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INVARIANT PROPER BAYES TESTS FOR EXPONENTIAL FAMILIES[1]

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1. Introduction. Throughout this paper the phrase "invariant Bayes test" used without further qualification will mean a test which is (proper) Bayes among all tests and which is also invariant.

In a variety of problems arising in normal multivariate analysis Kiefer and Schwartz (1965) (hereafter K-S (1965)) have constructed fully invariant proper Bayes tests. In K-S (1965) each problem and each test for a given problem are treated somewhat separately. There is little indication of a general method of constructing *a priori* distributions which yield invariant tests or of the requirements on the problem in order that the method be successful, or of the class of tests which can be constructed in this way.

The present paper is focused on testing problems concerning the parameter of an exponential family of probability densities when the problem remains invariant under a locally compact group. *A priori* distributions are constructed in such a way that the role of the transformation groups leaving the problem invariant is clarified. Verification that the resulting Bayes tests are fully invariant does not depend on an explicit computation of the tests.

The basic idea of this paper arose from the realization that the methods used in K-S (1965) are intimately related to Stein's method of obtaining the probability density of the maximal invariant under a group, G, as an integral over G with respect to Haar measure. Although Stein's representation of the probability density of the maximal invariant motivates the construction of the *a priori* distributions in Theorem 1 of Section 3, neither the construction itself nor the invariance of the resulting Bayes tests depend upon the validity of the representation.

However, when Stein's representation is valid, the Bayes tests obtained in this paper have an interesting interpretation, and this interpretation permits the characterization of a wide class of invariant Bayes tests (Theorem 2).

In Section 2, requisite notation and definitions are given, along with background material on Stein's representation. Section 3 gives the main general results on invariant Bayes tests. One specific example, the MANOVA problem, has been worked out in Schwartz (1966) using explicit computations rather than the general results based on Stein's representation, and that paper also contains a sketch of the general results. (Without the example and background provided by Schwartz (1966) the formulation of Section 3 below will probably seem unmotivated and difficult to follow!) Section 5 is devoted to a second example:

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the problem of testing independence of sets of variates. Section 6 discusses, briefly, the problem of testing equality of proportional covariance matrices for which it is known that every fully invariant test is inadmissible.

2. Notation, definitions and background. Throughout $(X, \mathcal{S}, \mu)$ will be a σ-finite measure space and G will be a locally compact group of transformations on X. Let λ be a fixed (left) Haar measure on the Borel subsets, $\mathcal{B}$, of G.

DEFINITION 1. Let G be a transformation group on X. The measure space $(X, \mathcal{S}, \mu)$ will be called invariant if $A \varepsilon \mathcal{S}$ and $g \varepsilon G$ imply $gA \varepsilon \mathcal{S}$ and $\mu(gA) = \mu(A)$.

DEFINITION 2. Let G be a group of measurable transformations on $(X, \mathcal{S}, \mu)$ and $\{P_\theta : \theta \varepsilon \Theta\}$ a family of probability measures on $\mathcal{S}$ with corresponding densities p_θ wrt μ. The family $\{p_\theta ; \theta \varepsilon \Theta\}$ is invariant if $\theta \varepsilon \Theta$ and $g \varepsilon G$ imply $P_\theta g^{-1} = P_{\theta'}$ for some $\theta' \varepsilon \Theta$.

If $\{p_\theta ; \theta \varepsilon \Theta\}$ is invariant then we shall always assume $\theta \neq \theta'$ implies p_θ and $p_{\theta'}$ are not densities for the same measure. Then, if the random variable X has density p_θ, define a map $g \rightarrow \bar{g}$ by stating that gX has density $p_{\bar{g}\theta}$. The set of all images $\bar{g}$ is a group, $\bar{G}$, called the induced group of transformations on Θ and the map $g \rightarrow \bar{g}$ is a homomorphism.

For $g \varepsilon G$ define λ_g on $\mathcal{B}$ by $\lambda_g(B) = \lambda(Bg)$. Then $(G, \mathcal{B}, \lambda_g)$ is invariant for each g and therefore $\lambda_g = (\Lambda(g))\lambda$. The function $\Lambda(g)$, so defined, is a homomorphism into the positive real numbers and is called the modular function. If $\Lambda(g) \equiv 1$, G is called unimodular.

Let $R \subset X$ contain exactly one representative per orbit. Define $F: X \rightarrow R$ by the condition that $F(x)$ and x belong to the same orbit so that F is a maximal invariant function. Under appropriate conditions there exists a measure γ on R such that if f is a probability density wrt μ, then f^* the probability density of F wrt γ is given by

$$f^*(y) = \int_G f(gy)\, d\lambda(g) = \int_G f(gx)\, d\lambda(g)\Lambda(g_0) \tag{1}$$

where $x = g_0 y$ with $y \varepsilon R$. If f_1 and f_2 are both pdf's wrt μ then from (1) the probability ratio of the maximal invariant is

$$f_1^*(y)/f_2^*(y) = \int_G f_1(gx)\, d\lambda(g) / \int_G f_2(gx)\, d\lambda(g) \tag{2}$$

for every x in the orbit of y.

The expressions (1) and (2) are Stein's representation of the probability density and ratio respectively of the maximal invariant F. (Note that (2) gives the probability ratio of every maximal invariant.) The most useful conditions, in terms of common applications, for the validity of Stein's representation have been given by Wijsman (1967) where, among other results, the validity of (2) is proved for all the usual normal examples. A somewhat different approach is followed in Schwartz (1968).

The validity of (1) and (2) requires among other things that $(X, \mathcal{S}, \mu)$ be invariant. Usually the probability density f will not be given with respect to an invariant measure, but instead with respect to another measure μ_1 (e.g., Lebesgue measure) which is absolutely continuous wrt μ. In order to apply Stein's representation as stated, it is then necessary to compute $d\mu_1/d\mu$ and to write the density wrt μ. Equation (3) below gives an alternate expression for the probability density f^*. If one is concerned only with the probability ratio f_1^*/f_2^*, then use of equation (3) avoids the explicit computation of an invariant measure μ and of $d\mu_1/d\mu$.

Equation (3) also suggests the way in which the methods used in K-S (1965) can be combined with Stein's representation to obtain a general method of constructing invariant proper Bayes tests for exponential problems.

