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A COMPACT TABLE FOR POWER OF THE t-TEST[1]

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1. Introduction. While most intermediate and advanced statistical textbooks discuss the power of the t-test, few if any provide tables that would enable the student to acquire a working knowledge of this important topic. The omission may be due in part to the fact that available tables giving good coverage run to many pages, and are therefore not suited for inclusion in the brief compendium at she back of a textbook. We present here a one-page table for t-power which covers any value of the (one-sided) significance level α in the range from .005 to .1 (double these values for the two-sided test); any value of the second-type error probability β in the range from .01 to .5; and any number f of degrees of freedom greater than 2.

Such a table should not only be compact but also convenient to use. In particular, it should not require high-powered interpolation since this would be almost prohibitively laborious in a triple-entry table. The problem is therefore to find a compact presentation in which the tabulated quantity will admit accurate interpolation over wide intervals by means of low order formulas—ideally, by means of linear interpolation. A second possible difficulty in the use of such a table stems from the great variety of statistical problems involving t-power. No matter how a table is designed, it will deal by direct entry with only one type of problem. Many of the more interesting applications will call for some sort of trial-and-error. Initial or trial values with which to enter the table are obtained by guess, or by some approximate method. Unless one is lucky, the solution corresponding to the initial values may have to be adjusted, calling for an iterative use of the table. A satisfactory presentation of t-power must therefore provide for trial values with which to enter the table; preferably, these should be accurate enough so that iteration will not be necessary.

The presentation provided here gives reasonably accurate answers without iteration and using only linear interpolation. In order to achieve these features in a one-page table of broad coverage, we found it necessary to reparametrize the problem. Our presentation uses not the error probabilities themselves but rather their normal transforms, say

$$\Phi^{-1}(1-\alpha) = u, \qquad \Phi^{-1}(1-\beta) = v.$$

We must therefore assume that the user has at hand a table of the normal distribution Φ. In terms of u and v, the asymptotic expansion recorded in Section 4

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makes it natural to write

$$u + v = \delta(1 - u^2/4f) - Au^2(u + v)/f^2. \tag{1}$$

Here δ is the noncentrality parameter of the t-statistic.

We may think of the coefficient A of the remainder term of order $1/f^2$ as implicitly defined by the relation (1) itself. Our basic method consists in finding A in the accompanying table, and then solving (1) for whichever parameter is being calculated. (As it stands, (1) applies to one-sided t-tests against alternatives to the right. The modifications called for by other problems are discussed in Section 3).

The justification for presenting t-power by means of (1) and a table of A rests on three facts. First, A turns out to be a smooth function of u, v and $1/f$; as discussed in Section 5, quadratic interpolation in the table gives four-decimal results in most cases, and even linear interpolation is accurate enough for most purposes. Second, the remainder term of which A is the coefficient is usually rather small, so that A itself need not be highly accurate to give good results. Third, this approach deals satisfactorily with the problem of trial values. If we drop the remainder term, the curtailed version of (1),

$$u + v = \delta(1 - u^2/4f), \tag{1*}$$

may be solved to give trial values that have an error of order $1/f^2$. These trial values are in most cases good enough so that no iterative use of the table is called for.

By virtue of the expansion on which it is based, our method works best if f is not too small. We do not cover $f = 1$ and $f = 2$. To illustrate the degree of precision available with other small values of f, we have chosen for the Examples values of f ranging from 3 to 13. As will be seen, in these problems linear interpolation without iteration typically gives probabilities correct to about four or five decimal places. This should be adequate for nearly all purposes.

More accurate results would of course have been obtained with larger values of f. In fact, for f above 15 or so, a four-decimal value of A will often provide more precise answers than given by any other table we have seen. For this or some other reason, the user may occasionally wish to push our method to the limit of its accuracy. Such refinement of the solution requires a cycle of iteration in which A is interpolated quadratically with respect to u and $1/f$.

2. Examples of one-sided tests. The eight Examples of the next two Sections illustrate a variety of t-power problems. It is our hope that the flexibility of (1) will encourage the use of unconventional significance levels and nonstandard test designs. While the details differ from one problem to the next, the general approach involves three steps.

(i). The curtailed equation (1^*) is solved to give the trial or entry values of the parameters. These values are labelled with subscript 0.

