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ROBUSTNESS OF SOME NONPARAMETRIC PROCEDURES IN LINEAR MODELS[1]

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1. Introduction and summary. For the random variables X_{ij} ($i = 1, \cdots, N$; $j = 1, \cdots, r$) consider the linear model

$$(1.1) \qquad X_{ij} = \mu + \beta_i + \tau_j + Y_{ij} \qquad (\sum \beta_i = 0, \qquad \sum \tau_j = 0),$$

where the τ's are treatment effects, the β's are nuisance parameters (block effects), and the Y_{ij}'s are error components. Nonparametric procedures for estimating and testing contrasts in the τ's, based on the Wilcoxon signed rank statistics, are due to Lehmann (1964), Hollander (1967) and Doksum (1967), among others. These rest on the assumption that the Y_{ij}'s are independent with a common continuous distribution. Since these procedures are actually based on the paired differences X^*_{ijk}, defined by (2.1), they are unaffected by the addition of a random variable V_i to β_i (or to Y_{ij}) for $i = 1, \cdots, N$.

The object of the present investigation is to show that these procedures are valid even if $Y_{i1}, \cdots, Y_{ir}$ are interchangeable random variables, for each $i(=1, \cdots, N)$. It may be noted that if in (1.1) the superimposed random variable V_i is absorbed in Y_{ij}, then of course $Y_{i1}, \cdots, Y_{ir}$ are interchangeable, but the interchangeability of $Y_{i1}, \cdots, Y_{ir}$ does not necessarily imply that $Y_{ij} = W_{ij} + V_i$, where W_{ij}'s are independent and identically distributed random variables (iidrv). In fact, in 'mixed model' experiments, interchangeability of $Y_{i1}, \cdots, Y_{ir}$ (of quite arbitrary nature) may arise when there is no block versus treatment interaction [cf. Koch and Sen (1968) for details]. It is also shown that the procedures mentioned above are robust against possible heterogeneity of the distributions of the error vectors $\mathbf{Y}_i = (Y_{i1}, \cdots, Y_{ir})$, $i = 1, \cdots, N$. This situation may arise when the block effects are not additive or the errors are heteroscedastic. Thus, in this paper the independence of the errors is replaced by within block symmetric dependence, while the additivity of the block effects and homoscedasticity of the errors are relaxed.

2. Some fundamental lemmas. Define $\Delta_{jk} = \tau_j - \tau_k$, $j \neq k = 1, \cdots, r$, and let

$$(2.1) \qquad X^*_{ijk} = X_{ij} - X_{ik}, \quad U_{ijk} = Y_{ij} - Y_{ik} \quad \text{for} \quad j \neq k = 1, \cdots, r \quad \text{and} \quad i = 1, \cdots, N.$$

Assume that $\mathbf{Y}_i$ has a continuous r-variate cumulative distribution function (cdf) $F_i(\mathbf{x})$ which is symmetric in its r arguments, for all $i = 1, \cdots, N$. This interchangeability of $Y_{i1}, \cdots, Y_{ir}$ implies that (i) the cdf $G_i(x)$ of U_{ijk} is inde-

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pendent of $j \neq k = 1, \cdots, r$, continuous and symmetric about zero, and also (ii) the bivariate (joint) cdf $G_i^*(x, y)$ of U_{ijk}, $U_{ijk'}$ is independent of $j \neq k \neq k'$ $(=1, \cdots, r)$, for all $i = 1, \cdots, N$. Let $h(x)$ be a real valued skew-symmetric function, i.e.,

$$h(x) + h(-x) = 0 \quad \text{for all} \quad x. \tag{2.2}$$

Define

$$\zeta_{i,0} = E\{h(U_{ijk})h(U_{ij'k'})\}, \qquad j \neq j' \neq k \neq k', \tag{2.3}$$

$$\zeta_{i,1} = E\{h(U_{ijk})h(U_{ijk'})\} \quad \text{and} \quad \zeta_{i,2} = E\{h^2(U_{ijk})\}, \qquad j \neq k \neq k'. \tag{2.4}$$

Then, we have the following.

Lemma 2.1. *If* (i) $Y_{i1}, \cdots, Y_{ir}$ *are interchangeable random variables and (ii)* $h(x)$ *satisfies* (2.2), *then* (i) $E\{h(U_{ijk})\} = 0$ *and* (ii) $\zeta_{i,0} = 0$.

