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THE ORTHANT PROBABILITIES OF FOUR GAUSSIAN VARIATES

By M. C. Cheng

University of Adelaide

1. Introduction. Let $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$ and $\mathbf{X}_4$ be four normal variates with zero means, unit variances and correlation matrix $\{\rho_{ij}\}$. The orthant probability i.e. the probability that all X_i's are positive has not in general been expressed in closed form. However, several series expansions have been given, see for examples [9], [11], [15]. In many cases, these expansions converge slowly, so that some alternative approach appears desirable, David [6].

When the correlation matrices of the quadrivariate are of certain specific forms (e.g. when $\rho_{ij} = \rho$ for all $i \neq j$), the orthant probabilities have been tabulated [1], [8], [18], and approximate expressions [2], [12], [16], [20], [21] for the orthant probabilities have also been given. In particular, Abrahamson [1] has shown that the general orthant probability of the quadrivariate can be expressed as a linear combination of six orthoscheme probabilities. She has also derived a relation between one particular orthoscheme probability and the orthant probability of the equicorrelated case.

David [6] and, earlier, Schläfli [19] derived a recurrence relation which permits the orthant probability of five normal variates to be expressed in terms of the orthant probabilities of one, two, three and four normal variates. In general, they showed that the $(2n + 1)$-dimensional orthant probability is expressible in terms of lower dimensional ones.

This paper describes a method which leads to closed form expressions for the orthant probabilities of the quadrivariate in terms of the inverse trigonometric functions, the dilogarithm function [10] and its real part [10], when the correlation matrices are of specific forms. The dilogarithm function and its real part have been studied and tabulated, see, for example, Lewin [10]. The orthant probability of five normal variates is deduced when the off diagonal elements of the correlation matrix are equal.

2. Orthant probabilities of the quadrivariate normal distribution. A method, similar to that used by Cheng [3], [4] will now be used to find the closed form expressions of the orthant probabilities for the quadrivariate normal distribution having certain specific forms of correlation matrices. The orthant probability of n Gaussian variates is denoted by P_n in each case.

(i) Consider the following correlation matrix.

$$C = \begin{bmatrix} 1 & a & ab & ab \\ a & 1 & ab & ab \\ ab & ab & 1 & a \\ ab & ab & a & 1 \end{bmatrix}, \tag{2.1}$$

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where $-\frac{1}{3} < a < 1$ and $|b| \leqq 1$. The orthant probability of the four Gaussian variates with the above correlation matrix C can then be denoted by $\Phi(a, b)$ i.e. $\Phi(a, b) = P[X_1 > 0, X_2 > 0, X_3 > 0, X_4 > 0/C]$. From the identity

$$(2.2) \qquad \Phi(a, b) = \Phi(0, 0) + \Phi(a, 0) - \Phi(0, 0) + \Phi(a, b) - \Phi(a, 0),$$

$\Phi(a, b)$ can be expressed as

$$(2.3) \qquad \Phi(a, b) = \Phi(a, 0) + \int_{\beta=0}^{b} (\partial/\partial\beta)\Phi(a, \beta)\, d\beta.$$

Denoting $\Phi(a, \beta) = \Phi^*(\rho_{12}, \rho_{13}, \rho_{14}, \rho_{23}, \rho_{24}, \rho_{34}) = \Phi^*$, we have

$$(2.4) \quad (\partial/\partial\beta)\Phi(a, \beta) = a\{(\partial/\partial\rho_{13})\Phi^* + (\partial/\partial\rho_{14})\Phi^* + (\partial/\partial\rho_{23})\Phi^* + (\partial/\partial\rho_{24})\Phi^*\}.$$

Applying Plackett's reduction formula [17], the integral in (2.3) can be evaluated as

$$\Phi(a, b) = \tfrac{1}{16} + \{\arcsin(a) + 2\arcsin(ab)\}/4\pi + [\arcsin(a)]^2/4\pi^2 - \pi^{-2}\int_0^{ab} (1 - x^2)^{-\frac{1}{2}} \arcsin g_1(x)\,dx$$

where

$$g_1(x) = \{x[1 - 2/(1 + a)]\,[1 - 2x^2/(1 + a)]^{-1}\}.$$

Using (A10) of the appendix yields

$$(2.5) \qquad \begin{aligned} \Phi(a, b) = \tfrac{1}{16} &+ \{\arcsin(a) + 2\arcsin(ab)\}/4\pi \\ &+ \{[\arcsin(a)]^2 - 2[\arcsin(ab)]^2\}/4\pi^2 \\ &+ \{2Li_2[f, \arccos(ab)] + \tfrac{1}{2}Li_2[-f^2] - Li_2[f^2, \arccos(a)]\}/\pi^2, \end{aligned}$$

where

$$(2.6) \qquad f = f(a, b) = \{(1 + a) - [(1 + a)^2 - 4a^2b^2]^{\frac{1}{2}}\}/2ab, \quad \text{with} \quad f(0, b) = f(a, 0) = 0;$$