(ii) The table is entered at these values, and linear interpolation with respect

TABLE OF A

$\beta =$	.01	.05	.10	.20	.30	.40	.50
$v =$	2.326348	1.644854	1.281552	.841621	.524401	.253347	0
f			$\alpha = .005$,	$u = 2.575829$			
3	.0983	.0412	.0076	−.0361	−.0695	−.0993	−.1280
4	.1705	.1037	.0653	.0165	−.0202	−.0524	−.0831
5	.2011	.1297	.0894	.0389	.0014	−.0312	−.0620
6	.2167	.1429	.1018	.0508	.0132	−.0193	−.0499
8	.2310	.1552	.1137	.0627	.0254	−.0066	−.0367
12	.2398	.1634	.1221	.0718	.0353	.0041	−.0252
24	.2436	.1681	.1277	.0787	.0433	.0131	−.0152
∞	.2431	.1699	.1309	.0837	.0497	.0206	−.0066
			$\alpha = .01$,	$u = 2.326348$			
3	.1693	.1090	.0737	.0282	−.0064	−.0370	−.0662
4	.2058	.1396	.1020	.0544	.0190	−.0119	−.0412
5	.2205	.1518	.1135	.0656	.0304	−.0001	−.0289
6	.2274	.1576	.1191	.0716	.0367	.0067	−.0215
8	.2329	.1626	.1244	.0775	.0434	.0141	−.0133
12	.2352	.1654	.1278	.0820	.0489	.0205	−.0060
24	.2347	.1662	.1297	.0854	.0534	.0260	.0004
∞	.2316	.1656	.1303	.0877	.0570	.0307	.0061
			$\alpha = .025$,	$u = 1.959964$			
3	.2157	.1555	.1209	.0769	.0439	.0153	−.0118
4	.2236	.1618	.1271	.0840	.0522	.0248	−.0010
5	.2253	.1632	.1290	.0869	.0561	.0296	.0048
6	.2251	.1634	.1297	.0883	.0583	.0325	.0083
8	.2237	.1627	.1298	.0897	.0606	.0357	.0124
12	.2210	.1614	.1294	.0906	.0625	.0385	.0161
24	.2172	.1593	.1285	.0911	.0641	.0410	.0195
∞	.2125	.1568	.1271	.0912	.0653	.0432	.0225
			$\alpha = .05$,	$u = 1.644854$			
3	.2200	.1637	.1320	.0924	.0633	.0382	.0148
4	.2168	.1614	.1310	.0934	.0660	.0426	.0207
5	.2136	.1593	.1298	.0936	.0674	.0449	.0239
6	.2110	.1576	.1288	.0936	.0681	.0463	.0259
8	.2073	.1553	.1274	.0934	.0689	.0479	.0283
12	.2031	.1527	.1258	.0931	.0695	.0493	.0305
24	.1986	.1500	.1240	.0926	.0700	.0506	.0325
∞	.1938	.1470	.1221	.0920	.0703	.0517	.0343
			$\alpha = .1$,	$u = 1.281552$			
3	.1999	.1522	.1261	.0941	.0709	.0512	.0329
4	.1929	.1477	.1232	.0934	.0719	.0535	.0364
5	.1885	.1449	.1214	.0929	.0724	.0548	.0383
6	.1854	.1430	.1202	.0926	.0727	.0557	.0398
8	.1815	.1406	.1187	.0921	.0730	.0566	.0413
12	.1776	.1382	.1171	.0916	.0732	.0575	.0428
24	.1736	.1357	.1155	.0910	.0733	.0583	.0442
∞	.1696	.1332	.1138	.0903	.0734	.0589	.0454

to u, v and $1/f$ used to find A. This linearly-interpolated value of A is denoted by A_1.

(iii) With A set equal to A_1, equation (1) is now solved. The resulting parameter value is labelled with subscript 1. In the examples, it is recorded to about one more decimal place than is reliable.

[The solution would normally stop at this point, but for purposes of illustration we append to each Example in square brackets the refinement that results from a cycle of iteration using quadratic interpolation of A with respect to u and $1/f$. These refined answers are labelled with subscript 2].

In this section we give five illustrations involving tests against alternatives to the right of the hypothesis.

