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AN EXTENSION OF PAULSON'S SELECTION PROCEDURE

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1. Introduction. Suppose that $\pi_1, \pi_2, \cdots, \pi_k$ are k populations in which we may observe the independent random variables $X_1, X_2, \cdots, X_k$, respectively. We assume that the distribution of X_i is a member of the one-dimensional Koopman-Darmois family with density $\exp\{P(x)Q(\theta_i) + R(x) + S(\theta_i)\}$, $i = 1, 2, \cdots, k$. Let $\tau = Q(\theta)$ and suppose that the ordered set of the τ-values of $\pi_1, \pi_2, \cdots, \pi_k$ are denoted by $\tau_{[1]} \leqq \tau_{[2]} \leqq \cdots \leqq \tau_{[k]}$. These τ-values are assumed to be unknown, and if $\tau_{[k]} > \tau_{[k-1]}$, we refer to the population associated with $\tau_{[k]}$ as the best population. In this paper we obtain a sequential procedure which guarantees that the probability of selecting the best population is at least a specified amount whenever $\tau_{[k]}$ exceeds $\tau_{[k-1]}$ by a specified quantity. This procedure is an extension of Paulson's [4] procedure for selecting the normal population with the greatest mean. The major difference between the two procedures is that Paulson's is truncated while the one obtained here is not.

An exhaustive discussion of the different aspects of the problem considered in this paper will be found in the monograph by Bechhofer, Kiefer and Sobel [1], who have given a different sequential procedure which guarantees the same probability requirements. The notations used in the present paper largely follow those used in the monograph. The procedure obtained in this paper solves the ranking problem when the measure of distance between two populations π_i and π_j is defined to be $|\tau_i - \tau_j|$. However in some applications this measure may not be appropriate.

In Section 2 we describe briefly the ranking problem and the proposed rule. A proof is given for the fact that the procedure guarantees the stated probability requirements. This procedure can be easily extended to the ranking of k stochastic processes belonging to the Koopman-Darmois family. The procedure so obtained will tend to be on the conservative side in the sense of "overprotection" due to inequalities on the probability of correct selection which are used in its construction. Certain modification of the procedure for values of $k \leqq 5$ so as to reduce the amount of overprotection (and consequently, the average sample size) are obtained in Section 3. In Section 4 the procedure is applied to the problem of selecting the Poisson process with the largest parameter when $k = 3$ and the *exact* probability of correct selection is obtained using relaxation techniques.

2. Formulation of the selection problem.

2.1. *Koopman-Darmois Populations and Some of Their Relevant Properties.* A univariate population is said to belong to the Koopman-Darmois family if

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its frequency function $f(x;\theta)$ can be written in the form

$$f(x;\theta) = \exp\{P(x)Q(\theta) + R(x) + S(\theta)\}, \tag{1}$$

where $f(x;\theta)$ represents the probability density at x for continuous variables or the probability of obtaining the observed value x for discrete variables and θ is the unknown real parameter on which the frequency function depends. Here $P(x)$ and $R(x)$ do not involve θ, while $Q(\theta)$ and $S(\theta)$ do not depend on x. We shall define $Q(\theta)$ to be the Koopman-Darmois parameter, and all of our statements regarding the ranking procedure will be made in terms of this parameter. The function $Q(\cdot)$ is assumed to be a continuous, strictly increasing function of θ. A detailed characterization of these populations and the usual regularity assumptions on the functions $P(\cdot)$, $Q(\cdot)$, $R(\cdot)$ and $S(\cdot)$ are contained in Bechhofer, Kiefer and Sobel [1]. We now state a result pertaining to the Koopman-Darmois family of distributions which we shall need in the sequel.

Let X_1 and X_2 be two independent random variables distributed according to the frequency functions belonging to the *same* univariate Koopman-Darmois family given by (1), and let their frequency functions be denoted by $f(X_1;\theta_1)$ and $f(X_2;\theta_2)$ respectively. Let $Z = P(X_1) - P(X_2)$, and $Z_1, Z_2, \cdots$ be a sequence of independently distributed random variables having the same distribution as that of Z. If $\theta_1 < \theta_2$, then

$$\Pr\{\sup_n \textstyle\sum_{i=1}^n Z_i \geqq a\} \leqq e^{-[Q(\theta_2)-Q(\theta_1)]a}, \tag{2}$$

where a is positive constant.