Proof. (i) follows trivially from the fact that G_i is symmetric about 0 and $h(x)$ satisfies (2.2). To prove (ii), denote by $\mathbf{t} = \{t_1 \leqq t_2 \leqq t_3 \leqq t_4\}$ the order statistics corresponding to Y_{ij}, $Y_{ij'}$, Y_{ik} and $Y_{ik'}$. Since the Y_{ij}'s are interchangeable variables, the conditional distribution of Y_{ij}, $Y_{ij'}$, Y_{ik}, $Y_{ik'}$, given $\mathbf{t}$, will be uniform on the 24 equally likely permutations of t_1, t_2, t_3 and t_4. Further, $h(x)$ satisfies (2.2). Hence, it is easy to show that for $j \neq k \neq j' \neq k'$,

$$E\{h(U_{ijk})h(U_{ij'k'}) \mid \mathbf{t}\} = E\{h(Y_{ij} - Y_{ik})h(Y_{ij'} - Y_{ik'}) \mid \mathbf{t}\} = 0. \tag{2.5}$$

Thus, writing $\zeta_{i,0}$ equivalently as

$$E_{\mathbf{t}}\{E[h(U_{ijk})h(U_{ij'k'}) \mid \mathbf{t}]\},$$

the proof follows from (2.5). Q.E.D.

Lemma 2.2. *If* $Y_{i1}, \cdots, Y_{ir}$ *are interchangeable variables and* $h(x)$ *satisfies* (2.2), *then* $\zeta_{i,1} \leqq \frac{1}{2}\zeta_{i,2}$, *where the equality sign holds only when* $h(x) = bx$ *with probability* 1. *If, in addition,* $h(x)$ *is monotonic,* $0 \leqq \zeta_{i,1} \leqq \frac{1}{2}\zeta_{i,2}$.

Proof. Define $Z_i = h(U_{i12}) + h(U_{i23}) + h(U_{i31})$. Then, by (2.3), (2.4) and Lemma 2.1, we obtain that

$$V(Z_i) = 3\zeta_{i,2}(1 - 2\zeta_{i,1}/\zeta_{i,2}) \geqq 0, \tag{2.6}$$

where the equality sign holds only when $Z_i \equiv 0$ a.e. Now (2.6) implies that $\zeta_{i,1} \leqq (\frac{1}{2})\zeta_{i,2}$. Also, by definition $U_{i12} + U_{i23} + U_{i31} = 0$. Hence, $Z_i \equiv 0$ a.e., along with (2.2) implies that with probability one

$$h(U_{i12}) + h(U_{i23}) = h(U_{i12} + U_{i23}), \tag{2.7}$$

for all U_{i12}, U_{i23}. (2.7) in turn implies that $h(x) = bx$, with probability 1. This completes the first part of the proof. Let now $\mathbf{t} = \{t_1 \leqq t_2 \leqq t_3\}$ be the order statistics corresponding to Y_{ij}, Y_{ik} and $Y_{ik'}$. Using then (2.2) and proceeding as in the proof of Lemma 2.1, one obtains that

$$\begin{aligned} &E\{h(U_{ijk})h(U_{ijk'}) \mid \mathbf{t}\} \\ &= E\{h(Y_{ij} - Y_{ik})h(Y_{ij} - Y_{ik'}) \mid \mathbf{t}\} \\ &= \tfrac{1}{3}[h(t_1 - t_2)h(t_1 - t_3) + h(t_3 - t_1)h(t_3 - t_2) - h(t_2 - t_1)h(t_3 - t_2)]. \end{aligned} \tag{2.8}$$

Assume that $h(x)$ is $\uparrow$ in x (otherwise, work with $-h(x)$). Then,

$$(2.9)\quad 0 \leqq h(t_2 - t_1) \leqq h(t_3 - t_1) \quad \text{and} \quad 0 \leqq h(t_3 - t_2) \leqq h(t_3 - t_1).$$

By (2.9), the left hand side of (2.8) is essentially non-negative, and integrating over the distribution of **t**, it follows that $\zeta_{i,1} \geqq 0$. Q.E.D.

Let now

$$(2.10)\quad \lambda(F_i) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G_i(x)G_i(y)\, dG_i^*(x, y) \quad \text{for} \quad i = 1, \cdots, N.$$

LEMMA 2.3. *If $Y_{i1}, \cdots, Y_{ir}$ are interchangeable random variables, $\frac{1}{4} \leqq \lambda(F_i) \leqq \frac{7}{24}$, where the upper bound $\frac{7}{24}$ is attained only when G_i is a uniform* cdf *over $(-a, a)$, $a > 0$.*

PROOF. We let $h(x) = G_i(x) - \frac{1}{2}$. As G_i is symmetric about 0, (2.2) is satisfied. Some straightforward computations yield that $\zeta_{i,2} = \frac{1}{12}$ and $\zeta_{i,1} = \lambda(F_i) - \frac{1}{4}$. Also $G_i(x)$ is $\uparrow$ in x. Hence, the lemma directly follows from Lemma 2.2. Q.E.D.

