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A NOTE ON THE WEAK LAW[1]

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Let $S_n = \sum_{k=1}^n X_k$, where X_k, $k \geqq 1$, are independent and identically distributed random variables with characteristic function φ. The object of this note is to prove the following generalization of weak law of large numbers.

Theorem. *Let $\alpha \in (\frac{1}{2}, \infty)$. Then, the following are equivalent.*

(i) *There exist constants a_n such that $S_n/n^\alpha - a_n \rightarrow_P 0$.*

(ii) *$|\log|\varphi(t)||^\alpha$ is differentiable at $t = 0$.*

(iii) $\lim_{t\to\infty} tP\{|X_1| > t^\alpha\} = 0$.

Proof. Note first that any of the conditions (i), (ii), (iii) holds for the sequence $\{X_n\}$ if and only if it holds for the symmetrized sequence $\{\hat{X}_n\}$. This is obvious for (ii). For (i) and (iii) it is an easy consequence of the symmetrization inequalities [1], p. 245. Also, if X_1 is symmetric we can take the centering constants a_n to be zero in (i). Therefore, we shall assume hereafter that X_1 is symmetric and φ is real.

First, we establish the equivalence of (i) and (ii). The logarithm of φ, as usual, is defined in a neighborhood of $t = 0$. We have $S_n/n^\alpha \rightarrow_P 0 \Leftrightarrow \varphi^n(t/n^\alpha) \rightarrow 1$, all $t \Leftrightarrow n \log \varphi(t/n^\alpha) \rightarrow 0$, all $t \Leftrightarrow \log \varphi(t/n^\alpha)/|t|^{1/\alpha}/n \rightarrow 0$, all $t \neq 0 \Leftrightarrow |\log \varphi(t/n^\alpha)|^\alpha/|t|/n^\alpha \rightarrow 0$, all $t \neq 0 \Leftrightarrow |\log \varphi(t)|^\alpha$ is differentiable at $t = 0$. For the last equivalence make use of the fact that the convergence is uniform on bounded intervals.

It remains to show that (i) and (iii) are equivalent. We shall essentially follow [2], p. 232. To show (iii) implies (i), let τ_n be the truncation at $\pm n^\alpha$. Define $T_n = \sum_{j=1}^n \tau_n(X_j)$. Then (iii) implies that $\lim_{n\to\infty} P(S_n \neq T_n) = 0$, and consequently it is enough to show $T_n/n^\alpha \rightarrow_P 0$. We have

$$P\{|T_n| > \epsilon n^\alpha\} \leqq E|T_n|^2 \epsilon^{-2} n^{-(2\alpha)} = \epsilon^{-2} n^{-(2\alpha-1)} \int_0^n t^{2\alpha}\, dG(t)$$
$$\leqq 2\alpha \epsilon^{-2} n^{-(2\alpha-1)} \int_0^n t^{2\alpha-1} [1 - G(t)]\, dt,$$

where $G(t) = P\{|X_1|^{1/\alpha} \leqq t\}$. By (iii) $\lim_{t\to\infty} t[1 - G(t)] = 0$, and this is easily seen to imply that the right hand expression has limit zero for each $\epsilon > 0$. (i) implies (iii) because

$$P\{|S_n| > n^\alpha\} \geqq \tfrac{1}{2} P\{\max_{1\leqq j\leqq n} |S_j| > n^\alpha\}$$
$$= \tfrac{1}{2}(1 - P\{\max_{1\leqq j\leqq n} |S_j| \leqq n^\alpha\})$$
$$\geqq \tfrac{1}{2}(1 - P\{\max_{1\leqq j\leqq n} |X_j| \leqq 2n^\alpha\})$$
$$\geqq \tfrac{1}{2}(1 - \exp(-n P\{|X_1| > 2n^\alpha\})).$$

This completes the proof of the theorem.

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It should be noted that (i) and (ii) are equivalent for all $\alpha > 0$. However, if $0 < \alpha \leqq \frac{1}{2}$ (i) (and consequently (ii)) can be true if and only if X_1 is degenerate. The centering constants a_n in (i) can always be taken to be medians of S_n/n^α, or, as is easy to see from the proof of (iii) $\Rightarrow$ (i) that they can be selected to be $a_n = n^{1-\alpha} \int_{-n^\alpha}^{n^\alpha} t\, dF(t)$, where F is the df of X_1.

Another consequence of the equivalence of (i) and (iii) is the following. For $\frac{1}{2} < \alpha < \infty$, S_n/n^α converges, on centering, in probability to zero if and only if $n^{-1} \sum_{j=1}^{n} |X_j|^{1/\alpha}$ converges, on centering, in probability to zero.

REFERENCES

[1] LOÈVE, MICHEL (1963). *Probability Theory* (3rd Edition). Van Nostrand, Princeton.
[2] FELLER, WILLIAM (1966). *An Introduction to Probability Theory and Its Applications*, **2**. Wiley, New York.