$Li_2[x]$ and $Li_2[r, \theta]$ (following Lewin's notation [10]) is the dilogarithm function and its real part defined in the appendix by (A5) and (A6) respectively. It can be shown that P_4 is given by (2.5) for the region $|b| \leqq 1$, $-\frac{1}{3} \leqq a \leqq 1$. Outside this region, C is not a correlation matrix and $\Phi(a, b)$ is complex valued.

David and Mallows [7] gave a series expansion of this orthant probability for $a = \frac{1}{2}$ and $b = \rho$.

By setting $b = 1$, we obtain readily the orthant probability of the equicorrelated case which has received considerable attention [8], [12], [18], [20], [21], [22]. We have

$$(2.7) \qquad \begin{aligned} P_4 = \Phi(a, 1) &= \Psi(a) \\ &= \tfrac{1}{16} + [3\arcsin(a)]/4\pi - [\arcsin(a)]^2/4\pi^2 \\ &\quad + \{2Li_2[m, \arccos(a)] + \tfrac{1}{2}Li_2[-m^2] - Li_2[m^2, \arccos(a)]\}/\pi^2 \end{aligned}$$

where

$$(2.8) \quad m = m(a) = [1 + a - (1 + 3a)^{\frac{1}{2}}(1 - a)^{\frac{1}{2}}]/2a, \quad \text{with} \quad m(0) = 0, \quad \text{and} \quad -\tfrac{1}{3} \leqq a \leqq 1.$$

The orthant probability of five normal variates when the off diagonal elements of the correlation matrix are equal can be deduced from either the recurrence relation due to Schläfli [19] and David [6], or the differential relationship connecting P_5 and P_3 due to Ruben [18]. In either case, it is found that

$$(2.9) \quad P_5 = \tfrac{1}{32} + [5 \arcsin (a)]/8\pi - 5 [\arcsin (a)]^2/8\pi^2 + 5\{2Li_2[m, \arccos (a)] + \tfrac{1}{2}Li_2[-m^2] - Li_2[m^2, \arccos (a)]\}/2\pi^2,$$

where $E[x_i x_k] = a, i \neq k; -\frac{1}{3} \leqq a \leqq 1$.

(ii) Abrahamson [1] has shown that the general orthant probability of four normal variates can be expressed as a linear combination of six orthoscheme probabilities of four normal variates whose correlation matrix is of the form

$$\begin{bmatrix} 1 & \rho_{12} & 0 & 0 \\ \rho_{12} & 1 & \rho_{23} & 0 \\ 0 & \rho_{23} & 1 & \rho_{34} \\ 0 & 0 & \rho_{34} & 1 \end{bmatrix}.$$

In order that the determinant of this matrix be positive, it is necessary that

$$\rho_{23}^2 < (1 - \rho_{12}^2)(1 - \rho_{34}^2).$$

Using the method demonstrated above, we obtain readily two integral expressions for the general orthoscheme probability $V(\rho_{12}, \rho_{23}, \rho_{34})$ as:

$$(2.10) \quad V(\rho_{12}, \rho_{23}, \rho_{34}) = \tfrac{1}{16} + \{\arcsin (\rho_{12}) + \arcsin (\rho_{23}) + \arcsin (\rho_{34})\}/8\pi + \tfrac{1}{4}\pi^{-2} \int_0^{\rho_{34}} (1 - \gamma^2)^{-\frac{1}{2}} \arcsin g_2(\gamma) d\gamma$$

where

$$g_2(\gamma) = [\rho_{12}\{(1 - \gamma^2)/(1 - \gamma^2 - \rho_{23}^2)\}^{\frac{1}{2}}];$$