If $\{f_\theta ; \theta \varepsilon \Theta\}$ is an invariant family of probability density function then $(f_\theta)^*$ depends only on θ^*, the orbit of θ. It is convenient to write f_{θ^*} for $(f_\theta)^*$ so that f_α is a probability density on x or R depending on whether $\alpha \varepsilon \Theta$ or $\alpha \varepsilon \Theta^*$.

Let μ_1 be a measure on $(X, \mathcal{S})$ absolutely continuous wrt μ (the latter invariant). Suppose $\{f_\theta ; \theta \varepsilon \Theta\}$ is an invariant family of probability density function wrt μ_1. If $x = g_0 y$ with $y \varepsilon R$ and θ, θ_1, and $\theta_2 \varepsilon \Theta$ then (1) implies

$$f_\theta^*(y) = \Lambda(g_0)(d\mu_1/d\mu)(x) \int_G f_{\bar{g}^{-1}\theta}(x)\, d\lambda(g) \tag{3}$$

and (2) implies

$$(f_{\theta_1^*}/f_{\theta_2^*})(y) = \int_G f_{\bar{g}^{-1}\theta_1}(x)\, d\lambda(g) / \int_G f_{\bar{g}^{-1}\theta_2}(x)\, d\lambda(g). \tag{4}$$

The proof which is given in Schwartz (1968) uses the fact that, since μ is invariant it follows from Lehmann (1959), p. 252, that

$$f_{\bar{g}^{-1}\theta}(x)(d\mu_1/d\mu)(x) = f_\theta(gx)(d\mu_1/d\mu)(gx) \quad \text{a.e. } [\mu]. \tag{4'}$$

3. Invariant Bayes tests. Let $\{P_\theta^m; \theta \varepsilon \Theta, m \varepsilon M\}$ be a family of probability densities wrt μ; it is *not* assumed that μ is invariant. The hypotheses to be tested concern the value of the parameter θ, while the parameter m may be thought of roughly as the sample size. In normal examples M is typically the interval $(p - 1, \infty)$ for which the density $W_m(z)$ defined on the positive definite $p \times p$ matrices by

$$W_m(z) = C|\Sigma|^{-m/2} z^{(m-p-1)/2} \exp\{\text{tr} - \tfrac{1}{2}\Sigma^{-1}z\}$$

is integrable. If m is an integer this is the usual central Wishart density.

ASSUMPTION 1. (i) For each m, $\{p_\theta^m; \theta \varepsilon \Theta\}$ is a G-invariant family and the action on Θ of each $g \varepsilon \bar{G}$, the induced group $\bar{G}$ of transformations, is the same for all m.

(ii) $p_\theta^m(x)$ has the form

$$p_\theta^m(x) = h_m(\theta)\, d_m(x) \exp\{\theta T(x)\} \tag{5}$$

where $T: X \to E^k$ is a measurable function into Euclidan k-dimensional space and Θ is a subset of the linear functionals on E^k.

For each $m \varepsilon M$ let P_m be the following statistical problem: Z is a random variable with density p_θ^m. On the basis of observing Z test

$$H_0 : \theta \varepsilon \Theta_0 \subset \Theta \quad \text{versus} \quad H_1 : \theta \varepsilon \Theta_1 \subset \Theta$$

We assume that P_m remains invariant under G.

Let P_m^* be the problem obtained from P_m after reduction by invariance. That is, P_m^* is the problem of testing

$$H_0^*: \theta^* \varepsilon \Theta_0^* \subset \Theta^* \quad \text{versus} \quad H_1^*: \theta^* \varepsilon \Theta_1^* \subset \Theta^*$$

on the basis of observing Z^*, the orbit of Z.

Let $\tilde{\Theta}$ denote the set of all linear functionals on E^k such that $\tilde{\theta} \varepsilon \tilde{\Theta}$, $\theta \varepsilon \Theta_j \Rightarrow (\tilde{\theta} + \theta) \varepsilon \Theta_j$ for $j = 0, 1$. For $\tilde{\theta} \varepsilon \tilde{\Theta}$ define $W_{\tilde{\theta}} : G \times \Theta \rightarrow \Theta$ by $W_{\tilde{\theta}}(g, \gamma) = \tilde{\theta} + \bar{g}^{-1}\gamma$. Then $W_{\tilde{\theta}}[G \times \Theta_j] \subset \Theta_j$. We shall find it convenient to construct measures $G \times \Theta$ and from these, by use of the map $W_{\tilde{\theta}}$, to induce *a priori* measures on the parameter set Θ. We shall only require that the constructed measures be finite and not necessarily normalized.

Also for $n \varepsilon M$ define $L_n : X \times \Theta \rightarrow E^1 \cup \{+\infty\}$ by

$$L_n(x, \gamma) = \int_G h_n(\bar{g}^{-1}\gamma) \exp\{(\bar{g}^{-1}\gamma)T(x)\}\, d\lambda(g).$$

LEMMA 1. *Under Assumption 1, let $\xi = \xi_0 + \xi_1$ be a probability measure on $\Theta_0 \cup \Theta_1$ with ξ_j supported on Θ_j* for $j = 0, 1$. *Suppose there exist $\tilde{\theta} \varepsilon \tilde{\Theta}$ and $n, m \varepsilon M$ such that*

$$\int_\Theta \int_G h_n(\bar{g}^{-1}\gamma)/h_m(\tilde{\theta} + \bar{g}^{-1}\gamma)\, d\lambda(g)\, d\xi(\gamma) < \infty. \tag{6}$$

Then for each $K \geqq 0$

$$\{x \mid \int_{\Theta_1} L_n(x, \gamma)\, d\xi_1(\gamma)/\int_{\Theta_0} L_n(x, \gamma)\, d\xi_0(\gamma) \leqq K\} \tag{7}$$

is a proper Bayes acceptance region for the problem P_m.

