/t)$ decreases to $\frac{1}{2}$.*

PROOF. By the central limit theorem, both limits are $\frac{1}{2}$. Monotonicity follows by applying Theorem 2.1, with $p_n = 1/t$, $m_n = nt$ and $r = -1$ for the first sequence and $r = 0$ for the second.

THEOREM 3.2. *If the mean number of successes in n independent trials is an integer k then the median is also k.*

PROOF. First suppose that all success probabilities are equal to k/n. By Theorem 2.1 of [1], or from Theorem 2.1 of this paper, the sequences $\{B(k-1; n, k/n)\}$ and $\{B(k; n, k/n)\}$ are, respectively, increasing and decreasing in n, so that

$$B(k-1; n, k/n) < B(k-1; kn, 1/n) < \tfrac{1}{2} < B(k; kn, 1/n) < B(k; n, k/n),$$

where the two inequalities in the middle follow from Theorem 3.1.

If the success probabilities are unequal, we apply the Hoeffding theorem (see proof of Theorem 2.1) to conclude that

$$P(\leqq k-1 \text{ successes}) < B(k-1; n, k/n) < \tfrac{1}{2} < B(k; n, k/n) < P(\leqq k \text{ successes}).$$

COROLLARY 3.1. *If the mean number of successes is between the integers k and $k+1$ then the median is k or $k+1$.*

PROOF. Use Theorem 3.2 and the fact that, for every j, the probability of $\leqq j$ successes is a decreasing function of each success probability.

The preceding fact, plus a result of Hoeffding {[6], Theorem 4} gives

THEOREM 3.3. *For $k = 0, \cdots, n-1$, define λ_k and θ_{k+1} to be, respectively, the solutions of*

$$\tfrac{1}{2} = \min_{r=0,1,\cdots,n-k-1} B(k; n-r, \lambda_k/(n-r)),$$

$$\tfrac{1}{2} = \max_{s=0,1,\cdots,k} B(k-s; n-s, (\theta_{k+1}-s)/(n-s)).$$

Let I_k' be the interval $[k, \lambda_k)$ and J_k' be the interval $(\theta_k, k]$. Then

$$B_n = \bigcup_{k=0}^{n-1} (I_k' \cup J_{k+1}')$$

is the set of all possible mean numbers of successes in n independent trials which uniquely determine the median.

The point of this theorem is that the analogue to statement (b), for the median, cannot be as explicit as is the statement for the mode, since clearly the λ_k's and θ_k's are not rational functions of n and k.

If we require that the success probabilities be *almost equal* then, for large n, the median cannot be smaller than the one corresponding to the case of equality. The precise statement of this remark is given in the next theorem which follows from Theorem 5.1 of [1].

THEOREM 3.4. *If λ, the mean number of successes in n independent trials, is between the integers k and $k+1$ and if each of the success probabilities is $\leqq \lambda/(n-1)$, and if n is sufficiently large so that*

$$\lambda > [(n-1)/(n-3)][k - 1/(n-k)],$$

then the median is $k+1$ whenever $B(k; n, \lambda/n) < \tfrac{1}{2}$.

4. Ramanujan's equation. Among the multitude of interesting problems Ramanujan discussed in his correspondence with Hardy (see [8]) is one concerned with the equation,

$$e^n/2 = 1 + n + (n^2/2!) + \cdots + y_n(n^n/n!), \tag{4.1}$$

which is equivalent to (1.3). In 1911 he proved that $y_n \to \frac{1}{3}$ as $n \to \infty$ and correctly conjectured that $\frac{1}{3} < y_n \leqq \frac{1}{2}$ for all non-negative integers n (obviously $y_0 = \frac{1}{2}$). Szegö (1928) and Watson (1928) showed that y_n decreases to $\frac{1}{3}$ and discussed more precise estimates conjectured by Ramanujan in 1913. These results were subsequently extended by various authors. Copson (1933) considered the case when n is a negative integer while Carlitz (1965) studied complex exponents.

The Poisson interpretation was first utilized by T. T. Cheng (1949) who gave a more elementary proof for the bounds on y_n while studying the error in the normal approximation to the Poisson distribution.

In this section we consider the binomial analogue of (4.1) as given by (1.4). Some values of $z_{k,n}$ are obvious: $z_{0,n} = z_{n,n} = z_{n,2n} = \frac{1}{2}$. Also, by Theorem 3.2, $0 < z_{k,n} < 1$ for all k and n. The following theorem yields sharper bounds. (In the remainder of this section we assume $k \geqq 1$.)

THEOREM 4.1. *For each k, and for $n \geqq 2k$, $z_{k,n}$ decreases to y_k as $n \to \infty$.*

PROOF. The assertion that the limit is y_k is an immediate consequence of the Poisson convergence theorem. Since $z_{k,2k} = \frac{1}{2}$, it suffices to prove that, if $p_0 \geqq \frac{1}{2}$ and $n - 1 \geqq 2k$, (which implies that $(n-1)p_0 \geqq k$), then

$$\begin{aligned} B(k-1; n-1, k/(n-1)) + (1-p_0)b(k; n-1, k/(n-1)) \\ < B(k-1; n, k/n) + (1-p_0)b(k; n, k/n). \end{aligned} \tag{4.2}$$

Now (4.2) is identical to (2.1); hence it is true by Theorem 2.2.

Since $z_{k,n} \to y_k > \frac{1}{3}$ as $n \to \infty$ we have the following

COROLLARY 4.1. $n > 2k \Rightarrow \frac{1}{3} < z_{k,n} < \frac{1}{2}$.

A weaker statement is:

COROLLARY 4.2. $n > 2k \Rightarrow z_{k,n} < \frac{1}{2}$ (i.e. *Theorem* 4.1 *can be regarded as providing a new proof of the Simmons inequality* [9]).

Note that by interchanging the roles of success and failure we can conclude that $\frac{1}{2} < z_{k,n} < \frac{2}{3}$ for $k < n < 2k$; but we cannot conclude that $z_{k,n}$ is decreasing in n for $n < 2k$. This is not surprising since the statement is false (for example, $z_{k,k} = \frac{1}{2} < z_{k,k+1}$; $z_{k,2k-1} > \frac{1}{2} = z_{k,2k}$). Presumably $z_{k,n}$ first increases and then decreases as n increases from k to $2k$, but we are unable to prove this.

We are also unable to generalize the statement that y_k is decreasing in k. We sought a result of the form

$$\begin{aligned} B(k-1; n, k/n) + B(k; n, k/n) < B(k; m, (k+1)/m) \\ + B(k+1; m, (k+1)/m) \end{aligned}$$

for some appropriate sequences of n's and m's, which, it can be shown, would imply that $z_{k,n} > z_{k+1,m}$. But the inequality is false when $n = 2k$, $m = 2k + 1$, as well as when $n = 3k$ or $4k$ and $m = 3(k+1)$ or $4(k+1)$. Of course, Theorem 4.1 plus the fact that $y_k > y_{k+1}$, implies that $z_{k,n} > z_{k+1,n}$ for all sufficiently large n. But we are unable to make this more precise.

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