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DOMAINS OF OPTIMALITY OF TESTS IN SIMPLE RANDOM SAMPLING

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0. Summary. This paper deals with the structure of sets Ω of distributions for which a particular test is the most powerful for testing a simple hypothesis $H:f = f_0$ vs. $K:f \varepsilon \Omega$, that is, with the domain of optimality of a test. The context is restricted to these Ω consisting of probabilities having continuous positive densities, and to one-sample tests.

The important concept is that of a family of tests, one for each significance level. This concept allows us to use the full power of the Neyman-Pearson Lemma.

The main results are:
(1) The domain of optimality of a test family Φ is essentially a multiplicatively-convex (convex in the logarithms) cone; hence there are distributions both "near to" and "far from" the null distribution for which Φ is optimal. (Theorems 1, 2, and 3).
(2) If Φ is uniformly most powerful for testing $H:f = f_0$ vs. $K:f \varepsilon \Omega$ with $n \geqq 2$ then the class of distributions has a monotone likelihood ratio. (Theorem 4).

1. Test families. In the usual nomenclature, a statistical "test" (e.g., t-test, Mann Whitney test) is, in fact, a family of tests, indexed by the size or significance level α. The idea of families of tests leads, it will be seen, to a number of useful converses to known theorems. The relationship among tests in a family is that the critical region expands with increasing size α of test.

We will require test function $\varphi_\alpha(x_1, \cdots, x_n)$ giving the probability of rejecting H when $X_1 = x_1, \cdots, X_n = x_n$, with $P\,[\text{Rej.}\, H \mid H] = \int_{E_n} \varphi_\alpha(x_1, \cdots, x_n) \cdot f_0(x_1) \cdots f_0(x_n)\, dx_1 \cdots dx_n \leqq \alpha$. Then a test family should be defined by (1) $\Phi = \{\varphi_\alpha \mid 0 \leqq \alpha \leqq 1, \varphi_\alpha \leqq \varphi_{\alpha'} \textit{ if } \alpha < \alpha'\}$. This guarantees the "expanding critical region".

We will speak of most powerful (MP) families, UMP families, etc., if each $\varphi_\alpha \varepsilon \Phi$ has the designated property.

2. Optimal families. To avoid all measure-theoretic problems, we restrict ourselves to test situations in which the (simple) null hypothesis and the alternative consist of probability measures from the Scheffé class Ω_3^* [3] of measures possessing continuous, strictly positive densities (with respect to a fixed measure μ, Lebesgue or otherwise) on the open interval (a, b) $(-\infty \leqq a < b \leqq \infty)$, and null outside (a, b).

For this class, it is clear that there is no test having power 1 and size < 1. Hence the full Neyman-Pearson lemma ([1], p. 65) applies. φ_α is a most powerful

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size α test of $H:f = f_0$ vs. $K:f = f_1$ if and only if φ_α is of exact size α and there exists a number $c_{1\alpha}$ such that (a.e. μ)

$$(2) \qquad \begin{aligned} \varphi_\alpha(x_1, \cdots, x_n) &= 1 \quad \text{if} \quad \prod f_1(x_i)/\prod f_0(x_i) > c_{1\alpha} \\ &= 0 \quad \text{if} \quad \prod f_1(x_i)/\prod f_0(x_i) < c_{1\alpha}. \end{aligned}$$

(The definition of φ_α on the set—possibly of positive measure—having likelihood ratio $= c_{1\alpha}$ is arbitrary, within the restriction that φ_α have exact size α.) We may suppose that (2) is satisfied for all (not just almost all) $(x_1, \cdots, x_n)$ satisfying either inequality.

Hence we have the result: Φ is a most powerful family if and only if each φ_α is a likelihood ratio (LR) test function (2). Such a family obviously satisfies the defining condition (1).

Furthermore, we have the following Fundamental Lemma. Φ is most powerful for testing $H:f = f_0$ vs. $K:f = f_1$ and for testing $H:f = f_0$ vs. $K:f = f_2$ if and only if $\prod_{i=1}^n [f_2(x_i)/f_0(x_i)] \equiv h(\prod_{i=1}^n [f_1(x_i)/f_0(x_i)])$ where h is a strictly increasing, continuous function on $(\min L_1, \max L_1)$ where $L_1 = \prod [f_1(x_i)/f_0(x_i)]$.

Proof. Suppose such an h exists. Then for any α it is clear that the LR test is the same for either testing problem.

