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A NOTE ON CHARACTERISTIC FUNCTIONS

BY K. G. BINMORE AND H. H. STRATTON

State University of New York at Albany and Royal Holloway College, Land University; and State University of New York at Albany

1. Let $F(t)$ be a probability distribution function. Let $\phi(x)$ be its characteristic function, given by

$$\phi(x) = \int_{-\infty}^{\infty} e^{ixt}\, dF(t),$$

and define

$$\psi(x) = 1 - \phi(x). \tag{1.1}$$

We obtain some elementary inequalities for $\psi(X)$ from which we deduce a number of facts about characteristic functions. To provide an application of these results, we prove the following theorem. For $\alpha = 1$, this theorem is contained in a theorem of Pitman [6] and the proof of Boas' Theorem 1 of [1] yields the case $0 < \alpha < 1$. See also Pitman [7].

THEOREM 1. *If $0 < \alpha < 2$, a necessary and sufficient condition that*

$$v^{\alpha} \int_{|t|>v} dF(t) = o(1) \qquad (v \to \infty) \tag{1.2}$$

is that

$$(1 - \Re\phi(u))/u^{\alpha} = o(1) \qquad (u \to 0+). \tag{1.3}$$

Boas' method, in fact, establishes that, for $0 < \alpha < 1$, (1.2) is equivalent to

$$(1 - \phi(u))/u^{\alpha} = o(1) \qquad (u \to 0+) \tag{1.4}$$

and so conditions (1.3) and (1.4) are equivalent for $0 < \alpha < 1$.

COROLLARY 1. *Let S_n denote the sum of n independent random variables each with distribution $F(t)$. If $\alpha > 0$, then (1.4) is a necessary and sufficient condition that*

$$n^{-1/\alpha} Sn \to_p 0 \qquad (n \to \infty). \tag{1.5}$$

REMARKS. (1) Theorem 1 remains true if the o in both (1.2) and (1.3) is replaced by O.

(2) If $\alpha = 2n + \beta$, where n is a positive integer and $0 < \beta < 2$, a necessary and sufficient condition for (1.2) is that

$$\Re\phi^{(2n)}(0) - \Re\phi^{(2n)}(u) = o(u^{\beta}) \qquad (u \to 0+).$$

(3) If $0 < \alpha < 1$ in Corollary 1, then condition (1.4) may be replaced by (1.3). If $\alpha = 1$, this is not true. If $\alpha > 1$, (1.3) implies the existence of the

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mean μ of the distribution $F(t)$ and (1.4) in Corollary 1 may be replaced by (1.3) as long as S_n is replaced by $S_n - n\mu$. If $\alpha = 1$, (1.3) does not imply the existence of μ. (See e.g. Lukacs [5], p. 29).

2. An elementary property of characteristic functions is that they are positive definite, i.e.

$$\sum_{j,k=1}^{n} \phi(t_j - t_k) z_j \bar{z}_k \geqq 0,$$

for any real $t_1, t_2, \cdots t_n$ and complex $z_1, z_2, \cdots z_n$. Since $\phi(0) = 1$, it follows that

$$\left|\sum_{j=1}^{n} z_j\right|^2 \geqq \sum_{j,k=1}^{n} \psi(t_j - t_k) z_j \bar{z}_k,$$

where $\psi(x)$ is defined by (1.1). But $\psi(-x) = \bar{\psi}(x)$, and so

$$\tfrac{1}{2}\left|\sum_{j=1}^{n} z_j\right|^2 \geqq \Re\left\{\sum_{j<k} \psi(t_j - t_k) z_j \bar{z}_k\right\}. \tag{2.1}$$

We take $n = 3$, $x = t_1 - t_2$, $y = t_2 - t_3$, and $x + y = t_1 - t_3$. If $x > 0$ and $y > 0$, the choice $z_1 = y$, $z_2 = -(x + y)$ and $z_3 = x$ yields

$$\Re\psi(x + y)/(x + y) \leqq \Re\psi(x)/x + \Re\psi(y)/y, \tag{2.2}$$

i.e. the function $x^{-1}\Re\psi(x)$ is subadditive on $(0, \infty)$. For any non-negative function $h(x)$ which is subadditive on $(0, \infty)$ we have that

$$yh(y) \leqq 2\int_0^y h(x)\,dx \qquad (y > 0). \tag{2.3}$$

In particular, therefore,

$$\Re\psi(y) \leqq 2\int_0^y x^{-1}\Re\psi(x)\,dx. \tag{2.4}$$

The choice $z_1 = 1$, $z_2 = -(1 + re^{i\theta})$ and $z_3 = re^{i\theta}$, where

$$r = \{\Re\psi(x)/\Re\psi(y)\}^{\frac{1}{2}} \quad \text{and} \quad \theta = \arg\{\psi(x + y) - \psi(x) - \psi(y)\},$$

yields that

$$|\psi(x + y) - \psi(x) - \psi(y)| \leqq 2\{\Re\psi(x)\cdot\Re\psi(y)\}^{\frac{1}{2}}. \tag{2.5}$$