EXAMPLE 1. *Noncentrality parameter.* With $f = 13$ and $\alpha = .06$, what value of δ corresponds to $\beta = .3$?

(i) Since the values of α, β and f are all given in this problem, no preliminary solution of (1^*) is needed in this case.

(ii) From a normal table we note that $u = 1.554774$ corresponds to $\alpha = .06$. Interpolating linearly for A with respect to u and $1/f$ at the given values of u and f, we find $A_1 = .0705$.

(iii) With this value of A, (1) becomes a linear equation in δ, the solution of which is $\delta_1 = 2.18274$.

[It is difficult to imagine needing a more precise value of δ, but for purposes of illustration let us consider the refinement of quadratic interpolation. One gets $A_2 = .07090$, leading to $\delta_2 = 2.182752$. In this problem, since $u^2(u + v)/f^2$ is about .03, four-decimal accuracy for A corresponds to a possible error of .0000015 in δ.]

EXAMPLE 2. *Power.* Consider the test with $\alpha = .07$ and $f = 6$. What is its power at $\delta = 4$?

(i) From a normal table we read out $u = 1.475791$ corresponding to $\alpha = .07$. To find an entry value for v, we solve (1^*) getting $v_0 = 2.1612$.

(ii) Linear interpolation in the table gives $A_1 = .1874$.

(iii) With this value, (1) is a linear equation for v, with solution $v_1 = 2.12044$, corresponding to power $1 - \beta = .98302$.

[If we now enter the table at v_1, and interpolate quadratically in u, we find $A_2 = .18569$. With this value, (1) gives the refined answer $v_2 = 2.120811$, corresponding to $1 - \beta_2 = .983031$. Further cycles of iteration would not change this answer.]

EXAMPLE 3. *Significance level.* With $f = 10$, a t-test is to have power .8 at $\delta = 3.5$. What must its significance level be?

(i) In this problem, (1^*) is a quadratic equation for u, with root $u_0 = 2.2251$.

(ii) Linear interpolation in the table gives $A_1 = .0830$.

(iii) With this value, (1) is a cubic equation for u, which is most easily solved numerically, giving $u_1 = 2.21617$, corresponding to $\alpha_1 = .01334$.

[To refine this solution, one may iterate, using three-point interpolation in the table, getting $A_2 = .0843$ and hence $u_2 = 2.216032$, corresponding to $\alpha_2 = .013345$.]

EXAMPLE 4. *Equal error probabilities.* Since the user of a test must be concerned with both error probabilities, it is often more reasonable to specify control of both simultaneously, rather than to fix α arbitrarily at a conventional value without regard for β. To illustrate how (1) can give relatively easy solution to such problems, let us find the test for which $\alpha = \beta$ when $\delta = 6$ and $f = 12$.

(i) Equation (1^*) is a quadratic in $u = v$, with solution $u_0 = v_0 = 2.5830$.

(ii) This entry point lies outside the tabular ranges of u and v, but A will tolerate the extrapolation. Linear extrapolation gives $A_1 = .2688$.

(iii) With this value, (1) becomes a cubic equation in $u = v$, with solution $u_1 = v_1 = 2.55932$, corresponding to $\alpha_1 = \beta_1 = .005244$.
[In refining this answer, it seems safer to use a three-point formula not only for u but also for v, because of the considerable extrapolation. One gets $A_2 = 2.6528$ and hence $u_2 = v_2 = 2.559626$, or $\alpha_2 = \beta_2 = .0052392$.]

EXAMPLE 5. *Sample Size.* The details of sample size determination depend somewhat on the statistical problem, but the methods can be illustrated on the problem of a single sample of size n. In this case $f = n - 1$ and $\delta = \Delta n^{\frac{1}{2}}$, and the problem is to find the smallest integer n for which given error probabilities α and β are attained at given Δ. Suppose for illustration that $\alpha = .04$, $\beta = .25$ and $\Delta = 1.6$.

(i) Proceeding as usual, we solve (1) curtailed of its last term, finding $n_0 = 4.1$.

(ii) This initial value suggests that $n = 4$ will not quite suffice, to verify which we put $n = 4$, getting $f = 3$. Linear interpolation in the table gives $A_1 = .0712$.