Conversely, if Φ is MP for f_1 vs. f_0 and f_2 vs. f_0, then φ_α is a LR test for any α, $0 < \alpha < 1$. It follows that $\prod_1^n [f_1(x_i)/f_0(x_i)] > \prod_1^n [f_1(y_i)/f_0(y_i)]$ if and only if $\prod_1^n [f_2(x_i)/f_0(x_i)] > \prod_1^n [f_2(y_i)/f_0(y_i)]$. For if $\prod_1^n [f_1(x_i)/f_0(x_i)] > \prod_1^n [f_1(y_i)/f_0(y_i)]$ while $\prod_1^n [f_2(x_i)/f_0(x_i)] \leqq \prod_1^n [f_2(y_i)/f_0(y_i)]$ there clearly exists an α such that the LR test $\varphi_\alpha(x_1, \cdots, x_n) = 1$ for $K:f = f_1$ while $\varphi_\alpha(y_1, \cdots, y_n) = 0$, but φ_α is the LR test for f_2. This is a contradiction unless $\prod_1^n [f_2(x_i)/f_0(x_i)] = \prod_1^n [f_2(y_i)/f_0(y_i)] = c_{1\alpha}$. In this case we can change $c_{1\alpha}$, thereby changing α, by a small amount, and have a contradiction.

This result will be summarized by saying that φ is MP for f_1 vs. f_0 and f_2 vs. f_0 if and only if f_1 and f_2 have the same likelihood ratio order (LRO).

Now, suppose f_1 and f_2 have the same LRO. Then, if for some $x_1, \cdots, x_n$, $L_1(x_1, \cdots, x_n) = \prod_1^n [f_1(x_i)/f_0(x_i)] = x$, define $h(x) = L_2(x_1, \cdots, x_n) = \prod_1^n [f_2(x_i)/f_0(x_i)]$. This defines h on (m, M) where $m = \min L_1$, $M = \max L_1$. h is clearly unambiguously defined (by the LRO property), increasing and continuous on (m, M), q.e.d.

We will have use for the following:

Corollary. *If Φ is* MP *for testing $H:f = f_0$ vs. $K:f = f_1$ and $H:f = f_0$ vs. $K:f = f_2$, then $f_1(x)/f_0(x) > f_1(y)/f_0(y)$ if and only if $f_2(x)/f_0(x) > f_2(y)/f_0(y)$, hence if and only if $f_2(x)/f_0(x) = h_1(f_1(x)/f_0(x))$ where h_1 is a strictly increasing, continuous function on* $(\min [f_1(x)/f_0(x)], \max [f_1(x)/f_0(x)])$.

Proof. Suppose $f_1(x)/f_0(x) > f_1(y)/f_0(y)$ while $f_2(x)/f_0(x) \leqq f_2(y)/f_0(y)$. Then $L_1(x, \cdots, x) > L_1(y, \cdots, y)$ while $L_2(x, \cdots, x) \leqq L_2(y, \cdots, y)$ which contradicts the fundamental lemma. The construction of h_1 is the same as that of h.

3. Multiplicative convexity of domains of optimality. By definition, f_1 is in the domain of optimality of Φ if and only if Φ is optimal for testing $H:f = f_0$ vs. $K:f = f_1$. We will say that the domain of optimality of Φ, $D(\Phi)$, is m-convex (multiplicatively convex) if $D(\Phi)$ contains the density $C_\theta f_1^\theta(x) f_2^{1-\theta}(x)$ $(0 \leqq \theta \leqq 1)$ whenever it contains f_1 and f_2. Note that, by Hölder's inequaltiy ([2], p. 156), $f_1^\theta f_2^{1-\theta}$ is integrable if f_1 and f_2 are integrable.

As an immediate consequence of the fundamental lemma we have

THEOREM 1. *For any* Φ, $D(\Phi)$ *is* m*-convex.*

PROOF. If $D(\Phi)$ is empty or contains only one point, there is nothing to prove. Suppose $D(\Phi)$ contains f_1 and f_2. Then, by the fundamental lemma, $L_2(x_1, \cdots, x_n) \equiv h(L_1(x_1, \cdots, x_n))$, where h is continuous and strictly increasing. Take $h_\theta(x) = C_\theta^n x^\theta (h(x))^{1-\theta}$, $0 \leqq \theta \leqq 1$ where $C_\theta^{-1} = \int_{-\infty}^{\infty} f_1^\theta(x) \cdot f_2^{1-\theta}(x)\, d\mu < \infty$. Then $h_\theta(x)$ is clearly increasing and continuous and

$$\prod_1^n [C_\theta f_1^\theta(x_i) f_2^{1-\theta}(x_i)/f_0(x_i)] \equiv h_\theta\left(\prod_1^n f_1(x_i)/\prod_1^n f_0(x_i)\right).$$

Hence $C_\theta f_1^\theta(x) f_2^{1-\theta}(x)$ is in $D(\Phi)$ whenever f_1, f_2 are, by the fundamental lemma.