A detailed proof of this result is given in [1]. Briefly, since $E(Z) < 0$, we have that for all positive a,

$$P[\sup_n \textstyle\sum_{i=1}^n Z_i \geqq a] \leqq e^{-t_0 a},$$

where t_0 is the non-zero root of the equation $E(e^{tZ}) = 1$. Also

$$\begin{aligned} E(e^{tZ}) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t[P(x_1)-P(x_2)]} f(x_1;\theta_1)f(x_2;\theta_2)\,dx_1\,dx_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\{P(x_2)[Q(\theta_2)-t] + P(x_1)[Q(\theta_1)+t] \\ &\qquad + S(\theta_1) + S(\theta_2) + R(x_1) + R(x_2)\}\,dx_1\,dx_2 \end{aligned}$$

and if we set $t = Q(\theta_2) - Q(\theta_1)$ we see that $E(e^{tZ}) = 1$. This result was first given by Girshick [3].

2.2. *Formulation of the ranking problem.* Let $\pi_1, \pi_2, \cdots, \pi_k$ be k populations belonging to the same Koopman-Darmois family given by (1) and let the θ-values of these frequency functions be denoted by $\theta_1, \theta_2, \cdots, \theta_k$. Let the ranked values of these parameters be $\theta_{[1]} \leqq \theta_{[2]} \leqq \cdots \leqq \theta_{[k]}$. Let $\tau = Q(\theta)$ and the corresponding τ-values be denoted by $\tau_{[1]} \leqq \tau_{[2]} \leqq \cdots \leqq \tau_{[k]}$. Since $Q(\cdot)$ has been assumed to be strictly increasing in θ, we shall have $Q(\theta_{[i]}) = \tau_{[i]}$, $i = 1, 2, \cdots, k$. Our problem is to obtain a procedure which will select the population associated with $\theta_{[k]}$ (and therefore $\tau_{[k]}$) such that

$$\Pr\{\text{correct selection of the ``best'' population}\} \geqq P^*, \tag{3}$$

whenever

$$\tau_{[k]} - \tau_{[k-1]} \geqq \delta^* > 0. \tag{4}$$

Here P^* is a preassigned positive constant lying between $1/k$ and 1, and δ^* is another specified positive constant. These constants are supplied to the statistician by the experimenter and will depend on his requirements of accuracy. The above formulation of the ranking problem is due to Bechhofer, Kiefer and Sobel [1].

2.3. *The proposed selection procedure.* Let X_{ij} denote the jth observation from the ith population ($i = 1, 2, \cdots, k; j = 1, 2, \cdots$.). It is assumed that the X_{ij}'s are mutually independently distributed. Let $Y_{im} = \sum_{j=1}^{m} P(X_{ij})$, $i = 1, 2, \cdots, k$; $m = 1, 2, \cdots$. According to the procedure we start by taking one observation from each population $\pi_1, \pi_2, \cdots, \pi_k$ and compute Y_{i1} for each i. If for any i, we find that

$$Y_{i1} \leqq \max_s Y_{s1} - (\delta^*)^{-1} \ln (k - 1)(1 - P^*)^{-1}, \tag{5}$$

we eliminate π_i from further consideration and take observations at the next stage on the remaining populations only. We carry on this procedure computing at each stage the sums Y_{im} on the populations not yet eliminated, and removing those populations π_j from further consideration for which

$$Y_{jm} \leqq \max_s Y_{sm} - (\delta^*)^{-1} \ln (k - 1)(1 - P^*)^{-1}. \tag{6}$$

We shall terminate the procedure at the stage when only one population has not been eliminated and select it as the best. We now prove that the above procedure guarantees (3).

For simplicity of notation, we relabel the populations so that π_i is associated with the value $\tau_{[i]}$. With this convention the ranked values of τ_i's can be denoted by $\tau_1 \leqq \tau_2 \leqq \cdots \leqq \tau_k$. Let Δ denote the condition $\tau_k \geqq \tau_j + \delta^*, j = 1, 2, \cdots, k - 1$. We then have

$$\begin{aligned} &\Pr\{\text{incorrect selection} \mid \Delta\} \\ &\quad = \Pr\{\pi_k \text{ is eliminated} \mid \Delta\} \\ &\quad \leqq \Pr\{\text{for at least one value of } i (i = 1, 2, \cdots, k-1),\ Y_{km} \leqq Y_{im} \\ &\qquad - (\delta^*)^{-1} \ln (k-1)(1 - P^*)^{-1} \text{ for some value of } m < \infty \mid \Delta\} \\ &\quad \leqq \textstyle\sum_{i=1}^{k-1} \Pr\{\sup_m (Y_{im} - Y_{km}) \geqq (\delta^*)^{-1} \ln (k-1)(1 - P^*)^{-1} \mid \Delta\}. \end{aligned} \tag{7}$$