(iii) With this value, (1) can now be solved for $\Delta_1 = 1.67$. As this is larger than the given $\Delta = 1.6$, we see that $n = 4$ is indeed not quite big enough, so that it will be necessary to take $n = 5$.

3. Alternatives to the left and two-sided tests. As mentioned above, formula (1) as it stands applies to t-tests against alternatives to the right. Thus, if the test statistic is $T(\delta)$ (this notation designating a t-random variable with noncentrality parameter δ), (1) applies directly to rejection regions of the form $T(\delta) > t$.

It is of course easy to modify (1) to deal with tests against alternatives to the left. Rejection if $T(\delta) < -t$ is equivalent to rejection if $-T(\delta) > t$, and $-T(\delta)$ has the same distribution as $T(-\delta)$. Thus, we need only change the sign of δ in (1) for it to apply to left-sided tests. For example, the left-sided test with $f = 13$, $\alpha = .06$, and $\beta = .3$ has $\delta_1 = -2.18274$ (Example 1), and the left-sided test with $f = 6$ and $\alpha = .07$ has power $1 - \beta_1 = .98302$ at $\delta = -4$ (Example 2).

Now let us consider a two-sided test, which rejects if either $T(\delta) < -t_L$ or $T(\delta) > t_R$. This rejection region is the union of the regions of the left-sided test $T(\delta) < -t_L$ and of the right-sided test $T(\delta) > t_R$. Let us distinguish the parameters of those two one-sided tests by means of subscripts L and R respectively, using α and β for the error probabilities of the two-sided tests. The total rejection probability or power $1 - \beta$ is the sum of the powers $1 - \beta_L$ and $1 - \beta_R$ of the two one-sided tests, so that $\beta = \beta_L + \beta_R - 1$. As a special case, with

$\delta = 0$ we have $\alpha = \alpha_L + \alpha_R$. Since (1) applies to each one-sided tests, (with the sign of δ changed in the left-sided test) we can write

$$(2) \qquad \begin{aligned} u_L + v_L &= -\delta(1 - u_L^2/4f) - A_L u_L^2(u_L + v_L)/f^2, \\ u_R + v_R &= \delta(1 - u_R^2/4f) - A_R u_R^2(u_R + v_R)/f^2. \end{aligned}$$

These equations apply to all parameter values, but our table covers only certain ranges of values. What two-sided tests does it cover? Clearly we shall require that α_L and α_R each fall within the range of α-values covered by the Table, or roughly that $1.28 < u_L, u_R < 2.58$. To discuss the coverage of β, let us note from (2) that, to a first approximation, $u_L + v_L = -\delta$ and $u_R + v_R = \delta$. Thus, v_L is approximately $-(u_L + u_R + v_R)$. Therefore if v_R is within tabular range, i.e., $0 < v_R < 2.33$, v_L will not be: in fact v_L will be less than about -2.56, and usually much less. Hence the value of A_L is not available from the table. Fortunately, for such large negative values of v_L the value of β_L will be quite near 1, so that β is nearly equal to β_R. Furthermore, because the normal density at v_L is so small in such cases, a rather crude value of v_L will give β_L to good accuracy. Therefore, in the first equation of (2), one can get along reasonably well without the remainder term. Similarly, when v_L is within tabular range, we shall have β nearly equal to β_L and can omit the A_R-term in (2). To summarize, the table will deal adequately with those two-sided tests for which α_L, α_R and β fall within tabular range.

It is traditional to consider among two-sided tests only those that are symmetrical, in the sense that $t_L = t_R$ and $\alpha_L = \alpha_R$. We shall begin with symmetrical examples, but then in Example 8 show that our method permits us to deal with the more interesting asymmetrical tests as well.

EXAMPLE 6. *Power.* Find the power at $\delta = 4$ of the (symmetrical) two-sided test with $\alpha = .14$ and $f = 6$.

By symmetry, $\alpha_L = \alpha_R = .07$. The calculation of β_R is identical with the calculation of β in Example 2, leading to $\beta_{R1} = .01698$. From (2) it appears that v_L is about -5.1, and inspection of a normal table shows β_L equals 1 to six decimal places. Thus the power of this test agrees with that of the one-sided test of Example 2, or $1 - \beta_1 = .98302$ at $\delta = 4$ (and of course also at $\delta = -4$). [Refinement of the solution proceeds as in Example 2.]