REMARK. Lemma 2.3 generalizes Theorem 2 of Lehmann (1964) to exchangeable random variables and also supplies condition under which the upper bound $\frac{7}{24}$ for $\lambda(F_i)$ may be attained.[2]

3. Robustness for interchangeable errors. It may be noted that if $V(Y_{ij}) = \sigma^2$ and $\text{Cov}(Y_{ij}, Y_{ik}) = \rho\sigma^2$, $j \neq k$, the classical ANOVA-test based on the variance-ratio criterion is a valid test for the null hypothesis $H_0 : \tau_1 = \cdots = \tau_r = 0$, when $F_1 = \cdots = F_N = F$ is a multinormal cdf. It is also asymptotically valid for any F having finite second order moments. Again, for the sequence of alternative hupotheses $\{H_N\}$:

$$(3.1)\quad H_N : \Delta_{jk} = N^{-\frac{1}{2}} a_{jk}\, ;\, a_{jk} = a_j - a_k, \qquad 1 \leqq j < k \leqq r, \qquad \sum_{j=1}^{r} a_j = 0,$$

(where $a_1, \cdots, a_r$ are all real and finite), $(r - 1)$ times the variance-ratio criterion has asymptotically (under $F_1 = \cdots = F_N = F$) a non-central chi-square distribution with $r - 1$ degrees of freedom and non-centrality parameter

$$(3.2)\qquad \sum_{j=1}^{r} a_j^2/[\sigma^2(1 - \rho)].$$

It is also known that the method of ranking after alignment [cf. Hodges and Lehmann (1962)] allows for the interchangeability of the error components, and it has been shown by Sen (1968b) that the efficiency of the non-parametric procedures based on aligned observations is not affected by the interchangeability of the errors. It will be shown here that the same is true for the procedures considered by Lehmann (1964) and Doksum (1967). For this, define $c(u)$ as 1 or 0 according as u is > 0 or not, and let

$$(3.3)\quad W_{N,jk} = \binom{N}{2}^{-1} \sum_{1 \leqq i < i' \leqq N} c(X_{ijk}^* + X_{i'jk}^*), \qquad 1 \leqq j < k \leqq r.$$

[2] The author is grateful to Professor Wassily Hoeffding for pointing out that the existence of F_i for which the corresponding G_i is uniform on $(-a, a)$, $a > 0$, is dubious. Our conjecture is that there exists no such cdf F_i for which the corresponding G_i is uniform on $(-a, a)$, $a > 0$. In fact, for independent errors, the conjecture is proved to be true by Puri and Sen (1968).

Assume that $F_1 = \cdots = F_N = F (\Rightarrow G_1 = \cdots = G_N = G$ and $\lambda(F_1) = \cdots = \lambda(F_N) = \lambda(F)$,) and that $G(x)$ has a continuous density function $g(x)$ satisfying the conditions of Theorem 1 of Lehmann (1964). Then, it is easy to show that

$$(3.4) \quad \lim_{N\to\infty} \{N^{\frac{1}{2}} E[W_{N,jk} - \tfrac{1}{2} \mid H_N]\} = 2a_{jk} \textstyle\int_{-\infty}^{\infty} g^2(x)\, dx, \quad \text{for all } 1 \leqq j < k \leqq r.$$

Using then Theorem 7.1 of Hoeffding (1948), our Lemmas 2.1 and 2.3 and following some routine steps, we arrive at the following.

THEOREM 3.1. *If* (i) $F_1 = \cdots = F_N = F$, (ii) $F(\mathbf{x})$ *is symmetric in its* r *arguments, and* (iii) $\{H_N\}$ *in* (3.1) *holds, then* $\{N^{\frac{1}{2}}(W_{N,jk} - \frac{1}{2}) - 2a_{jk} \int_{-\infty}^{\infty} g^2(x)\, dx, 1 \leqq j < k \leqq r\}$ *has asymptotically a* $\frac{1}{2}r(r-1)$*-variate normal distribution with null mean vector and dispersion matrix* $\mathbf{\Gamma} = ((\gamma_{jk,j'k'}))$ *given by*

$$(3.5) \quad \begin{aligned} \gamma_{jk,j'k'} &= \tfrac{1}{3}, & & j = j', k = k', j \neq k, \\ &= 4\lambda(F) - 1, & & j = j', k \neq k', j \neq k \neq k', \\ &= 1 - 4\lambda(F), & & j \neq j', j = k', j \neq j' \neq k, \\ &= 0 & & j \neq j' \neq k \neq k'. \end{aligned}$$