Therefore using (2) we have

$$\begin{aligned} &\Pr\{\text{incorrect selection} \mid \Delta\} \\ &\quad \leqq \textstyle\sum_{i=1}^{k-1} \exp\{-(\tau_k - \tau_i)(\delta^*)^{-1} \ln (k-1)(1 - P^*)^{-1}\} \\ &\quad \leqq (k-1) \exp\{-\ln (k-1)(1 - P^*)^{-1}\} \\ &\quad \leqq 1 - P^*, \end{aligned} \tag{8}$$

and thus the assertion is proved. We observe that using this procedure no upper bound can be given to the maximum number of observations which will be taken before the procedure terminates. (Bechhofer, Kiefer and Sobel [1] have shown that in general no procedure using a finite number of stages can guarantee (3) for all the members of the Koopman-Darmois family when the measure of distance between two populations π_i and π_j is taken to be $|\tau_i - \tau_j|$). It can be shown, however, that the above procedure terminates with probability one. Finally we should remark that the procedure extends with obvious modifications to the case of ranking Koopman-Darmois processes as well.

3. Certain simple modifications of the selection procedure when $k = 2, 3, 4$ or 5. From the preceding section we observe that the above procedure guarantees the probability requirement (3) but because of the number of inequalities used in its construction the exact probability of correct selection will always exceed the nominal probability requirement. The amount of overprotection for low values of P^* may be considerable. It is possible to achieve slightly better approximations to the probability of correct selection than is given in (8) by strengthening one of the inequalities given there for $k \leqq 5$. This improvement in turn leads to certain modifications of the procedure which result in some reduction in the amount of overprotection. We describe below the modifications which are essentially based on the following inequality.

Let $\{X_i\}$, $\{Y_i\}$, $\{Z_i\}$ be three independent sequences of random variables such that $\{X_i\}$ and $\{Y_i\}$ have the same distribution. Then

$$\Pr\{\sup_i(X_i - Z_i) > c \text{ and } \sup_i(Y_i - Z_i) > c\} \geqq \Pr^2\{\sup_i(X_i - Z_i) > c\}. \tag{9}$$

In order to prove (9) it is enough to note that for all M

$$\begin{aligned}
&\Pr\{\max_{0<i\leqq M}(X_i - Z_i) > c \text{ and } \max_{0<i\leqq M}(Y_i - Z_i) > c\} \\
&\quad = E[\Pr\{\max_{0<i\leqq M}(X_i - Z_i) > c \text{ and } \max_{0<i\leqq M}(Y_i - Z_i) > c \mid \{Z_i\}\}] \\
&\quad = E[\Pr^2\{\max_{0<i\leqq M}(X_i - Z_i) > c \mid \{Z_i\}\}] \\
&\quad \geqq E^2[\Pr\{\max_{0<i\leqq M}(X_i - Z_i) > c \mid \{Z_i\}\}] \\
&\quad = \Pr^2\{\max_{0<i\leqq M}(X_i - Z_i) > c\}.
\end{aligned}$$

Now replacing $(\delta^*)^{-1} \ln (k - 1)(1 - P^*)^{-1}$ by c in (7), we obtain

$$\Pr\{\text{incorrect selection} \mid \Delta\} \leqq \Pr\{\text{for at least one value of } i = 1, 2, \cdots, k-1, Y_{km} \leqq Y_{im} - c \mid \Delta\}. \tag{10}$$

Denoting the event $\{\sup_m(Y_{im} - Y_{km}) \geqq c \mid \Delta\}$ by A_i and applying Bonferroni's inequality to the right hand side of (10), we have

$$\Pr\{\text{incorrect selection} \mid \Delta\} \leqq \textstyle\sum_{i=1}^{k-1} \Pr(A_i) - \sum_{i<j} \Pr(A_iA_j) + \sum_{i<j<l} \Pr(A_iA_jA_l). \tag{11}$$

In Section 2, we considered only the first term in the right hand side of (11). Here we shall obtain some improvement by considering the additional terms. First considering the case $k = 3$, we have from (9) and (11) that

(12) $\Pr\{\text{incorrect selection} \mid \Delta\}$

$$\leqq \sum_{i=1}^{2} \Pr(A_i) - \Pr(A_1A_2) \leqq \sum_{i=1}^{2} \Pr(A_i) - \Pr^2(A_i).$$

Now from (2) and (7), we have

(13) $$\Pr(A_i) \leqq e^{-\delta^* c},$$

and since $2\Pr(A_i) - \Pr^2(A_i)$ is a strictly increasing function of $\Pr(A_i)$ it follows that

(14) $$\Pr\{\text{incorrect selection}\} \leqq 2e^{-\delta^* c} - e^{-2\delta^* c}.$$

Therefore by taking

(15) $$c = -(\delta^*)^{-1} \ln (1 - (P^*)^{\frac{1}{2}}),$$

we guarantee a probability of correct selection of at least P^*.