PROOF. For $j = 0, 1$ let Q_j be the measure on $G \times \Theta_j$ defined by

$$Q_j(A) = \int\int \chi_A(g, \gamma) h_n(\bar{g}^{-1}\gamma)/h_m(\tilde{\theta} + \bar{g}^{-1}\gamma)\, d\lambda(g)\, d\xi_j(\gamma).$$

Then Q_j is a finite measure for $j = 0, 1$ by (6) and for the problem P_m the Bayes acceptance region corresponding to the measure induced on Θ from $Q = Q_0 + Q_1$ by $W_{\tilde{\theta}}$ has the form

$$\int_{G\times\Theta} p^m_{W_{\tilde{\theta}}(g,\gamma)}(x)\, dQ_1(g, \gamma)/\int_{G\times\Theta} p^m_{W_{\tilde{\theta}}(g,\gamma)}(x)\, dQ_0(g, \gamma) \leqq K.$$

Substituting from (5) and the definitions of $W_{\tilde{\theta}}$ and Q_j, the left hand side of this last expression becomes

$$[\int_\Theta \int_G h_m(\tilde{\theta} + \bar{g}^{-1}\gamma)\, d_m(x) \exp\{(\tilde{\theta} + \bar{g}^{-1}\gamma)T(x)\} h_n(\bar{g}^{-1}\gamma)/h_m(\tilde{\theta} + \bar{g}^{-1}\gamma)\, d\lambda(g)\, d\xi_1(\gamma)]$$
$$\cdot[\int_\Theta \int_G h_m(\tilde{\theta} + \bar{g}^{-1}\gamma)\, d_m(x) \exp\{(\tilde{\theta} + \bar{g}^{-1}\gamma)T(x)\} h_n(\bar{g}^{-1}\gamma)/h_m(\tilde{\theta} + \bar{g}^{-1}\gamma)\, d\lambda(g)\, d\xi_0(\gamma)]^{-1}$$
$$= [\int_\Theta \int_G h_n(\bar{g}^{-1}\gamma) \exp\{(\tilde{\theta} + \bar{g}^{-1}\gamma)T(x)\}\, d\lambda(g)\, d\xi_1(\gamma)]$$
$$\cdot[\int_\Theta \int_G h_n(\bar{g}^{-1}\gamma) \exp\{(\tilde{\theta} + \bar{g}^{-1}\gamma)T(x)\}\, d\lambda(g)\, d\xi_0(\gamma)]^{-1}$$
$$= [\int_\Theta \int_G h_n(\bar{g}^{-1}\gamma) \exp\{(\bar{g}^{-1}\gamma)T(x)\}\, d\lambda(g)\, d\xi_1(\gamma)]$$
$$\cdot[\int_\Theta \int_G h_n(\bar{g}^{-1}\gamma) \exp\{(\bar{g}^{-1}\gamma)T(x)\}\, d\lambda(g)\, d\xi_0(\gamma)]^{-1}$$
$$= \int_\Theta L_n(x, \gamma)\, d\xi_1(\gamma)/\int_\Theta L_n(x, \gamma)\, d\xi_0(\gamma).$$

The first equation is obtained from obvious cancellations. The second equation makes essential use of the exponential structure in the elimination of $\tilde{\theta}$. The final expression, obtained from the definition of L_n, proves the lemma, since all values of the critical constant K can be obtained by considering the measure $qQ_0 + (1 - q)Q_1$ for $0 \leqq q \leqq 1$.

Lemma 1 provides conditions under which the acceptance region (7) is Bayes. We are interested in conditions which insure that the region (7) is invariant.

ASSUMPTION 2. μ is absolutely continuous wrt a σ-finite invariant measure τ on $\mathcal{S}$.

LEMMA 2. *Under Assumptions* 1 *and* 2 *the acceptance region* (7) *is invariant.*

PROOF.

$$\begin{aligned}
&[\textstyle\int_\Theta L_n(g_0x, \gamma)\, d\xi_1(\gamma)][\textstyle\int_\Theta L_n(g_0x, \gamma)\, d\xi_0(\gamma)]^{-1} \\
&\quad = [\textstyle\int_\Theta \int_G h_n(\bar{g}^{-1}\gamma) \exp \{(\bar{g}^{-1}\gamma)(T(g_0x))\}\, d\lambda(g)\, d\xi_1(\gamma)]^{-1} \\
&\qquad \cdot[\textstyle\int_\Theta \int_G h_n(\bar{g}^{-1}\gamma) \exp \{(\bar{g}^{-1}\gamma)(T(g_0x))\}\, d\lambda(g)\, d\xi_0(\gamma)]^{-1} \\
&\quad = [\textstyle\int_\Theta \int_G p^n_{\bar{g}^{-1}\gamma}(g_0x)(d\mu/d\tau)(g_0x)\, d\lambda(g)\, d\xi_1(\gamma) \\
&\qquad \cdot[\textstyle\int_\Theta \int_G p^n_{\bar{g}^{-1}\gamma}(g_0x)(d\mu/d\tau)(g_0x)\, d\lambda(g)\, d\xi_0(\gamma)]^{-1} \\
&\quad = [\textstyle\int_\Theta \int_G p^n_{\bar{g}_0^{-1}\bar{g}^{-1}\gamma}(x)(d\mu/d\tau)(x)\, d\lambda(g)\, d\xi_1(\gamma)] \\
&\qquad \cdot[\textstyle\int_\Theta \int_G p^n_{\bar{g}_0^{-1}\bar{g}^{-1}\gamma}(x)(d\mu/d\tau)(x)\, d\lambda(g)\, d\xi_0(\gamma)]^{-1} \\
&\quad = [\textstyle\int_\Theta \int_G p^n_{(\bar{g}\bar{g}_0)^{-1}\gamma}(x)\, d\lambda(gg_0)\, d\xi_1(\gamma)\Delta(g_0^{-1})] \\
&\qquad \cdot[\textstyle\int_\Theta \int_G p^n_{(\bar{g}\bar{g}_0)^{-1}\gamma}(x)\, d\lambda(gg_0)\, d\xi_0(\gamma)\Delta(g_0^{-1})]^{-1} \\
&\quad = [\textstyle\int_\Theta \int_G p^n_{\bar{g}^{-1}\gamma}(x)\, d\lambda(g)\, d\xi_1(\gamma)][\textstyle\int_\Theta \int_G p^n_{\bar{g}^{-1}\gamma}(x)\, d\lambda(g)\, d\xi_0(\gamma)]^{-1} \\
&\quad = [\textstyle\int_\Theta L_n(x, \gamma)\, d\xi_1(\gamma)][\textstyle\int_\Theta L_n(x, \gamma)\, d\xi_0(\gamma)]^{-1}.
\end{aligned}$$

In the first equation the definition of L_n is written out. The second equation is obtained by multiplying numerator and denominator by $d_n(g_0x)(d\mu/d\tau)(g_0x)$. The third equation follows from (4′) (with μ_1 and μ in (4′) replaced by μ and τ respectively), which is valid because of Assumption 2. The remaining equations are self-explanatory.