Theorem 3.1 and Lemma 2.3 show that the results derived by Doksum (1967) in his Lemmas 2.1 through 2.4 remain true even when the errors are not all independent, but are within block symmetric dependent. Further, the use of Theorem 3.1 as in (2.6) of [4] generalizes Theorem 1 of Lehmann (1964) to exchangeable error components. Since the main results of Lehmann (1964) are based on his Theorems 1 and 2, and that of Doksum (1967) on his Lemmas 2.3 and 2.4, it follows from our Lemma 2.3 and Theorem 3.1 that *the Lehmann-Doksum procedures remain valid for within block exchangeable error components.*

Now, we note that the variance of the cdf G is $2\sigma^2(1 - \rho) = \sigma^2(G)$, say. As such, using (3.2) and generalizing Theorem 3.1 in the same manner as in Lemma 2.3 of Doksum (1967), the asymptotic relative efficiency (ARE) of the Doksum-test with respect to the classical ANOVA test, can again be shown to be equal to e', defined by (2.11) and (2.12) of Doksum (1967). It is thus clear that the ARE is unaffected by the within block symmetric dependence of the error components. The same conclusion also applies to Lehmann's procedure, as the variance of $N^{-\frac{1}{2}} \sum_{i=1}^{N} X_{ijk}^*$ is also equal to $\sigma^2(G)$. This shows that Theorem 4 of Lehmann (1964) is also true for exchangeable errors.

In order to apply these procedures in practice, one needs consistent estimates of (i) $\lambda(F)$ and, in the case of Lehmann's procedure, (ii) $\int_{-\infty}^{\infty} g^2(x)\, dx$. Since X_{ijk}^*, $i = 1, \cdots, N$, are iidrv's with a cdf $G(x - \Delta_{jk})$ symmetric about Δ_{jk}, the estimate of $\int_{-\infty}^{\infty} g^2(x)\, dx$, based on X_{ijk}^*'s, considered by Lehmann (1964) remains valid even for exchangeable errors. For $\lambda(F)$, we consider the following simple estimator due to Puri and Sen (1967). Let $L_{j;kq}$ be the Spearman rank covariance between (X_{ijk}^*, X_{ijq}^*), $i = 1, \cdots, N$. That is

$$(3.6) \quad L_{j;kq} = \textstyle\sum_{i=1}^{N} (R_{ijk} - (N+1)/2)(R_{ijq} - (N+1)/2)/[N(N+1)^2],$$

where R_{ijk} stands for the rank of X^*_{ijk} among all $X^*_{ijk}, \cdots, X^*_{Njk}$. Also, let

$$L = [2/r(r-1)(r-2)] \sum_{j=1}^{r} \sum_{k<q(\neq j)}^{r} L_{j;kq} \tag{3.7}$$

be the average over all possible distinct $j \neq k \neq q = 1, \cdots, r$. Then, from the results of Puri and Sen (1967) it follows that L in (3.7) consistently estimates $\lambda(F) - \frac{1}{4}$, no matter whether $\boldsymbol{\tau}$ is $\mathbf{0}$ or not and the errors are independent or exchangeable within each block. From computational aspects, this estimator also appears to be simpler than the original estimator proposed by Lehmann (1964).

4. Robustness for heteroscedastic errors. Let us define

$$\bar{G}_N(x) = N^{-1} \sum_{i=1}^{N} G_i(x), \tag{4.1}$$
$$\bar{G}_N^*(x, y) = N^{-1} \sum_{i=1}^{N} G_i^*(x, y) \quad \text{and} \quad \bar{g}_N(x) = (d/dx)\{\bar{G}_N(x)\}.$$

Assume that (a) there exists a continuous (bivariate) cdf $\Phi^*(x, y)$ such that

$$\lim_{N\to\infty} \bar{G}_N^*(x, y) = \Phi^*(x, y), \qquad \text{at all points of continuity of the latter,} \tag{4.2}$$

and (b) $\bar{G}_N(x)$, defined by (4.1), also converges (as $N \to \infty$) to an absolutely continuous cdf $\Phi(x)$ (having a continuous density function $\phi(x)$), in such a way that

$$|\bar{G}_N(x) - \Phi(x)| \leqq w(x)/N^{\beta}, \quad \text{for some } \beta > \tfrac{1}{2}, \tag{4.3}$$

where $\int_{-\infty}^{\infty} w(x)\, dG_i(x) < \infty$, uniformly in $i = 1, \cdots$.

For the justification of (4.3), we consider the following models, though it may hold for other models too.