EXAMPLE 7. *Noncentrality parameter.* Find δ for the (symmetrical) two-sided test with $\alpha = .2$, $\beta = .5$ and $f = 9$. (These extreme values of α and β are chosen to illustrate the situation that is most difficult for our method.)

By symmetry $\alpha_L = \alpha_R = .1$, and the power is the same at δ and at $-\delta$. Without loss of generality we shall seek the positive solution, for which β_L is near 1 and hence β_R is near .5, so that v_R is near 0.

(i) If we omit the remainder terms and add the equations (2), we get $v_L + v_R = -2.563104$. Solving this in conjunction with $\beta_L + \beta_R = 1.5$ gives the entry value $v_{R0} = -.0135$.

(ii) Linear extrapolation in v and linear (harmonic) interpolation in f gives $A_{R1} = .0410$.

(iii) With this value, and pretending that $A_L = 0$ as explained above, we re-solve (2) to find $\delta_1 = 1.32980$. Of course, $-\delta_1$ is also a solution.
[To refine the solution we iterate, using quadratic interpolation in $1/f$, to find $A_{R2} = .0410$, unchanged. We may also refine the value $A_L = 0$, using the series (5) below to get the approximation $A_L = -.0908 - .0183 = -.1091$. With these refinements, (2) gives $\delta = 1.32991$.]

EXAMPLE 8. *Asymmetrical test.* It is often said that a one-sided test should be used if only the alternatives on one side of the hypothesis are possible or of interest, while the (symmetric) two-sided test should be used if alternatives on both sides must be considered. The latter test pays equal attention to each side, but in practice one may well be interested in both sides although not equally so. In such cases the reasonable choice is an *asymmetrical* two-sided test.

To illustrate the design of such a test, let us suppose that $f = 11$ and $\alpha = .05$, and that we desire $\beta = .15$ and $\beta = .05$ at equally-distant alternatives to the left and right respectively, say at $-\delta$ and at δ. We seek left-sided and right-sided tests such that $\alpha_L + \alpha_R = .05$. Since the region on the left will make a negligible contribution to the power at δ, we may replace the condition $\beta = .05$ at δ by the condition $\beta_R = .05$ at δ. Similarly, the condition $\beta = .15$ at $-\delta$ may be replaced by $\beta_L = .15$ at δ. We therefore must solve simultaneously the three equations

$$\alpha_L = \alpha_R = .05,$$
$$u_L + 1.036433 = \delta(1 - u_L^2/44) - u_L^2(u_L + 1.036433)A_L/121,$$
$$u_R + 1.644854 = \delta(1 - u_R^2/44) - u_R^2(u_R + 1.644854)A_R/121.$$

(i) Proceeding as usual, we first ignore the remainder terms. Using a normal table to translate the first condition into terms of u_L and u_R, it is easy to solve numerically for $u_{L0} = 2.23710$, corresponding to $\alpha_{L0} = .01264$, and $u_{R0} = 1.78218$, corresponding to $\alpha_{R0} = .03736$.

(ii) Entering the table at these values, and interpolating linearly, one finds $A_{L0} = .1031$ and $A_{R0} = .1569$.

(iii) With these values we re-solve the equations, now finding $\alpha_{L1} = .012648$, etc.
[The refinement of quadratic interpolation gives $\alpha_{L2} = .0126492$].

4. An asymptotic expansion for A. Asymptotic expansion of β in inverse powers of f has been a key method for dealing with t-power since the pioneering paper of Johnson and Welch in 1940. There is a tabulation in [1] of the coefficients of the series for β expressed in terms of the critical value of the t-test and of the expectation of chi. It is a straightforward matter to rework that series to obtain the expression

$$(3)\quad u + v - \delta(1 - u^2/4f) = \delta[(-6u^2 + 5u^4)/96f^2 + (4u^2 + 6u^4 - u^6)/128f^3] + \delta^2[-u^3/24f^2 + (-2u^3 + u^5)/48f^3] + O(1/f^4).$$

