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BOOK REVIEWS

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JAROSLAV HÁJEK AND ZBYNĚK ŠIDÁK, *Theory of Rank Tests*. Academic Press, New York and London. Academia Publishing House of the Czechoslavak Academy of Sciences, Prague, 1967. $8.00. 297 pp.

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Some of the main features of this monograph are concisely described in the Preface: "This book is designed for specialists, teachers and advanced students in statistics. We assume the reader to be acquainted with the basic facts about the theory of testing hypotheses, measure theory, stochastic processes, and the central limit theorem. ... Striving for compactness and lucidity of the theory, we concentrated on contiguous alternatives and on problems concerning location and scale parameters. In this respect the results obtained are almost complete. The two most serious gaps still left are the absence of an effective method for the estimation of the type of a density, and the failure to carry out an adequate asymptotic treatment of the alternatives for the hypothesis of independence. Relatively little space has been given to the non-contiguous alternatives and to the famous Chernoff-Savage theorem, and no space at all to the interesting investigations on the possibility of employing rank tests for estimation problems [started by Lehmann and Hodges in 1963]."

Within these self-imposed bounds the authors have written an admirable monograph. Much of its content appears for the first time in book form. A number of new results are included. The material is skillfully organized. Proofs are carefully and elegantly carried out. The mathematical development is supplemented by lucid discussion of the relation of theory to applications. Each chapter ends with a section of problems and complements which contain much additional material. The reviewer noticed only a few trivial errors, mostly misprints, which are easily corrected by the reader.

Rank tests are applicable when the hypothesis implies invariance of the distribution of the observations under some group of permutations and this invariance is not shared by the alternatives under consideration. They have proved to be important both for practical and for theoretical reasons. They are "exact" and are quick and simple to apply (all of this subject to rather obvious qualifications). They can be used when only ranking data are available. When applied to numerical observations, rank tests may seem to discard much information contained in the sample. However, it is now known that for many parametric alternatives there are rank tests which are highly efficient in large samples compared with the best tests for those alternatives. In fact, as is shown in this book, the results of the theory of rank tests can be utilized to seek asymptotically optimum tests within the class of all tests. Moreover, for some of the common parametric models there are rank tests which are asymptotically as efficient for the models as the best tests assuming those models. Indeed, they are sometimes superior to standard tests based on the assumption of normality in the sense that not only their significance level but also their power is less affected by deviations from normality. (This robustness property of rank tests is not emphasized in the book; some of the relevant results by Hodges and Lehmann and by Chernoff and Savage are mentioned.)

Rank tests form a subclass of the class of permutation tests. Just as rank tests, permutation tests are exact (similar) for testing the same kind of wide hypotheses. A permutation test which is optimal against a class of alternatives is, in general, more powerful than the corresponding optimal rank test, although often the two have the same asymptotic power against the specified alternatives. The relation between permutation tests and rank tests is briefly studied in Chapter II of this book. It may be added that in typical cases an optimal permutation test is asymptotically equivalent to the corresponding optimal parametric test (as shown by the reviewer in 1952) and therefore suffers from a lack of robustness when the latter does. Besides, an optimal permutation test takes much more labor to apply than a rank test.

The chapter headings are: I. Introduction. II. Elementary theory of rank tests. III. Selected rank tests. IV. Exact distributions of test statistics under the hypotheses and their computation. V. Limiting distributions of test statistics under the hypotheses. VI. Limiting distributions of test statistics under the alternatives. VII. Asymptotic optimality and efficiency of rank tests. Some of the main topics under study are described in what follows.

Throughout the book it is assumed that the distributions of the observed random variables are absolutely continuous with respect to Lebesgue measure. The only exception is a brief but informative section on the treatment of ties. For the study of tests concerning location and scale parameters, the sole condition that the density be absolutely continuous and have finite Fisher information turns out to be sufficient.

The main hypotheses that are considered are the following. H_0 : randomness; H_1 : symmetry (random sample from a distribution symmetric about zero); H_2 : independence (in a random sample from a bivariate distribution); H_3 : random blocks. The main categories of rank tests are linear rank tests, the related rank tests of χ^2 type, tests based on exceeding observations, and tests of Kolmogorov-Smirnov type.