(I) *Replication model.* For some (fixed) positive integer $b(\geqq 1)$, consider a sequence of positive integers $n_1, \cdots, n_b$ such that $\sum_{j=1}^{b} n_j = N$, $n_j = n_j(N)$, where as N increases

$$N^{-1} n_j(N) = k_j + o(N^{-\frac{1}{2}}), \qquad 0 < k_j < 1, \quad \text{for all} \quad j = 1, \cdots, b. \tag{4.4}$$

Consider then the model in which the set $\{F_i(\mathbf{x}), i = 1, \cdots, N\}$ is composed of b distinct subsets, where the jth subset contains n_j cdf's which are all identical to the cdf $F_j(\mathbf{x})$, for $j = 1, \cdots, b$. Then, of course, as $N \to \infty$,

$$\bar{G}_N^*(x, y) = N^{-1} \sum_{j=1}^{b} n_j G_j^*(x, y) \to \sum_{j=1}^{b} k_j G_j^*(x, y) = \Phi^*(x, y), \tag{4.5}$$

and writing $\Phi(x) = \sum_{j=1}^{b} k_j G_j(x)$,

$$|\bar{G}_N(x) - \Phi(x)| = |\sum_{j=1}^{b} [(n_j/N) - k_j] G_j(x)| = o(N^{-\frac{1}{2}}), \tag{4.6}$$

uniformly in $x(-\infty < x < \infty)$, by (4.4). Thus, both (4.2) and (4.3) hold. The physical interpretation for this model is that we have an initial set of b blocks with cdf's $F_1, \cdots, F_b$ respectively, and conceptually, we replicate the blocks a large number of times keeping the proportion of replications for these as $k_1, \cdots, k_b$ respectively.

(II) *Outlier model.* Suppose out of the N cdf's $\{F_1, \cdots, F_N\}$, we have a subset

$\{F_{i_j}, j = 1, \cdots, n\}$ (where $i_1, \cdots, i_n$ are any n distinct integers out of $1, \cdots, N$) which corresponds to outliers, for which the cdf's are different from the rest of the cdf's which are all identical to a common cdf $F(\mathbf{x})$. It then follows that if n/N tends to 0 as $N \to \infty$, then (4.2) holds (with $\Phi^* = G^*$), while if $n/N = o(N^{-\frac{1}{2}})$, then (4.3) holds (with $\Phi = G$), where G and G^* are the univariate and bivariate marginals of $F(\mathbf{x})$.

(III) *Gross error model.* Consider a finite number (say, d) of cdf's $F_1^*, \cdots, F_d^*$ each of which is symmetric in its r arguments, but they are, otherwise, quite arbitrary. As in model (I), partition the set of N cdf's $\{F_i\}$ into b subsets of $n_1, \cdots, n_b$ identical cdf's, where n_j's satisfy (4.4). Consider then the following error contamination model where the cdf F_j for the jth set is given by

$$(4.7)\quad F_j(\mathbf{x}) = (1 - \epsilon_j)F_1^*(\mathbf{x}) + \epsilon_{j2}F_2^*(\mathbf{x}) + \cdots + \epsilon_{jd}F_d^*(\mathbf{x}), \quad j = 1, \cdots, b,$$

where $\epsilon_j = \epsilon_{j2} + \cdots + \epsilon_{jd}$, $0 < \epsilon_j, \epsilon_{j2}, \cdots, \epsilon_{jd} < 1$, and are usually quite small. The univariate and bivariate marginals of the cdf F_k^* are denoted by G_k and G_k^*, respectively, for $k = 1, \cdots, d$. Let then

$$(4.8)\qquad \Phi(x) = \textstyle\sum_{j=1}^{b} k_j[(1 - \epsilon_j)G_1(x) + \epsilon_{j2}G_2(x) + \cdots + \epsilon_{jd}G_d(x)],$$

$$(4.9)\quad \Phi^*(x, y) = \textstyle\sum_{j=1}^{b} k_j[(1 - \epsilon_j)G_1^*(x, y) + \epsilon_{j2}G_2^*(x, y) + \cdots + \epsilon_{jd}G_d^*(x, y)].$$

Then, proceeding as in (4.5) and (4.6), it follows that (4.2) and (4.3) also hold for this model.