From this expression it may readily be seen that

$$(4)\qquad \delta = u + v + u^2(u + v)/4f + O(1/f^2).$$

When this value for δ is substituted into the right-hand side of (3), one sees that the two terms have the common factor $-u^2(u + v)/f^2$, which makes it natural to define A as we have done in (1). Furthermore, the substitution gives the two initial terms of an asymptotic expansion for A as so defined:

(5) $$A = (4uv - u^2 + 6)/96 + (8uv - u^4 + 2u^2 - 6)/192f + O(1/f^2)$$

We have used this expression in various ways. The first term was used to compute the tabular values at $f = \infty$. Using both terms, one can get an approximation for larger f for values of A outside the tabular range, as was done in Example 7. The expression was used to help with the interpolation of A for intermediate values of f, as explained in Section 6. Finally, it serves to guide our discussion of interpolation in the next Section.

5. Interpolation. One of the most attractive features of A is its near-linearity as a function of v. Since both the constant term and the $1/f$ term of the expansion (5) are linear in v, it appears that the error of linear interpolation of A with respect to v will be of order $1/f^2$. A numerical investigation of the table shows that in fact the error does not exceed about $.02/f^2$. Unless one has both a very small f and a need for high precision, one may safely use linear interpolation with respect to v. (Quadratic interpolation gives four-decimal accuracy in all cases, aside from the inevitable rounding errors.)

The interpolation of A with respect to u is not quite so attractive. We see from (5) that the constant term is a quadratic function of u, suggesting the use of three-point interpolation even for very large f. Furthermore since the $1/f$ term is a quartic function of u, one might expect to need five points unless f is fairly large. Fortunately, the nonquadratic component of the $1/f$ term, $u^4/192f$, is so small that the error of its quadratic interpolation is negligible, and as a practical matter the maximum error of three-point interpolation of A with respect to u is about $.01/f^2$. Therefore, three-point interpolation may be used except when f is very small and maximum precision is sought.

Even linear interpolation of A with respect to u is reasonably good, thanks to the smallness of the nonlinear terms of (5). Roughly speaking, for f above about 5, the error of linear interpolation is bounded by $.0004 + .005/f$. Linear interpolation is good to two decimal places even at $f = 3$.

The fact that A has an expansion in powers of $1/f$ suggests that A be interpolated harmonically with respect to f. Accordingly, we have chosen for the table the f values 6, 8, 12, 24, ∞, giving unit spacing on the scale $24/f$. It turns out that three-point (harmonic) interpolation may be relied on in all cases, and even linear (harmonic) interpolation will lead to an error less than .0007.

In judging the significance of these errors, one must remember that A is the coefficient of a remainder term that is usually itself small. An error of order $1/f^2$ in A will appear as an error of order $1/f^4$ in u, v or δ. An error in the third decimal place of A is likely to be reflected in α or β by an error in the fourth or fifth decimal place. As the eight Examples show, even linear interpolation in the Table of A gives good results, even for small values of f.

The smoothness of A is made more attractive by comparison with other quantities that might be used for a one-page table. For example, if one carries (4) one step farther, one finds that it contains the term $5u^5/96f^2$. This quantity is not negligible if one seeks reasonably accurate values of δ, and its interpolation with respect to u requires a six-point formula if the u-interval is wide.

6. Calculation of the Table. The Table of A was compiled from a variety of sources. At $\alpha = .05$, for example, the values of A for very small f were derived from D. B. Owen's excellent five-decimal table of δ [5], and we should like to express our thanks to Professor Owen for helpful correspondence. As f increases, the error in A derived from this table increases like f^2, and this source is not adequate above $f = 6$, where the possible error in A approaches .0001. Starting at the other end, the values at $f = \infty$ were calculated from (5). The values at $f = 24$ and $f = 12$ were readily obtained to the necessary accuracy by the series method explained in [1]. Finally, the values at $f = 8$ were interpolated, use being made of both terms of (5) as well as the values previously computed. The procedure was similar at the other values of α, except that at $\alpha = .1$ the table in Section 6 of [4] was used for very small f, and except that at $\alpha = .005$ and $\alpha = .01$, the interpolation method was used for $f = 12$ instead of $f = 8$. As indicated, the fifth decimal place for intermediate f is not reliable, and no doubt some of these entries may be misrounded. However, when 70 of the least reliable values were checked by high-order interpolation based on Table B-9 of [3], we were led to change the rounding of only one entry.

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