By applying the further inequality $\Pr(A_iA_jA_k) \leqq \Pr(A_iA_j)$, for $k = 4$ and 5 we obtain similar results, which are shown in Table 1. This method does not lead to an improvement when $k > 5$.

The entry against the modified procedure when $k = 2$ was not derived by us; for this case elimination of one population is equivalent to making a terminal decision, and the Bechhofer-Kiefer-Sobel procedure [1] when applied to this case leads to a stopping boundary $(1/\delta^*) \ln (P^*/(1 - P^*))$.

4. Determination of the exact operating characteristics of the selection procedure in a special case. It is possible in some special cases of the Koopman-Darmois family to derive the exact operating characteristics of the procedure when $k = 2$; see Bechhofer, Kiefer and Sobel [1] for these computations. When $k \geqq 3$ the problem becomes much more complicated because it then typically involves computation of multi-dimensional absorption probabilities. One way out is to perform Monte-Carlo computations. This has been done to some considerable extent by Ramberg [5] for Paulson's procedure. In this section we give a numerical method for the determination of the exact values of the

TABLE 1

k	Value of c for which the ith population will be eliminated when $Y_{im} \leqq \max_s Y_{sm} - c$	
	Original Procedure	Modified Procedure
2	$(\delta^*)^{-1} \ln (1-P^*)^{-1}$	$(\delta^*)^{-1} \ln P^*(1-P^*)^{-1}$
3	$(\delta^*)^{-1} \ln 2(1-P^*)^{-1}$	$-(\delta^*)^{-1} \ln (1-(P^*)^{\frac{1}{2}})$
4	$(\delta^*)^{-1} \ln 3(1-P^*)^{-1}$	$-(\delta^*)^{-1} \ln (\frac{3}{4} - \frac{1}{4}(1 + 8P^*)^{\frac{1}{2}})$
5	$(\delta^*)^{-1} \ln 4(1-P^*)^{-1}$	$-(\delta^*)^{-1} \ln (1 - (\frac{1}{2}(1 + P^*))^{\frac{1}{2}})$

operating characteristics of the procedure when $k = 3$ and the three populations are Poisson processes. The computation for these processes appears to be reasonable because at any moment of time a jump can take place for one process only.

The Poisson process $X(t)$ is given by

$$\Pr\{X(t) = x\} = e^{-\theta t}(\theta t)^x/x!, \qquad x = 0, 1, 2, \cdots, \tag{16}$$
$$= \exp(x \ln \theta t - \theta t - \ln x!).$$

We now define $\tau = Q(\theta) = \ln \theta$. Suppose that we have three Poisson processes $X_1(t)$, $X_2(t)$ and $X_3(t)$ in the least favorable configuration and let their parameters be θ, θ and θe^{δ^*}, respectively ($\delta^* > 0$). We then have

$$\tau_3 - \tau_1 = \delta^*. \tag{17}$$

Let

$$Y_i(t) = X_i(t) - X_3(t), \qquad i = 1, 2, \tag{18}$$

$$\delta = e^{\delta^*}, \tag{19}$$

and

$$p = 1/(2 + \delta). \tag{20}$$

Then each jump of the stochastic vector $(Y_1(t), Y_2(t))$ takes place in accordance with the following probability distribution:

(21)

Magnitude of Jump in $Y_1(t)$	$Y_2(t)$	*Probability*
1	0	p
0	1	p
-1	-1	δp

Now let

$$a = (\delta^*)^{-1} \ln (k-1)(1-P^*)^{-1} \text{ if } (\delta^*)^{-1} \ln (k-1)(1-P^*)^{-1} \text{ is an integer} \tag{22}$$
$$= [(\delta^*)^{-1} \ln (k-1)(1-P^*)^{-1}] + 1 \quad \text{otherwise.}$$