(IV) *Convergence model.* Suppose that for the sequence of cdf's $\{F_i(\mathbf{x})\}$, $\lim_{i\to\infty} F_i(\mathbf{x}) = F(\mathbf{x})$ exists ($\Rightarrow \lim_{i\to\infty} G_i(x) = G(x)$ and $\lim_{i\to\infty} G_i^*(x, y) = G^*(x, y)$ also exist), and that

$$(4.10)\qquad |G_i(x) - G(x)| \leqq i^{-\beta}w(x),$$

for some $\beta > \frac{1}{2}$, and all $i = 1, 2, \cdots$, where $\int_{-\infty}^{\infty} w(x)\, dG_i(x) < \infty$, uniformly in i. Then,

$$(4.11)\qquad |\bar{G}_N(x) - G(x)| \leqq w(x)[N^{-1} \textstyle\sum_{i=1}^{N} i^{-\beta}].$$

Now, we note that for $\beta > 1$, $\sum_{i=1}^{N} i^{-\beta} < \infty$, and for $\beta = 1$, $N^{-1}\sum_{i=1}^{N} i^{-1} \sim N^{-1}\log N = o(N^{-\beta'})$, for $\beta' > \frac{1}{2}$. Finally, for $\frac{1}{2} < \beta < 1$, $N^{-1}\sum_{i=1}^{N} (i/N)^{-\beta} \sim 1/(1 - \beta)$ is finite. Hence, from (4.11) we obtain that (4.3) holds with $\Phi(x) = G(x)$. (4.2) holds more trivially. Now, (4.10) holds in particular for the *heteroscedastic model* where

$$(4.12)\quad G_i(x) = G(x/\delta_i), \quad i = 1, 2, \cdots, \qquad \text{where } \delta_i\text{'s are all positive,}$$

and it is assumed that there exists a positive δ, such that $\delta_i/\delta = 1 + O(i^{-\beta})$, $\beta > \frac{1}{2}$, and that G has a continuous and bounded density g.

A consequence of (4.3) is that for any real and finite a,

$$(4.13)\quad N^{\frac{1}{2}}[\bar{G}_N(x + N^{-\frac{1}{2}}a) - \bar{G}_N(x)] - N^{\frac{1}{2}}[\Phi(x + N^{-\frac{1}{2}}a) - \Phi(x)] = w(x)\cdot o(1),$$

and hence, it follows that

(4.14) $\lim_{N\to\infty} \int_{-\infty}^{\infty} N^{\frac{1}{2}}[\bar{G}_N(x + N^{-\frac{1}{2}}a) - \bar{G}_N(x)]\, d\bar{G}_N(x) = a \int_{-\infty}^{\infty} \varphi^2(x)\, dx.$

It then follows from the results of Sen (1968a) that under (3.1) and (4.3) ($\Rightarrow$ (4.14)),

(4.15) $\lim_{N\to\infty} \{N^{\frac{1}{2}} E[(W_{N,jk} - \frac{1}{2}) \mid H_N]\} = 2a_{jk} \int_{-\infty}^{\infty} \varphi^2(x)\, dx, \quad 1 \leqq j < k \leqq r,$

where $W_{N,jk}$'s are defined by (3.3). Further, by a direct generalization of Theorem 2.1 of Sen (1968a), it follows that under (4.2), (4.3) and $\{H_N\}$ in (3.1), the stochastic vector $\{[N^{\frac{1}{2}}(W_{N,jk} - \frac{1}{2}) - 2a_{jk} \int_{-\infty}^{\infty} \varphi^2(x)\, dx],\ 1 \leqq j < k \leqq r\}$ has asymptotically a $r(r-1)/2$-variate normal distribution with null mean vector and dispersion matrix $\Gamma^* = (\gamma^*_{jk,j'k'})$, where $j < k$ and $j' < k'$, and

$$
(4.16)\qquad \begin{aligned}
\gamma^*_{jk,j'k'} &= \tfrac{1}{3}, && j = j',\ k = k' \\
&= 4\lambda(\Phi) - 1, && j = j',\ k \neq k' \text{ or } j \neq j',\ k = k', \\
&= 1 - 4\lambda(\Phi), && j = k',\ j' \neq k \text{ or } j \neq k',\ k = j', \\
&= 0, && j \neq k \neq j' \neq k',
\end{aligned}
$$

and

$$(4.17)\qquad \lambda(\Phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(x)\Phi(y)\, d\Phi^*(x, y),$$

with Φ and Φ^* defined by (4.3) and (4.2), respectively.