$$(2.11) \quad V(\rho_{12}, \rho_{23}, \rho_{34}) = \tfrac{1}{16} + \{\arcsin (\rho_{12}) + \arcsin (\rho_{23}) + \arcsin (\rho_{34})\}/8\pi + [\arcsin (\rho_{12}) \arcsin (\rho_{34})]/4\pi^2 + \tfrac{1}{4}\pi^{-2} \int_0^{\rho_{23}} (1 - \gamma^2)^{-\frac{1}{2}} \arcsin g_3(\gamma) d\gamma$$

where

$$g_3(\gamma) = [\gamma\rho_{12}\rho_{34}/\{(1 - \gamma^2 - \rho_{12}^2)(1 - \gamma^2 - \rho_{34}^2)\}^{\frac{1}{2}}].$$

Equation (2.10) was first derived by van der Vaart [23]. From (2.10) and (2.11) it is readily verified that

$$V(\rho_{12}, \rho_{23}, -\rho_{34}) = \tfrac{2}{16} + \{\arcsin(\rho_{12}) + \arcsin(\rho_{23})\}/4\pi - V(\rho_{12}, \rho_{23}, \rho_{34}) \tag{2.12}$$

$$V(\rho_{12}, -\rho_{23}, \rho_{34}) = \arcsin(\rho_{23})/4\pi + V(\rho_{12}, \rho_{23}, \rho_{34}), \tag{2.13}$$

$$V(-\rho_{12}, \rho_{23}, -\rho_{34}) = V(\rho_{12}, \rho_{23}, \rho_{34}) - \{\arcsin(\rho_{12}) + \arcsin(\rho_{34})\}/4\pi. \tag{2.14}$$

Abrahamson [1] has shown that

$$V[a, -\{(1-a)/2\}^{\frac{1}{2}}, -\tfrac{1}{2}] = \Psi(a)/6, \quad \text{where } \Psi(a) \text{ is now given by (2.7).} \tag{2.15}$$

Using the method shown above, it is found that for $|b| < 1, 0 < a^2 \leqq 1$,

$$V[(1-a^2)^{\frac{1}{2}}, a^2b, (1-a^2)^{\frac{1}{2}}] = \Phi_1(a, b),$$

where

$$\Phi_1(a, b) = \tfrac{1}{16} + [\arcsin(1-a^2)^{\frac{1}{2}}]/4\pi + [\arcsin(1-a^2)^{\frac{1}{2}}]^2/4\pi^2 + \tfrac{1}{4}\pi^{-2}\int_0^{a^2b}(1-x^2)^{-\frac{1}{2}}\arccos g_4(x)\,dx$$

where

$$g_4(x) = \{x(1-1/a^2)(1-x^2/a^2)^{-1}\}.$$

Using (A13) of the appendix, we obtain

$$\begin{aligned}\Phi_1(a, b) = {} & \tfrac{1}{16} + \{\arcsin(1-a^2)^{\frac{1}{2}} + \tfrac{1}{2}\arcsin(a^2b)\}/4\pi \\ & + \{[\arcsin(1-a^2)^{\frac{1}{2}}]^2 - \tfrac{1}{2}[\arcsin(a^2b)]^2\}/4\pi^2 \\ & + \{2Li_2[c, \arccos(a^2b)] + \tfrac{1}{2}Li_2[-c^2] \\ & - Li_2[c^2, \arccos(2a^2-1)]\}/4\pi^2,\end{aligned} \tag{2.16}$$

where $c = c(b) = [1-(1-b^2)^{\frac{1}{2}}]/b$, with $c(0) = 0$. At $a = 0$ and $b = \pm 1$, the correlation matrix is singular. By the continuity theorem [5], V is a continuous function of a and b in the closed region $|a| \leqq 1$, $|b| \leqq 1$. Since $\Phi_1(a, b)$ is continuous in the same region, and equals V in the open region $|b| < 1$, $0 < a^2 \leqq 1$, then they must also be equal in the closed region $|a| \leqq 1$, $|b| \leqq 1$.

From (2.12), (2.13) and (2.14) closed form expressions can also be found for cases:

$$V[-(1-a^2)^{\frac{1}{2}}, a^2b, -(1-a^2)^{\frac{1}{2}}] \quad \text{and} \quad V[(1-a^2)^{\frac{1}{2}}, a^2b, -(1-a^2)^{\frac{1}{2}}]$$

where $|a| \leqq 1$, $|b| \leqq 1$;

$V[a, -\{(1-a)/2\}^{\frac{1}{2}}, \tfrac{1}{2}]$, $V[a, \{(1-a)/2\}^{\frac{1}{2}}, -\tfrac{1}{2}]$ and $V[a, \{(1-a)/2\}^{\frac{1}{2}}, \tfrac{1}{2}]$

where $-\tfrac{1}{3} \leqq a \leqq 1$.