We summarize the results thus far in the following theorem:

THEOREM 1. *Under Assumptions* 1 *and* 2 *let* $\xi = \xi_0 + \xi_1$ *be a probability measure with* ξ_j *supported on* Θ_j *for* $j = 0, 1$. *If there exists* $\tilde{\theta} \in \tilde{\Theta}$ *and* $n, m \in M$ *such that* (6) *holds, then* (7) *is an invariant proper Bayes acceptance region for the problem* P_m.

Although the method used in proving Theorem 1 was suggested by Stein's representation and especially by equation (3), the actual proof does not depend on Stein's representation. If, however, equation (3) is applicable then the test (7) has an interesting interpretation.

As in Section 2 let $R \subset X$ contain exactly one representative of each orbit and let $F: X \to R$ be defined by $(F(x))^* = x^*$.

ASSUMPTION 3. For all $m \varepsilon M$ and $\theta \varepsilon \Theta$, $f_{\theta^*}^m$ is given by equation (3).

Suppose $x = g_0 y$ and $y \varepsilon R$. Then it is easily verified (cf. the computations used in the proof of Lemma 2) that

$$[\int_\Theta L_n(x, \gamma)\, d\xi_1(\gamma)][\int_\Theta L_n(x, \gamma)\, d\xi_0(\gamma)]^{-1}$$
$$= \int_\Theta \int_G \Delta(g_0) p^n_{\bar{g}^{-1}\gamma}(x)\, (d\mu/d\tau)(x)\, d\lambda(g)\, d\xi_1(\gamma)$$
$$\cdot [\int_\Theta \int_G \Delta(g_0) p^n_{\bar{g}^{-1}\gamma}(x) (d\mu/d\tau)(x)\, d\lambda(g)\, d\xi_0(\gamma)]^{-1}$$

Using equation (3), this last expression becomes

$$[\int_\Theta p^n_{\gamma^*}(y)\, d\xi_1(\gamma)][\int_\Theta p^n_{\gamma^*}(y)\, d\xi_0(\gamma)]^{-1}.$$

Hence the acceptance region (7) is identical to the region

$$\{x \mid y = F(x); [\int_\Theta p^n_{\gamma^*}(y)\, d\xi_1(\gamma)][\int_\Theta p^n_{\gamma^*}(y)\, d\xi_0(\gamma)]^{-1} \leqq K\}. \tag{8}$$

For $j = 0, 1$ let ξ_j^* be the measure induced from ξ_j on Θ^* by the map $\gamma \to \gamma^*$. Then (8) is precisely the inverse image under F of

$$\{y \mid y \varepsilon R; [\int_{\Theta^*} p^n_{\gamma^*}(y)\, d\xi_1^*(\gamma^*)][\int_{\Theta^*} p^n_{\gamma^*}(y)\, d\xi_0^*(\gamma^*)]^{-1} \leqq K\} \tag{9}$$

which explicitly shows that the image under F of the invariant region (7) is a Bayes acceptance region for the reduced problem P_n^* corresponding to the *a priori* measure $\xi^* = \xi_0^* + \xi_1^*$.

Conversely, starting with a set of the form (9), by a measurable choice of representatives $V: \Theta^* \to \Theta$ where $(V(\gamma^*))^* = \gamma^*$ one may induce measures ξ_j on Θ_j from ξ_j^*. Then (8) is the inverse image under F of (9) and, under Assumption 3, (7) and (8) are identical. If, in addition, Assumptions 1 and 2 hold and there exists $\tilde{\theta} \varepsilon \tilde{\Theta}$ and $m \varepsilon M$ such that (6) holds for $\xi = \xi_0 + \xi_1$ (with ξ_j obtained from ξ_j^* by means of V) then the inverse image under F of the region (9) (which is Bayes for P_n^*) is Bayes for P_m.

When ξ_j is obtained from ξ_j^* by means of V the condition (6) can be written as

$$\int_{\Theta^*} \int_G [h_n(\bar{g}^{-1}V(\gamma^*))][h_m(\tilde{\theta} + \bar{g}^{-1}V(\gamma^*))]^{-1}\, d\lambda(g)\, d\xi^*(\gamma^*) < \infty. \tag{9'}$$

(We remark, incidentally, that any method of "lifting" the measure ξ^* to Θ' such that $*: \Theta \to \Theta^*$ gives ξ^* back again, can be used in the preceding derivation.)

Again, we summarize:

THEOREM 2. *Let* $\xi^* = \xi_0^* + \xi_1^*$ *be a measure on* Θ^* *and* (9) *the corresponding Bayes acceptance region for the problem* P_n^*. *Let* $V: \Theta^* \to \Theta$ *be a measurable choice of representatives. If under Assumptions* 1, 2 *and* 3 *for* $m \varepsilon M$ *there exists* $\tilde{\theta} \varepsilon \tilde{\Theta}$ *such that* (9′) *holds, then the inverse image under* F *of the region* (9) *is an* (*invariant*) *proper Bayes acceptance region for* P_m.

Since every Bayes acceptance region for P_n^* has the form (9) we have

COROLLARY 1. *If for some* $m \varepsilon M$ *the assumptions of Theorem* 2 *are satisfied* *for every* ξ^*, *then every Bayes acceptance region for* P_n^* *is mapped by* F^{-1} *into a Bayes acceptance region for* P_m.

In Schwartz (1966) it was shown that the conclusion of Corollary 1 applies, for appropriate n and m, to the MANOVA problem and in Section 5 a comparable result will be given for the problem of testing independence of sets of variates.