The linear rank tests are those based on so-called linear rank statistics, typified by $S_c = \sum_{i=1}^{N} c_i a(R_i)$ for testing H_0, where R_i is the rank of X_i in the sample $(X_1, \cdots, X_N)$. They include the locally most powerful rank tests against alternatives of regression in location or scale. A version of the Wald-Wolfowitz-Noether-Hájek theorem, which asserts the asymptotic normality of S_c under

H_0, is proved under conditions suitable for applications to locally most powerful rank tests. The proof is akin to, but different from, Hájek's proof of 1961. The well-known Kolmogorov-Smirnov two-sample tests are generalized in two ways. The original K-S tests (one-sided and two-sided) are considered as tests against difference in location and appear as special cases of K-S type tests against regression alternatives, which were introduced by Hájek (Proc. Bernoulli-Bayes-Laplace Seminar, 1965). Against an alternative with density $\prod f(x_i - \Delta c_i)$, $\Delta > 0$, the (one-sided) K-S type statistic is $K_c^+ = \max_{1 \leqq k \leqq N} (k\bar{c} - \sum_{i=1}^{k} c_{D_i})/\{\sum_{i=1}^{N} (c_i - \bar{c})^2\}^{\frac{1}{2}}$, where $\bar{c} = N^{-1} \sum c_i$ and the "anti-ranks" D_i are given by $X_{D_1} < X_{D_2} < \cdots < X_{D_N}$. A related type of statistic is proposed against the alternative of two samples from distributions which differ in scale only. The limit distributions of these statistics are derived along the lines of Hájek's paper cited above. Proofs of the needed results on convergence in distribution of stochastic processes with continuous sample functions are included.

The alternatives under which limit distributions of rank test statistics are derived are contiguous, in the sense of Le Cam (1960), to distributions in the corresponding hypothesis class. The needed results from Le Cam's theory of contiguity are proved. They are first used to obtain the asymptotic distribution of S_c (as above) under regression alternatives with densities

$$(1) \quad \prod_{i=1}^{N} f_0(x_i - d_i), \ \max_{1 \leqq i \leqq N} (d_i - \bar{d})^2 \to 0, \quad I(f_0) \sum_{i=1}^{N} (d_i - \bar{d})^2 \to b^2, \quad 0 < b < \infty,$$

where $I(f_0) = \int (f_0'/f_0)^2 f_0 \, dx$, generalizing a theorem of Hájek (1962) for the special case $c_i = d_i$. Analogous results are obtained for alternatives to H_1 (symmetry) and for χ^2 type statistics under k-sample alternatives. For alternatives to independence of the form $\prod h_\Delta(x_i, y_i)$ with $h_\Delta(x, y) = \int f(x - \Delta z) \cdot g(y - \Delta z) \, dM(z)$, only a conjecture is stated. The asymptotic power $B(\alpha, f_0, b)$ of the K_c^+ test (defined above) of size α under alternatives (1) is expressed in terms of a Brownian bridge process with non-zero mean function, but its evaluation is "a rather complex task". The authors succeed in obtaining an explicit expression for the "local" asymptotic power as b (the constant in (1)) tends to zero. An approximation for $B(\alpha, f_0, b)$ in terms of the limit of the asymptotic efficiency as $b \to 0$ is discussed and is numerically compared, for the case $f_0(x) = \frac{1}{2}e^{-|x|}$, with the exact values of $B(\alpha, f_0, b)$ and with the asymptotic power of the best test.

For classes of alternatives such as (1) the rank vector is shown to be asymptotically sufficient in the sense of Le Cam (1960). This imples that a rank test of asymptotically maximin power has this property in the class of all tests. One interesting new result is Theorem VII, 1.4, which shows the asymptotic optimality (in the class of all tests) of a χ^2 type rank test against k-sample location alternatives. All these optimality results depend on the knowledge of the density f_0 (or rather of the type of the density) in terms of which the alternative is defined. A test which is independent of f_0 and is asymptotically most powerful against one-sided alternatives of the form (1) for any f_0 with finite Fisher information has been constructed by Hájek in 1962. Here a somewhat different construction is used, and an asymptotically optimal test like that of Theorem VII, 1.4, but independent of f_0 is also obtained. Unfortunately these tests "may be quite bad for moderate sample sizes".

It should be emphasized that this brief summary does not give an adequate idea of the richness of the material contained in the book.