(If $\Re\psi(x) = 0$ or $\Re\psi(y) = 0$, the inequality follows trivially from (2.1) with $n = 3$). From (2.5) we deduce that

$$|(\psi(x + y)|^{\frac{1}{2}} \leqq |\psi(x)|^{\frac{1}{2}} + |\psi/y)|^{\frac{1}{2}}, \tag{2.6}$$

i.e. the function $|\psi(x)|^{\frac{1}{2}}$ *is subadditive on* $(-\infty, \infty)$. A similar argument establishes that $\{\Re\psi(x)\}^{\frac{1}{2}}$ is subadditive on $(-\infty, \infty)$. As an application of (2.6), we have the following.

$$\lim_{t\to 0} |\psi(t)|/t^2 = \sup_{-\infty<t<\infty} |\psi(t)|/t^2. \tag{2.7}$$

(See Theorem 7.1.11 of Hille and Phillips [3], p. 250). The following well-known result is a corollary of (2.7). *If* $\phi(x) = 1 + o(x^2)(x \to 0)$, *then* $\phi(x) \equiv 1$.

In view of the inequality of the arithmetic and geometric means, we obtain further from (2.5) that

$$|\psi(x+y) - \psi(x) - \psi(y)| \leqq \Re\psi(x) + \Re\psi(y).$$

This, in turn, implies that $\Re\psi(x) + \Re\psi(y) \geqq \frac{1}{2}\Re\psi(x+y)$, an inequality which is well-known for the case $x = y$ (see e.g. Lukacs [5], p. 60).

3. Proof of Theorem 1. Using the notation

$$Q_\alpha(v) = v^\alpha p(|x| > v) = v^\alpha \int_{|t|>v} dF(t),$$

we have the following. *Suppose* $0 < \alpha < 2$, *and*

$$\limsup_{v\to\infty} Q_\alpha(v) = l. \tag{3.1}$$

Then

$$\limsup_{u\to 0+} u^{-\alpha}\Re\psi(u) \leqq 2^{3-\alpha} l/\alpha(2-\alpha). \tag{3.2}$$

From (2.4) we have that

$$\begin{aligned}
\Re\psi(u)/u^\alpha &\leqq 2u^{-\alpha} \int_0^u \Re\psi(x)/x\, dx \\
&= 2u^{-\alpha} \int_{-\infty}^{\infty} dF(t) \int_0^u (1 - \cos xt)/x\, dx \\
&= 2u^{-\alpha} \int_{-\infty}^{\infty} dF(t) \int_0^{ut} (1 - \cos y)/y\, dy \\
&= 2 \int_0^\infty (1 - \cos y) y^{-(1+\alpha)} Q_\alpha(y/u)\, dy.
\end{aligned}$$

Since $0 < \alpha < 2$, $(1 - \cos y)/y^{1+\alpha} \,\varepsilon\, \mathfrak{L}(0, \infty)$. Hence

$$\begin{aligned}
\limsup_{u\to 0+} u^{-\alpha}\Re\psi(u) &\leqq 2l \int_0^\infty (1 - \cos y)/y^{1+\alpha}\, dy \\
&\leqq l \int_0^2 y^{1-\alpha}\, dy + 4l \int_2^\infty y^{-1-\alpha}\, dy \\
&= 2^{3-\alpha} l/\alpha(2-\alpha).
\end{aligned}$$

(For an exact formula for $\int_0^\infty y^{-\alpha-1}(1 - \cos y)\, dy$ see Feller's "An introduction to probability theory" vol. II p. 542).

From the Truncation Inequality (see e.g. Loève [4], p. 196), we have that

$$\int_{|t|>v} dF(t) \leqq 14v \int_0^{1/v} \Re\psi(u)\, du \qquad (v > 0).$$

It follows immediately that, *if* $\alpha > 0$, *and*

$$\limsup_{u\to 0+} u^{-\alpha}\Re\psi(u) = m,$$

then

$$\limsup_{v\to\infty} Q_\alpha(v) \leqq 14m.$$

Theorem 1 is now an immediate consequence of the above results.

Before proving Corollary 1, we require a lemma.