In the special case that G acts transitively on Θ_0 then, with $\theta_0 \varepsilon \Theta_0$, (9) has the form

$$\{y \mid y \varepsilon R, \textstyle\int_{\Theta^*} p_{\gamma^*}^n(y)/p_{\theta_0^*}^n(y)\, d\xi_i^*(\gamma^*) \leqq K\}. \tag{10}$$

Since the probability ratio does not depend on the choice of a maximal invariant function, (10) may be rewritten as

$$\{x^* \mid \textstyle\int_{\Theta^*} (p_{\gamma^*}^n/p_{\theta_0^*}^n)(x^*)\, d\xi_1^*(\gamma^*) \leqq K\}. \tag{11}$$

Let $\xi_1 = \xi_1^* V^{-1}$ as before. If both equation (2), which is Stein's representation of the probability ratio, and Assumption (2) are valid, then according to equation (4), (11) is the image under the map $*$ of

$$\{x \mid \textstyle\int_{\Theta^*} [\int_G p_{\bar{g}^{-1}V(\gamma^*)}^n(x)\, d\lambda(g)][\int_G p_{\bar{g}^{-1}\theta_0}^n(x)\, d\lambda(g)]^{-1}\, d\xi_1(\gamma) \leqq K\}. \tag{12}$$

We conclude that if Assumption 3 is replaced by the assumption that G acts transitively on Θ_0 and that equation (2) is valid for all $\theta_1, \theta_2 \varepsilon \Theta$ then the conclusions of Theorem 2 and Corollary (1) still hold.

4. Further comments. We conclude the general discussion with some miscellaneous comments which may aid in interpreting the results of Section 3.

(a) The generality in considering an arbitrary locally compact group may be more apparent than real. Invariance of an exponential family may imply that the induced group, $\bar{G}$, is a group of linear-affine transformations (though G need not be). Compare Lehmann and Stein (1953).

(b) In the MANOVA example and in testing independence M is typically chosen to be the open interval $(p - 1, \infty)$ corresponding to the range of values of m in the density function

$$c|\Sigma|^{-m/2}|s|^{\frac{1}{2}(m-p-1)} \exp\{-\tfrac{1}{2} \operatorname{tr} \Sigma^{-1}s\}.$$

(If m is an integer this is the usual central Wishart density and we shall continue to refer to m as the error degrees of freedom even when m is not an integer.) Corollary 1 applies to these examples and its conclusion can be roughly restated as follows: If a procedure is Bayes among invariant tests when the error degrees of freedom is n then it is Bayes among all tests whenever the errors degrees of freedom, m, is such that $m - n > k$ (where k depends on p).

(c) In the MANOVA example each positive definite $p \times p$ matrix corresponds to a suitable choice of $\tilde{\theta}$ in Lemma 1. In the problem of testing independence each positive definite matrix belonging to the null hypothesis corresponds to a suitable choice of $\tilde{\theta}$. It should be noted that, for given $\xi = \xi_0 + \xi_1$, different choices of $\tilde{\theta}$ (satisfying (6)) alter the *a priori* distribution $Q_0 + Q_1$ but do not affect the test statistic in (7).

(d) There is a traditional method (usually applied when $\bar{G}$ acts transitively on both Θ_0 and Θ_1) of constructing invariant *improper* Bayes tests by choosing an invariant measure as the *a priori* distribution. The method of Section 3 is similar but involves the crucial modification of representing a parameter point θ by $\theta = \tilde{\theta} + \bar{g}^{-1}\gamma$ (in contradistinction to $\theta = \bar{g}^{-1}\gamma$). This was illustrated explicitly for the MANOVA problem in Schwartz (1966).

(e) For the problems occurring in normal multivariate analysis the Bayes test corresponding to a given *a priori* distribution is essentially unique and therefore admissible. Of course the traditional method of using improper *a priori* distributions does not yield a proof of admissibility of the resulting test.

(f) It often happens that, for each m, $\{p_\theta^m; \theta \varepsilon \Theta\}$ is an invariant family under some larger group, $G_1 \supset G$, of transformations on X but that Θ_0 and Θ_1 are not invariant under $\bar{G}_1$. Assume that the measure τ of Assumption 2 is G_1-invariant. Let λ_1 denote (left) Haar measure on G_1 and suppose that $\lambda_1\{g_1 \mid \bar{g}_1^{-1}\theta \not\varepsilon \Theta_1\} = 0$ for all $\theta \varepsilon \Theta_1$. Define $L_n': X \times \Theta_1 \rightarrow E^1 \cup \{+\infty\}$ by

$$L_n'(x, \gamma) = \int_{G_1} h_n(\bar{g}_1^{-1}\gamma) \exp\{(\bar{g}_1^{-1}\gamma)T(x)\}\, d\lambda_1(g_1).$$

If in the proof of Lemma 1 we substitute, instead of the measure Q_1 on $G \times \Theta_1$, the measure Q_1' on $G_1 \times \Theta_1$ defined by

$$Q_1'(A) \int\int \chi_A(g_1, \gamma) h_n(\bar{g}_1^{-1}\gamma)/h_m(\tilde{\theta} + \bar{g}_1^{-1}\gamma)\, d\lambda_1(g_1)\, d\xi_1(\gamma),$$

then the measure induced on Θ from Q_1' by the map $(g_1, \gamma) \rightarrow \tilde{\theta} + \bar{g}_1^{-1}\gamma$ assigns all measure to Θ_1. Let Q_0 be the measure defined in Lemma 1. If Q_1' is finite the measure on Θ which is the sum of the measures induced by Q_1' and Q_0 is a proper *a priori* distribution. The corresponding Bayes acceptance region has the form

$$\{x \mid \int_{\Theta_1} L_n'(x, \gamma)\, d\xi_1(\gamma) / \int_{\Theta_0} L_n(x, \gamma)\, d\xi_0(\gamma) \leqq K\}. \tag{7'}$$

We note that if G acts transitively on Θ_0 and G_1 acts transitively on Θ_1, the family of regions (7′) (for different K) is independent of $\xi = \xi_0 + \xi_1$. This is the case for both examples of the next chapter and in both instances the region (7′) turns out to be the likelihood ratio test. Of course, if (7′) is a proper Bayes test, it is *a fortiori* Bayes among invariant tests and when Corollary 1 applies it can be obtained by a construction of the type used in Section 3 (but it may be hard to guess directly the *a priori* distribution of this type which yields (7′)).