Now, by virtue of (4.2) and (4.3), we may also write

$$(4.18)\qquad \lambda(\Phi) = \lim_{N\to\infty} \lambda(\bar{F}_N), \quad \text{where}$$
$$\lambda(\bar{F}_N) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bar{G}_N(x)\bar{G}_N(y)\, d\bar{G}_N^*(x, y).$$

Since $\bar{G}_N(x) - \frac{1}{2}$ satisfies (2.2), proceeding as in Lemmas 2.2 and 2.3, we obtain that

$$(4.19)\qquad \tfrac{1}{4} \leqq \lambda(\bar{F}_N) \leqq \tfrac{7}{24}, \qquad \text{uniformly in } F_1, \cdots, F_N \text{ and } N.$$

(4.18) and (4.19) in turn imply that

$$(4.20)\qquad \tfrac{1}{4} \leqq \lambda(\Phi) \leqq \tfrac{7}{24}.$$

Let us now denote by $\sigma_i^2 = V(Y_{ij})$ and $\rho_i\sigma_i^2 = \mathrm{Cov}\,(Y_{ij}, Y_{ik}), j \neq k, i = 1, \cdots, N$, and let

$$(4.21)\qquad (\bar{\sigma}_N^*)^2 = N^{-1} \textstyle\sum_{i=1}^N \sigma_i^2(1 - \rho_i).$$

Then, we note that (i) $N^{-\frac{1}{2}} \sum_{i=1}^N X^*_{ijk}$ has the variance $2(\bar{\sigma}_N^*)^2$, and (ii)

$$(4.22)\qquad \lim_{N\to\infty} [2(\bar{\sigma}_N^*)^2] = \sigma^2(\Phi), \quad \text{the variance of the cdf } \Phi(x).$$

We shall now consider consistent estimates of $\sigma^2(\Phi)$, $\int_{-\infty}^{\infty} \varphi^2(x)\, dx$ and $\lambda(\Phi)$, which are required to apply these procedures in practice. First, define

$$(4.23)\qquad s_N^2 = [(N-1)(r-1)]^{-1} \textstyle\sum_{i=1}^N \sum_{j=1}^r (X_{ij} - X_{i\cdot} - X_{\cdot j} + X_{\cdot\cdot})^2,$$

where

$X_{i\cdot} = r^{-1}\sum_{j=1}^{r} X_{ij}$, $X_{\cdot j} = N^{-1}\sum_{i=1}^{N} X_{ij}$ and $X_{\cdot\cdot} = (Nr)^{-1}\sum_{i=1}^{N}\sum_{j=1}^{r} X_{ij}$.

Looking at (1.1), we observe that in (4.23) we may replace X_{ij}'s by Y_{ij}'s. Then by straightforward expansion and use of Markov's law of large numbers for the independent vectors $[(Y_{ij}, Y_{ik}), j, k = 1, \cdots, r]$, $i = 1, \cdots, N$, it follows that if $E|Y_{ij}|^{2+\delta} < \infty$, for some $\delta > 0$, then

$$|s_N^2 - (\bar{\sigma}_N^*)^2| \to_p 0 \quad \text{as} \quad N \to \infty. \tag{4.24}$$

Thus, from (4.21) through (4.24), it follows that $2s_N^2$ consistently estimates $\sigma^2(\Phi)$.

Second, as in (3.3), we define $W(X_{ijk}^* - a, i = 1, \cdots, N)$ based on $X_{ijk}^* - a$, $i = 1, \cdots, N$, and let

$$W_N^{(1)} = \tfrac{1}{2} - \tau_{\alpha/2}\cdot(3N)^{-\frac{1}{2}} \quad \text{and} \quad W_N^{(2)} = \tfrac{1}{2} + \tau_{\alpha/2}\cdot(3N)^{-\frac{1}{2}}, \tag{4.25}$$

where α $(0 < \alpha < 1)$ is prefixed and τ_α is the upper $100\alpha\,\%$ point of the standard normal distribution. Let then

$$\hat{\Delta}_{L(jk),N} = \inf\{a: W(X_{ijk}^* - a, i = 1, \cdots, N) \leqq W_N^{(2)}\}, \tag{4.26}$$

$$\hat{\Delta}_{U(jk),N} = \sup\{a: W(X_{ijk}^* - a, i = 1, \cdots, N) \geqq W_N^{(1)}\}. \tag{4.27}$$

[For the expressions of $\hat{\Delta}_{L(jk),N}$ and $\hat{\Delta}_{U(jk),N}$ in terms of the order statistics of $(X_{ijk}^* + X_{i'jk}^*)/2$, $1 \leqq i \leqq i' \leqq N$, see Lehmann (1963).] Using then Theorem 2.1 of Sen (1968a) and proceeding precisely on the same line as in Theorems 1 and 2 of Sen (1966), it follows that under (4.3) [$\Rightarrow$ (4.14)]

$$\hat{B}_{jk} = [W_N^{(2)} - W_N^{(1)}]/[\hat{\Delta}_{U(jk),N} - \hat{\Delta}_{L(jk),N}] \tag{4.28}$$