Representing $(Y_1(t), Y_2(t))$ as a random walk in the two-dimensional plane, it is easy to see that an incorrect decision is made when $Y_1(t) = a$ or $Y_2(t) = a$. A qualitative description of the random walk goes as follows:

"A two-dimensional random walk W starts at the origin (0, 0); in any one transition, it can move one step to the right with probability p, one step vertically upward with probability p and one step back and diagonally downwards with probability δp. When the process hits either of the lines $Y_1(t) = a$, $Y_2(t) = a$ it is absorbed. If W hits one of the lines $Y_1(t) = -a$ or $Y_1(t) - Y_2(t) = -a$, it can move thereafter one step up and one step down (on or parallel to the line

$Y_1(t) = -a$), with probabilities $1/(1 + \delta)$ and $\delta/(1 + \delta)$, respectively. Moreover if W hits one of the lines $Y_2(t) = -a$ or $Y_1(t) - Y_2(t) = a$, it can move thereafter one step to the right and one step to the left (on or parallel to the line $Y_2(t) = -a$), with probabilities $1/(1 + \delta)$ and $\delta/(1 + \delta)$, respectively."

It can be seen that the absorption of this random walk at the line $Y_1(t) = a$ or $Y_2(t) = a$ before the occurrence of any of the following events gives rise to an incorrect selection in the corresponding ranking problem:

E_1 : W hits the line $Y_1(t) = -a$ and then the point $(-a, -a)$;

E_2 : W hits the line $Y_2(t) = -a$ and then the point $(-a, -a)$;

E_3 : W hits the line $Y_1(t) - Y_2(t) = -a$ and then the line $Y_2(t) = -a$;

E_4 : W hits the line $Y_1(t) - Y_2(t) = a$ and then the line $Y_1(t) = -a$.

Let $P(x, y)$ denote the probability of absorption at $Y_1(t) = a$ or $Y_2(t) = a$ before E_1, E_2, E_3, or E_4 occur when the random walk starts from the point (x, y). ($P(0, 0)$ then gives the desired probability of incorrect selection.) It is clear that $P(x, y)$ satisfies the following difference equation, (see Feller [2], p. 314):

$$(23) \quad P(x, y) = pP(x + 1, y) + pP(x, y + 1) + \delta pP(x - 1, y - 1)$$

with boundary conditions:

$$(24) \qquad P(a, y) = 1, \qquad y = 0, \cdots, a,$$

TABLE 2

Comparison of Exact Probability of Correct Selection in the Least Favorable Configuration With the Nominal Probability for Selecting the Best of Three Poisson Processes

$\delta^* = 1$		
a	Nominal value of P^* corresponding to a	Exact Probability of correct selection in the least favorable configuration
1	.400	.576
2	.729	.794
3	.900	.913
4	.963	.966
6	.995	.995

$\delta^* = 2$		
a	Nominal value of P^* corresponding to a	Exact Probability of correct selection in the least favorable configuration
9	.669	.767
11	.778	.831
15	.900	.915
19	.955	.960

$$(25) \qquad P(x, a) = 1, \qquad x = 0, \cdots, a,$$

$$(26) \qquad P(-a, y) = h(y), \qquad y = -a, -a + 1, \cdots, 0,$$

$$(27) \qquad P(y - a, y) = h(y), \qquad y = 1, \cdots, a,$$

$$(28) \qquad P(x, -a) = h(x), \qquad x = -a, -a + 1, \cdots, 0; \quad \text{and}$$

$$(29) \qquad P(x, x - a) = h(x), \qquad x = 1, \cdots, a,$$

where $h(t) = (\delta^{-2a} - \delta^{t-a})/(\delta^{-2a} - 1)$.

It is difficult to obtain analytical expressions for the solution of the difference equation, but a numerical solution for particular values of δ^* and P^* using standard relaxation methods can be easily obtained. This technique consists of initially setting $P(x, y) = 0$ and then applying (23) to obtain a new set of values for each $P(x, y)$ using (24)–(29). Successive iterations are then performed until a convergence is obtained for the successive values of $P(0, 0)$. It can be easily shown that the solution to the above difference equation is unique and this method will provide a convergence to the unique solution.

In Table 2 we compare the nominal value of P^* with the exact probability of correct selection for several choices of a and $\delta^* = .2$ and 1. The table also shows the amount of overprotection resulting from use of the procedure.

The expected number of transitions for arriving at a terminal decision as well as the probability of correct selection in other configurations can be obtained in a similar way. Also for any particular value of θ the expected time to termination can be easily calculated.

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