The situation $\lambda_1\{g_1 \mid \bar{g}_1\theta \not\varepsilon \Theta_1\} = 0$ for all $\theta \varepsilon \Theta_1$ might be loosely described by saying G_1 almost leaves Θ_1 invariant. In testing independence the hypothesis of non-independence is almost invariant under all linear transformations. Similarly, in the MANOVA problem the alternative hypothesis $EY \neq 0$ is almost invariant under all linear-affine transformations.

5. Testing the independence of sets of variates. In this section the general results of Section 3 are applied to the problem of testing independence. Some further notation is required.

If A is a square matrix, the determinant of A, the trace of A, the transpose of

A and the exponential of the trace of A will be denoted by $|A|$, $\operatorname{tr} A$, A' and $\operatorname{etr} A$ respectively. The $p \times p$ identity matrix will be denoted by I_p.

The random matrices $U(p \times s)$ and $Z(p \times p)$ are independent and distributed as follows: The columns of U are independent normal p-vectors with common unknown covariance matrix Σ and the expectation of U is unknown. Z has a Lebesgue probability density $W_m(z)$ on the positive definite $p \times p$ matrices given by

$$W_m(z) = C|\Sigma|^{-m/2}|z|^{(m-p-1)/2} \operatorname{etr}\{-\tfrac{1}{2}\Sigma^{-1}z\} \tag{13}$$

where C is a normalizing constant and $m > p - 1$. (If m is an integer Z is the usual central Wishart variable. Consideration of all real $m > p - 1$ will permit a more unified statement of the results.)

The covariance matrix Σ is decomposed as

$$\Sigma = \begin{Bmatrix} \Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1k} \\ \Sigma_{12}' & \Sigma_{22} & \cdots & \Sigma_{2k} \\ \vdots & \vdots & & \vdots \\ \Sigma_{1k}' & & \cdots & \Sigma_{kk} \end{Bmatrix}$$

where Σ_{ij} is $p_i \times p_j$ with $\sum_{i=1}^{k} p_i = p$. The hypotheses to be tested are

$$H_0 : \Sigma_{ij} = 0 \quad \text{for } i \neq j \qquad \text{versus} \qquad H_1 : \Sigma_{ij} \neq 0 \quad \text{for some pair } (i, j).$$

We have reduced the usual problem statement by sufficiency. In the usual situation $s = 1$ and U is a multiple of the sample mean. However, for any s, Lemma 3.1 of K-S (1965) will apply to all of the *a priori* distributions constructed below. The inport of this lemma is that any test which is Bayes wrt an *a priori* distribution of appropriate form when $s = 0$ (i.e., U absent) is also Bayes for all values of s. Henceforth we treat the case where U is absent; the general case then follows from an application of the lemma.

Let $GL(p)$ denote the full linear group of non-singular $p \times p$ matrices and let

$$\begin{aligned} S_p &= \{p \times p \text{ symetric matrices}\}, \\ P_p^+ &= \{p \times p \text{ positive definite symmetric matrices}\} \\ E^k &= \text{Euclidean } k\text{-dimensional space}. \end{aligned}$$

The problem remains invariant under the closed subgroup G of $GL(p)$ consisting of all matrices C of the form

$$C = \begin{pmatrix} A_1 & & & \\ & A_2 & & 0 \\ & & \ddots & \\ 0 & & & A_k \end{pmatrix} \tag{14}$$

where $A_i \varepsilon GL(p_i)$. G is isomorphic to the direct product $GL(p_1) \times \cdots \times$

$GL(p_k)$, so that Haar measure on G is the corresponding product of the Haar measures on $GL(p_i)$. The action of an element $A \varepsilon G$ on $z \varepsilon P_p^+$ is given by AzA'.

We proceed now to investigate the validity of the assumptions made in proving Theorems 1 and 2 of Section 3. Letting $\Gamma = \Sigma^{-1}$ and using Γ to denote both a parameter and the linear functional on S_p defined by $\Gamma z = -\frac{1}{2} \operatorname{tr} \Gamma z$, it is readily seen that (13) has the form (5) of Section 3 and that Assumption 1 is satisfied.

Letting dz denote Lebesgue measure on P_p^+, it is well-known, e.g. Anderson ((1958), p. 162), that the measure $|z|^{-(p+1)/2}\, dz$ is invariant under all transformations of the form $z \rightarrow CzC'$ with $C \varepsilon GL(p)$. Hence Assumption 2 of Section 3 is valid.

G acts transitively on H_0. Also it follows from Wijsman (1967) that the probability ratio of the maximal invariant under G is given by Stein's representation (equation 2). As noted in the final paragraph of Section 3, the transitivity of G on H_0 and the validity of equation 2 may be substituted for Assumption 3 in Theorem 2 and Corollary 1.

Let $\xi = \xi_0 + \xi_1$ be a probability measure with ξ_j supported on H_j for $j = 0, 1$. It remains to investigate the conditions under which (6) and (9′) hold. Note first that if $\tilde{\Gamma}$ belongs to H_0 (or its closure) then $\Gamma \varepsilon H_j$ implies $\tilde{\Gamma} + \Gamma \varepsilon H_j$ for $j = 0, 1$. (Clearly this would not be so for $\tilde{\Gamma} \varepsilon H_1$). We hereafter assume that $\tilde{\Gamma} \varepsilon H_0$.

The specific form which the left hand side of (6) assumes in the present case is

$$(15) \quad \int_{P_p^+} \int_G |C^{-1'}\gamma C^{-1}|^{n/2} |\tilde{\Gamma} + C^{-1'}\gamma C^{-1}|^{-(m/2)} \textstyle\prod_{i=1}^k |A_i|^{-p_i} \prod_{i=1}^k dA_i\, d\xi(\gamma)$$

where C is given by (14) and dA_i denotes Lebesgue measure on $E^{p_i^2}$. The measure $\prod_{i=1}^k |A_i|^{-p_i} \prod_{i=1}^k dA_i$ is both left and right invariant and therefore inverse invariant. It is also transpose invariant. Hence, the inner integral in (15) is identical to

$$(16) \quad \int_G |C\gamma C'|^{n/2} |\tilde{\Gamma} + C\gamma C'|^{-(m/2)} \textstyle\prod |A_i|^{-p_i} \prod dA_i .$$

There is $C_0 \varepsilon G$ such

$$(17) \quad C_0 \gamma C_0' = \begin{pmatrix} I_{p_1} & & \psi \\ & \ddots & \\ \psi' & & I_{p_k} \end{pmatrix} = \gamma^* \quad \text{(say)}$$

where ψ is an array (non-rectangular for $k > 2$) required to fill out the matrix. The matrix C_0 is not uniquely determined by (17), nor is γ^*; if H is any orthogonal matrix belonging to G then $HC_0\gamma C_0'H'$ also has the form specified in (17). However $|\gamma^*|$ is uniquely determined by (17). For our purposes any C_0 satisfying (17) will do.