Intuitively it is more plausible that if Φ is MP for f_1 vs. f_0 and for f_2 vs. f_0, it is MP for any mixture $\theta f_1 + (1 - \theta) f_2$ vs. f_0. That this is false can easily be seen from the following discrete counterexample, which could easily be "continuized":

Take $n = 2$; $f_0(x) = \frac{1}{4}$ on $x = 1, 2, 3, 4$; $f_1(1) = 19C_1$, $f_1(2) = 20C_1$, $f_1(3) = 12C_1$, $f_1(4) = 30C_1$, $f_2(1) = 5C_2$, $f_2(2) = 10C_2$, $f_2(3) = 2C_2$, $f_2(4) = 22C_2$, where C_1 and C_2 are appropriate normalizing constants. Then it is easily shown that $L_1(1, 2) > L_1(3, 4)$ and $L_2(1, 2) > L_2(3, 4)$ but, with $L(x, y) = \frac{1}{2}[f_1(x) + f_2(x))\frac{1}{2}(f_1(y) + f_2(y)]/f_0(x)f_0(y)$, $L(1, 2) < L(3, 4)$, so that the LRO of $\frac{1}{2}f_1 + \frac{1}{2}f_2$ differs from that of f_1 and f_2. (It is routine to verify that the LRO's of f_1 and f_2 are identical.) Hence by the fundamental lemma, the f_1, f_2 optimal Φ is not optimal for $\frac{1}{2}f_1 + \frac{1}{2}f_2$.

$D(\Phi)$ is not only m-convex, it is essentially a convex cone; the meaning of this is defined in the following theorems.

THEOREM 2. *If* Φ *is* MP *for testing* f_1 *vs.* f_0, *it is* MP *for testing* $H:f = f_0$ vs. $K:f = f_\theta = C_\theta f_1^\theta f_0^{1-\theta}$, $0 < \theta \leqq 1$.

PROOF. Take $h(x) = C_\theta^n x^\theta$. h is continuous and increasing and $h\,(L_1(x_1, \cdots, x_n)) \equiv L_\theta(x_1, \cdots, x_n)$, with the obvious definitions of the likelihoods L_1 and L_θ. Hence the fundamental lemma applies.

If $\Phi = \{\varphi_\alpha\}$, define $\Phi^c = \{\varphi_\alpha^c\}$ where $\varphi_\alpha^c = 1 - \varphi_{1-\alpha}$.

THEOREM 3. *If there exists a* $\theta < 0$ *for which* $f_\theta = C_\theta f_1^\theta f_0^{1-\theta}$ *is a density, and* Φ *is* MP *for testing* $H:f = f_0$ *vs.* $K:f = f_1$, *then* Φ^c *is* MP *for testing* $H:f = f_0$ vs. $K:f = f_\theta$.

PROOF. The LRO of f_θ is the opposite of the LRO of f_1. Hence Φ^c is the LR test family.

These two results demonstrate that the most powerful nature of a test is a question of "direction" not "distance"; there is no test which is most powerful only for those distributions "far from" or "moderately far from" the null distribution. (If the densities in Ω are bounded, it follows that every MP test is locally most powerful, in that, if Φ is MP for $H:f = f_0$ vs. $K:f = f_1$, for each $\epsilon > 0$ there is a density f_θ such that Φ is MP for $H:f = f_0$ vs. $K:f = f_\theta$ with $\sup_x |f_0(x) - f_\theta(x)| < \epsilon$. We need only take θ sufficiently near 0.)

4. Montone likelihood ratios and UMP tests. In the case $n = 1$, the corollary to the fundamental lemma summarizes the nature of MP tests fairly completely. (In this case, the domains of optimality are also (additively) convex; take $h_\theta(x) = \theta x + (1 - \theta)h(x)$). When $n \geqq 2$, however, the independence assumption leads to sharp restrictions on the possibility of uniformly most powerful (UMP) tests; indeed, such test families can occur only in already known cases.

Recall that a family $\{f_\theta\}$ of densities indexed by a real parameter θ has a monotone likelihood ratio (MLR) if there exists a function $T(x_1, \cdots, x_n)$ such that if $\theta_1 < \theta_2 \prod f_{\theta_2}(x_i)/\prod f_{\theta_1}(x_i)$ is an increasing function of $T(x_1, \cdots, x_n)$. ([1], p. 68). If $\{f_\theta\}$ has a MLR there exists a UMP test family for $H:\theta \leqq \theta_0$ vs. $K:\theta > \theta_0$, namely:

$$\begin{aligned} \varphi_\alpha(x_1, \cdots, x_n) &= 1 && \text{if } T(x_1, \cdots, x_n) > C_{1\alpha} \\ &= C_{2\alpha} && \text{if } T(x_1, \cdots, x_n) = C_{1\alpha} \\ &= 0 && \text{if } T(x_1, \cdots, x_n) > C_{1\alpha}. \end{aligned}$$

We now are in a position to prove the converse.