is a translation invariant consistent estimator of $\int_{-\infty}^{\infty}\varphi^2(x)\,dx$, for all $1 \leqq j < k \leqq r$. [This generalizes the results of [6] to non-identically distributed random variables.] Hence, we propose the following consistent estimator of $\int_{-\infty}^{\infty}\varphi^2(x)\,dx$:

$$\hat{B} = \binom{r}{2}^{-1}\sum_{1\leqq j<k\leqq r}\hat{B}_{jk}. \tag{4.29}$$

Finally, we propose to show that L, defined by (3.6) and (3.7), consistently estimates $\lambda(\Phi) - \frac{1}{4}$, under (4.2) and (4.3). For this, following Hoeffding (1948), p. 318, we write

$$L_{j;kq} = (N+1)^{-1}[(N-2)K_{j;kq} + 3t_{j;kq}], \tag{4.30}$$

where $t_{j;kq}$ is the Kendall's rank correlation between (X_{ijk}^*, X_{ijq}^*), $i = 1, \cdots, N$ (thus $|t_{j;kq}| \leqq 1$), and

$$K_{j;kq} = [2\tbinom{N}{3}]^{-1}\sum{}'' s(X_{ijk}^* - X_{i'jk}^*)s(X_{ijq}^* - X_{i''jq}^*), \tag{4.31}$$

where the summation $\sum''$ extends over all distinct i, i' and i'', and $s(u)$ is 1, 0 or -1 according as u is $>$, $=$ or < 0. Since, $K_{j;kq}$ is a U-statistic with a bounded kernel, on using (5.14) through (5.18) of Hoeffding (1948), it follows that its variance is bounded above by $1/N$. Hence, by Chebyshev's lemma, $|K_{j;kq} - E(K_{j;kq})| \to_p 0$, as $N \to \infty$. It has also been shown elsewhere [Sen (1968c)]that if U_N is a U-statistic based on independent $X_1, \cdots, X_N$ with cdf's $F_1, \cdots, F_N$, and if $\bar{F}_N = N^{-1} \sum_{i=1}^N F_i$ and $\theta(\bar{F}_N)$ be the expectation of U_N computed under the assumption that $F_1 = \cdots = F_N = \bar{F}_N$, then

$$|E(U_N \mid F_1, \cdots, F_N) - \theta(\bar{F}_N)| = O(N^{-1}), \tag{4.32}$$

for all $F_1, \cdots, F_N$ for which U_N has a finite second moment. Hence, using (4.32) and following some routine steps, we obtain that

$$|E(K_{j;kq}) - [\lambda(\bar{F}_N) - \tfrac{1}{4}]| \to 0, \quad \text{as} \quad N \to \infty. \tag{4.33}$$

Thus, from (4.30), (4.33), (4.18), (3.6) and (3.7), it follows that under (4.2) and (4.3),

$$|L - [\lambda(\Phi) - \tfrac{1}{4}]| \to_p 0 \quad \text{as} \quad N \to \infty. \tag{4.34}$$

We shall now study the ARE of the procedures under consideration. Proceeding on the same line as in Lemmas 2.3 and 2.4 of Doksum (1967) and Theorems 2, 3 and 4 of Lehmann (1964), it follows on using (4.20), (4.21) and (4.22) that the ARE of the Lehmann-Doksum procedures with respect to the classical parametric procedures is

$$e' = e(\Phi)\{r/[2 + 6(r - 2)(4\lambda(\Phi) - 1)]\}, \tag{4.35}$$

where

$$e(\Phi) = 12\sigma^2(\Phi)[\textstyle\int_{-\infty}^{\infty} \varphi^2(x)\, dx]^2 \tag{4.36}$$

is the ARE of the Wilcoxon signed rank test with respect to the Student's t-test. By (4.20), the second factor on the right hand side of (4.35) is bounded below by 1 (though as pointed out by Hollander (1967), it is quite close to 1), while the first factor has been studied extensively by various workers and has well known bounds.

A special case considered below is of some interest. Suppose now that

$$F_i(\mathbf{x}) = F(\delta_i^{-1}\mathbf{x}), \qquad \delta_i > 0, \qquad \text{for all} \quad i = 1, 2, \cdots, \tag{4.37}$$

and that (4.2) and (4.3) hold for the sequence of scale factors $\{\delta_1, \cdots, \delta_N\}$. Then, for model (II) or (IV), $e(\Phi)$ equals to $e(G)$, where G is the univariate marginal of $F(\mathbf{x})$. For model (I), proceeding as in Theorem 2.2 of Sen (1968a), it is seen that under the conditions stated there

$$e(\Phi) \geqq e(G), \tag{4.38}$$

where the equality sign holds only when δ_i's are all equal. This illustrates the robust efficiency of the procedures for heteroscedastic errors.

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