Since $\tilde{\Gamma} \varepsilon H_0$ the substitution of $\tilde{\Gamma}^{\frac{1}{2}} CC_0$ for C in (16) (where $\tilde{\Gamma}^{\frac{1}{2}}$ is the positive square root of $\tilde{\Gamma}$) transforms (16) into

$$(18) \quad |\tilde{\Gamma}|^{(n-m)/2} \int_G |C\gamma^* C'|^{n/2} |I_p + C\gamma^* C'|^{-(m/2)} \textstyle\prod_{i=1}^k |A_i|^{-p_i} \prod_{i=1}^k dA_i .$$

Lemma 3. *If $C \varepsilon G$ and γ^* has the form* (17), *then*

$$|I_p + C\gamma^*C'| \geqq |\gamma^*||I_p + CC'| = |\gamma^*| \prod_{i=1}^k |I_{p_i} + A_iA_i'|.$$

Proof. We prove by induction on k. If $k = 1$ then $p_1 = p$ and $\gamma^* = I_p$ so that the result is immediate. Write

$$\gamma^* = \begin{pmatrix} I_{p_1} & V \\ V' & U \end{pmatrix}$$

where U is $(p - p_1) \times (p - p_1)$ and is of the form (17) so that the induction hypothesis may be applied to U. Let C_1 be the lower right hand $(p - p_1) \times (p - p_1)$ minor of C. We have

$$\begin{aligned}
|I_p + C\gamma^*C'| &= \begin{vmatrix} I_{p_1} + A_1 A_1' & A_1VC_1' \\ C_1V'A_1' & I_{p-p_1} + C_1UC_1' \end{vmatrix} \\
&= |I_{p_1} + A_1A_1'|\,|I_{p-p_1} + C_1UC_1'| \\
&\quad \cdot |I_{p_1} - (I_{p_1} + A_1A_1')^{-1}A_1VC_1'(I_{p-p_1} + C_1UC_1')^{-1}C_1V'A_1'| \\
&\geqq |I_{p_1} + A_1A_1'|\,|I_{p-p_1} + C_1UC_1'|\,|I_{p_1} - VUV'| \\
&\geqq |I_{p_1} + A_1A_1'|\,|I_{p-p_1} + C_1C_1'|\,|U|\,|I_{p_1} - VUV'| \\
&= |I_{p_1} + A_1A_1'|\,|I_{p-p_1} + C_1C_1'|\,|\gamma^*| \\
&= |I_p + CC'|\,|\gamma^*|,
\end{aligned}$$

which completes the proof.

Lemma 4. *If q_i, $i = 1, \cdots, k$, are non-negative and $\sum q_i = 1$ then, for all $w > 0$,*

$$|I_p + C\gamma^*C'|^w \geqq \prod_{i=1}^k |I_{p_i} + A_iA_i'|^{wq_i}.$$

Proof. It is easily checked that $|I_p + C\gamma^*C'| \geqq |I_{p_i} + A_iA_i'|$, for all $i = 1, \cdots, k$, from which the result follows immediately.

Lemma 5. *Suppose $n >$ max $p_i - 1$. Let ξ be a probability measure on P_p^+ and let $b =$ sup $\{b' \mid |\gamma^*|^{(n-b')/2}$ is ξ-integrable$\}$. Let $b_0 =$ min $[b, m]$. If*

(19A) $$(m - b_0) > \sum_{i=1}^k [p_i - 1 + n - b_0]^+$$

or

(19B) $$m = b_0 \text{ and } m - n > \max p_i - 1$$

then (13) *is finite.*

Proof. We first note that (19A) or (19B) is satisfied iff there exist $q_i \geqq 0$ with $\sum_1^k q_i = 1$ such that

(19C) $$q_i(m - b_0) + b_0 - n > p_i - 1, \qquad i = 1, \cdots, k.$$

Since γ is a correlation matrix $|\gamma^*| \leqq 1$, so that $b_0 \geqq n$. Choose b_1 as follows: If $b_0 = n$ then $b_1 = n$; if $b_0 > n$ choose b_1 such that $n < b_1 < b_0$ and $q_i(m - b_1) + (b_1 - n) > p_i - 1$ for $i = 1, \cdots k$.

Using Lemmas 3 and 4 the integral of (18) is bounded above by

$$\int |\gamma^*|^{(n-b_1)/2}|CC'|^{n/2} \prod_{i=1}^k |A_i|^{-p_i}[|I_p + CC'|^{b_1/2}|I_p + C\gamma^*C'|^{(m-b_1)/2}]^{-1} \prod_{i=1}^k dA_i$$
$$\leqq |\gamma^*|^{(n-b_1)/2} \int |CC'|^{n/2} \prod_{i=1}^k |A_i|^{-p_i} dA_i$$
$$\cdot [|I_p + CC'|^{b_1/2} \prod_{i=1}^k |I_{p_i} + A_iA_i'|^{q_i(m-b_1)/2}]^{-1}$$
$$= |\gamma^*|^{(n-b_1)/2} \prod_{i=1}^k \int | A_iA_i'|^{(n-p_i)/2}[|I_{p_i} + A_iA_i'|^{(b_1+q_i(m-b_1))/2}]^{-1} dA_i$$

From (3.7) of K-S (1965) and corresponding considerations near $|A_iA_i'| = 0$ the ith factor in the last displayed expression is finite iff $p_i - 1 < n$ and $b_1 + q_i(m - b_1) - (n - p_i) > 2p_i - 1$. Since $n > \max p_i - 1$ and (19C) holds each factor is finite. Finally $|\gamma^*|^{(n-b_1)/2}$ is ξ-integrable and therefore (15) is finite.

We remark on one special case of (19A). If $b = b_0 = n$ then (19A) becomes $m - n > p - k$. Since $b_0 \geqq n$ this shows that

$$\max p_i - 1 < n < m - p + k \tag{20}$$

is a sufficient condition for the finiteness of (15) for all ξ.