THEOREM 4. *If* Φ *is* UMP *for testing* $H:f = f_0$ *vs.* $K:f$ *in* Ω, *where* $\Omega \cup \{f_0\} \subset \Omega_3^*$, *and the sample size* $n \geqq 2$, *then the class of densities* $\Omega \cup \{f_0\}$ *has a* MLR, *with respect to an appropriate parameterization.*

PROOF. If $\Omega = \{f_1\}$, then, with $T(x_1, \cdots, x_n) = \prod [f_1(x_i)/f_0(x_i)]$, the theorem follows trivially. Suppose f_1 and f_2 are in Ω. Then by the fundamental lemma, $\prod [f_2(x_i)/f_0(x_i)] \equiv h(\prod [f_1(x_i)/f_0(x_i)])$, with h continuous and strictly increasing. Also, by the corollary, $f_2(x)/f_0(x) \equiv h_1(f_1(x)/f_0(x))$, h_1 continuous and strictly increasing. We first prove that for $n \geqq 2$, this requires that $h(x) = Cx^\theta$ for appropriate C and θ. We use the notation $L_a^1(x) = f_a(x)/f_0(x)$ with an appropriate index a.

Fix f_1; there exist x_L, x_U such that $L_1^1(x_L) < 1$, $L_1^1(x_U) > 1$, else $f_1(x) \geqq f_0(x)$ for all x or $f_1(x) \leqq f_0(x)$ for all x, either of which is a contradiction. Hence, by the continuity of L_1^1 and the intermediate value theorem there exists an x_0 (between x_L and x_U) such that $L_1^1(x_0) = 1$. Then, if $y = L_1^1(x)$

$$\begin{aligned} h(y) &= h(L_1^1(x)L_1^1(x_0) \cdots L_1^1(x_0)) \\ &= L_2^1(x)(L_2^1(x_0))^{n-1} \\ &= h_1(L_1^1(x))K^{n-1} \\ &= h_1(y)K^{n-1}. \end{aligned}$$

Hence, $h_1(y) = C'h(y)$, $C' = K^{1-n}$. (Of course, $K \neq 0$).

Furthermore, if $y_1 = L_1^1(x_1)$, $y_2 = L_1^1(x_2)$

$$\begin{aligned} h(y_1 y_2) &= h(L_1^1(x_1)L_1^1(x_2)L_1^1(x_0) \cdots L_1^1(x_0)) \\ &= h_1(L_1^1(x_1))h_1(L_1^1(x_2))K^{n-2} \\ &= C'h(y_1)C'h(y_2)K^{n-2} \\ &= (K^{-n})h(y_1)h(y_2). \end{aligned}$$

Let $z = \ln y$, $g(z) = \ln h(\exp z)$; then

$$\begin{aligned} g(z_1 + z_2) &= \ln h(\exp (z_1 + z_2)) \\ &= -n \ln K + \ln h(y_1) + \ln h(y_2) \\ &= (-n \ln K) + g(z_1) + g(z_2) \end{aligned}$$

for all z_1, z_2 in an interval containing 0. This well-known functional equation has as its only continuous solution $g(z) = az + b$; here $b = n \ln K$. It follows that the only continuous solution for h is (3) $h(y) = Cy^\theta$, $C = e^b$, $\theta = a$. Since h is increasing, $\theta > 0$. (Thus $D(\Phi)$ is the "line" through f_0 and f_1.)

Now for fixed f_1 and arbitrary f in Ω, we have $L(x_1, \cdots, x_n) = h(L_1(x_1, \cdots, x_n)) = C(\theta)(L_1(x_1, \cdots, x_n))^\theta$, since C is clearly determined by θ.

Hence, associated with each distribution f in Ω is a real parameter θ. We assign $\theta = 0$ to f_0, $\theta = 1$ to f_1, so equation (3) holds for these densities as well.

If we take $T(x_1, \cdots, x_n) = L_1(x_1, \cdots, x_n)$, then for $\theta_1 < \theta_2$

$$\begin{aligned} \prod [f_{\theta_2}(x_i)/f_{\theta_1}(x_i)] &= L_{\theta_2}(x_1, \cdots, x_n)/L_{\theta_1}(x_1, \cdots, x_n) \\ &= C(\theta_2)(T(x_1, \cdots, x_n))^{\theta_2}/C(\theta_1)(T(x_1, \cdots, x_n))^{\theta_1} \\ &= [C(\theta_2)/C(\theta_1)]T^{(\theta_2-\theta_1)}, \end{aligned}$$

an increasing function of T. This completes the proof.

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