Lemma 1. *If* $\alpha < 3$, *and* $Q_\alpha(v) \to 0 (v \to \infty)$, *then*

$$u^{1-\alpha} \int_{-1/u}^{1/u} t\, dF(t) - u^{-\alpha}\, \mathfrak{I}_m\, \phi(u) \to 0 \qquad (u \to 0+).$$

PROOF. We show that, *if* (3.1) *holds and* $\alpha < 3$, *then*

$$(3.3) \quad \limsup_{u\to 0+} \left|u^{1-\alpha}\int_{-1/u}^{1/u} t\,dF(t) - u^{-\alpha}\,\mathcal{I}m\,\phi(u)\right| \leqq ((7-2\alpha)/(6-2\alpha))l.$$

We have that

$$u^{-\alpha}\,\mathcal{I}m\,\phi(u) = u^{-\alpha}\int_{-1/u}^{1/u} \sin ut\,dF(t) + u^{-\alpha}\int_{|t|>1/u} \sin ut\,dF(f),$$

and so

$$(3.4) \qquad \limsup_{u\to 0+} \left|u^{-\alpha}\,\mathcal{I}m\,\phi(u) - u^{-\alpha}\int_{-1/u}^{1/u} \sin ut\,dF(t)\right| \leqq l.$$

But, writing $F(t) + F(-t) = f(t)$, we have

$$\begin{aligned}\left|u^{1-\alpha}\int_{-1/u}^{1/u} t\,dF(t) - u^{-\alpha}\int_{-1/u}^{1/u} \sin ut\,dF(t)\right| &= \left|u^{-\alpha}\int_0^{1/u} (ut - \sin ut)\,dF(t)\right| \\ &= \left|u^{-\alpha}\int_0^{1/u} dF(t)\int_0^{ut} (1-\cos y)\,dy\right| \\ &= \left|u^{-\alpha}\int_0^1 (1-\cos y)\,dy\int_{y/u}^{1/u} dF(t)\right| \\ &\leqq \int_0^1 (1-\cos y)\,y^{-\alpha}Q_\alpha(y/u)\,dy.\end{aligned}$$

Since $\alpha < 3$, $(1-\cos y)/y^\alpha \,\varepsilon\, \mathcal{L}(0, 1)$. Hence

$$(3.5) \quad \limsup_{u\to 0+} \left|u^{1-\alpha}\int_{-1/u}^{1/u} t\,dF(t) - u^{-\alpha}\int_{-1/u}^{1/u} \sin ut\,dF(t)\right| \leqq l\int_0^1 (1-\cos y)/y^\alpha\,dy \leqq l/2(3-\alpha).$$

Inequality (3.3) now follows from (3.4) and (3.5).

PROOF OF COROLLARY 1. Heyde and Rohatgi [2] have shown that, for $0 < \alpha < 2$, (1.5) is equivalent to the following pair of conditions:

$$(3.6) \qquad u^{1-\alpha}\int_{-1/u}^{1/u} t\,dF(t) \to 0 \qquad (u \to 0+);$$

$$(3.7) \qquad Q_\alpha(v) \to 0 \qquad (v \to \infty).$$

For $0 < \alpha < 2$, the corollary is therefore obtained from Theorem 1, Lemma 1 and the above result. If $\alpha \geqq 2$, we noted in section 2 that (1.4) cannot hold unless $\phi(x) \equiv 1$, and this is also true of (1.5) by the Central Limit Theorem.

REMARKS. (1) Heyde and Rohatgi note that, for $\alpha < 1$, (3.6) is implied by (3.7). We note that, if $\alpha > 1$, then (3.7) implies the existence of μ. An easy argument, similar to that employed in the proof of Lemma 1, shows that, if $\alpha > 1$ and (3.7) holds, then

$$(3.8) \qquad u^{1-\alpha}\mu - u^{-\alpha}\,\mathcal{I}m\,\phi(u) \to 0 \qquad (u \to 0+).$$

This result, together with Lemma 1, imply that (3.6) may be replaced by the condition $\mu = 0$, as long as $\alpha > 1$.

(2) From Boas' proof of his Theorem 3 in [1], we have that, for $0 < \alpha < 2$, a necessary and sufficient condition that $\int_{-\infty}^{\infty} |t|^\alpha\,dF(t)$ be finite is that

$$(3.9) \qquad \mathcal{R}\psi(x)/x^{\alpha+1} \,\varepsilon\, \mathcal{L}(0, 1).$$

Taking $h(x) = x^{-\alpha-1}\mathcal{R}\psi(x)$ in (2.3), we see that (3.9) implies that $x^{-\alpha}\mathcal{R}\psi(x) \to 0(x \to 0+)$. It is also true that, for $0 < \alpha < 2$,

$$\psi(x)/x^{\alpha+1} \; \varepsilon \; \mathcal{L}(0, 1) \tag{3.10}$$

implies that $x^{-\alpha}\psi(x) \to 0 \; (x \to 0+)$. When $0 < \alpha < 1$, there is nothing to prove (see comment after Theorem 1). When $\alpha \geqq 1$, then μ exists and $x^{-1}\psi(x) \to -i\mu \; (x \to 0+)$. Therefore, from (3.10), $\mu = 0$. This deals with the case $\alpha = 1$. For $\alpha > 1$, we need to show that $x^{-\alpha} \, \mathcal{I}_m \, \psi(x) \to 0 \; (x \to 0+)$. This follows from (3.8).

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