When $n > \max p_i - 1$ and either (19A) or (19B) is satisfied then (6) is satisfied and Theorem 1 applies and yields

THEOREM 3. *Let ξ be a probability measure on $H_0 \cup H_1$ and let $n > \max p_i - 1$. Suppose either* (19A) *or* (19B) *is satisfied. Then, writing $\Gamma = \tilde{\Gamma} + C^{-1'}\gamma C^{-1}$, the* a priori *measure on $G \times P_p^+$ defined by*

$$|C^{-1'}\gamma C^{-1}|^{n/2}|\tilde{\Gamma} + C^{-1'}\gamma C^{-1}|^{-(m/2)} \prod_{i=1}^k |A_i|^{-p_i} \prod_{i=1}^k dA_i \, d\xi(\gamma)$$

is finite and the corresponding Bayes test is invariant under G.

Similarly, Theorem 2 may be applied. For a maximal invariant in the parameter space we may choose a function $\rho(\Sigma)$ whose range is contained in the set of positive definite matrices having the form (17) and such that Σ and $\rho(\Sigma)$ belong to the same orbit. If ξ^* is a probability measure on the range of ρ let $b^* = \sup \{b' \mid |\rho|^{n-b'/2}$ is ξ^*-integrable$\}$. If for a given m, letting $b_0^* = \min [m, b^*]$ we have

(19*) Either (19A) or (19B) is satisfied with b_0 replaced by b_0^*,

then (9′) of Section 3 is satisfied. We then have

THEOREM 4. *Let ξ^* be a probability measure on the range space of the maximal invariant ρ. Let $n > p - 1$ and let φ be the Bayes acceptance region corresponding to ξ^* for the reduced problem with n degrees of freedom. If for a given m,* (19*) *is satisfied then φ is a proper Bayes acceptance region for the original (unreduced) problem with m degrees of freedom. In particular φ is a proper Bayes acceptance region whenever $m > n + p - k$.*

6. Remarks on the problem of testing independence. (a) All of the Bayes tests constructed in Section 5 are essentially unique and therefore admissible. For a fixed ξ the resulting test statistic does not depend on the choice of $\tilde{\Gamma} \varepsilon H_0$ in the representation $\Gamma = \tilde{\Gamma} + C^{-1'}\gamma C^{-1}$ although two different choices of $\tilde{\Gamma}$ result in two different *a priori* distributions.

(b) The problem of testing independence offers a second example for the application of Remark (f) of Section 4.

Although $GL(p)$ does not leave the independence problem invariant it almost leaves H_1 invariant. If $\Gamma \varepsilon P_p^+$ then the Haar measure of $\{g \mid g \varepsilon GL(p), g\Gamma g' \not\varepsilon H_1\}$ is zero. Hence, under H_1 we may write $\Gamma = \tilde{\Gamma} + g^{-1'}\gamma g^{-1}$ where $\tilde{\Gamma} \varepsilon H_0$, $\gamma \varepsilon P_p^+$ and $g \varepsilon GL(p)$. Under H_0, γ is restricted to H_0 and g is restricted G.

Suppose $GL(p) \times P_p^+$ is assigned the *a priori* measure

$$|g\gamma g'|^{n/2}|\tilde{\Gamma} + g\gamma g'|^{-(m/2)}|gg'|^{-p/2}\, dg\, d\xi_1(\gamma) \tag{21}$$

where dg is Lebesgue measure on E^{p^2} and ξ_1 is a finite measure. Since $GL(p)$ acts transitively on P_p^+, the measure on the parameter space induced from (21) by $(g, \gamma) \rightarrow \tilde{\Gamma} + g^{-1'}\gamma g^{-1}$ is independent of ξ and by the previous paragraph is supported on H_1. From (3.7) of K-S (1965) it is finite if $p - 1 < n < m + p - 1$.

Similarly $G \times H_0$ can be assigned the measure

$$|C^{-1'}\gamma C^{-1}|^{n/2}|\tilde{\Gamma} + C^{-1'}\gamma C^{-1}|^{-(m/2)} \textstyle\prod_{i=1}^{k} |A_i|^{-p_i}\, dA_i\, d\xi_0(\gamma) \tag{22}$$

where C is defined in (14) and ξ_0 is a finite measure on H_0. Again, since G acts transitively on H_0, the measure induced on the parameter space from (22) is independent of ξ_0. It is supported on H_0 and is finite if $p - 1 < n < m + p - 1$ since it is finite even under the less restrictive condition $\max p_i - 1 < n < m + \max p_i - 1$.

The Bayes test corresponding to the sum of the measures induced from (21) and (22) is the likelihood ratio test.

(c) In the special case $p = k$, (19C) reduces to $m - n > 0$. Clearly this is also a necessary condition for the finiteness of (15). For $k < p$ it is not known whether (19C) (and therefore (19A) and (19B)) might be weakened, though it appears somewhat doubtful.

7. Testing the equality of proportional covariance matrices. Let Z_1 and Z_2 be independent Wishart matrices with parameters Σ_1 and $\Sigma_2 = k\Sigma_1$ respectively with $\Sigma_1 \varepsilon P_p^+$ unknown. The problem of testing $k = 1$ vs. $k \neq 1$ remains invariant under all linear transformations. The best invariant test is inadmissible. (See K-S (1965), Section 7 (ii).) Therefore this test cannot be Bayes and it may be of interest to determine which of the assumptions of Section 3 fail.

Writing $\Gamma_1 = \Sigma_1^{-1}$, $\Gamma_2 = k^{-1}\Gamma_1$, the problem is to test $k^{-1} = 1$. If $\tilde{\Gamma} \varepsilon P_p^+$ then $\tilde{\Gamma} + \Gamma_1$ is not proportional to $\tilde{\Gamma} + \Gamma_2$ for all Γ_1 of the form $k\Gamma_2$. In terms of the notation used in Section 3 the set $\tilde{\Theta}$ consists only of the linear functional which is identically zero.

All of the other assumptions are satisfied. Indeed, if H_1 is enlarged (say) to the hypothesis $\Sigma_1 \neq \Sigma_2$, then the results of Section 3 can be applied.

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