(iii) David and Mallows [7] have given a series expansion for the orthant probability of four normal variates whose correlation matrix is

$$\begin{bmatrix} 1 & \frac{1}{2}b & \frac{1}{2}b & 0 \\ \frac{1}{2}b & 1 & 0 & \frac{1}{2}b \\ \frac{1}{2}b & 0 & 1 & \frac{1}{2}b \\ 0 & \frac{1}{2}b & \frac{1}{2}b & 1 \end{bmatrix},$$

where $|b| < 1$. It can be shown that the orthant probability P_4 is then given by $\Psi_1(b)$, where

$$\Psi_1(b) = \tfrac{1}{16} + \pi^{-2} \int_0^{b/2} (1 - x^2)^{-\frac{1}{2}} \arccos [-x/(1 - 2x^2)]\, dx.$$

$(A14)$ of the appendix yields

$$\begin{aligned} (2.17) \quad \Psi_1(b) = \tfrac{1}{16} &+ [\arcsin (b/2)]/2\pi - [\arcsin (b/2)]^2/2\pi^2 \\ &+ \{2Li_2[c, \arccos (b/2)] + \tfrac{1}{2}Li_2[-c^2] - Li_2[-c^4]/4\}/\pi^2, \end{aligned}$$

where $c = c(b) = [1 - (1 - b^2)^{\frac{1}{2}}]/b$, with $c(0) = 0$, $|b| < 1$. For $b = 1$, it can be shown that $\Psi_1(1) = \frac{1}{6}$ which agrees with the result obtained by Plackett [17]. By a continuity argument, the orthant probability P_4 is given by (2.17) for $|b| \leqq 1$.

(iv) Closed form expressions for the orthant probabilities of the quadrivariate normal distribution can be obtained for at least two other cases. The first has correlation matrix

$$\begin{bmatrix} 1 & a & b & ab \\ a & 1 & ab & b \\ b & ab & 1 & a \\ ab & b & a & 1 \end{bmatrix},$$

where $|a| \leqq 1$, $|b| \leqq 1$. Cheng [3] has shown that the orthant probability P_4 is given by $\Phi_2(a, b)$ where

$$\begin{aligned} (2.18) \quad \Phi_2(a, b) = \tfrac{1}{16} &+ \{\arcsin (a) + \arcsin (b) + \arcsin (ab)\}/4\pi \\ &+ \{[\arcsin (a)]^2 + [\arcsin (b)]^2 - [\arcsin (ab)]^2\}/4\pi^2. \end{aligned}$$

David and Mallows [7] obtained the closed form expressions for the cases $a = \pm\frac{1}{2}$, $b = \rho$. This orthant probability was applied by Cheng [3] to the clipping loss problem in signal detection.

The second case has correlation matrix

$$\begin{bmatrix} 1 & b & a & ab \\ b & 1 & ab & a \\ a & ab & 1 & a^2b \\ ab & a & a^2b & 1 \end{bmatrix},$$

where $|a| \leqq 1$, $|b| \leqq 1$. Cheng [4] has shown that P_4 is given by $\Phi_3(a, b)$ where

$$\begin{aligned}\Phi_3(a, b) &= \tfrac{1}{16} \\ &+ \{\arcsin(a) + \arcsin(ab) + \tfrac{1}{2}\arcsin(b) + \tfrac{1}{2}\arcsin(a^2 b)\}/4\pi \\ &+ \{[\arcsin(a)]^2\}/4\pi^2 - \tfrac{1}{2}\{[\arcsin(a^2 b)]^2\}/4\pi^2 \\ &+ \{2Li_2[c, \arccos(a^2 b)]\}/4\pi^2 \\ &+ \{\tfrac{1}{2} Li_2[-c^2] - Li_2[c^2, \arccos(2a^2 - 1)]\}/4\pi^2; \\ &\qquad c = c(b) = [1 - (1 - b^2)^{\frac{1}{2}}]/b, \quad \text{with} \quad c(0) = 0.\end{aligned} \tag{2.19}$$

This orthant probability has been applied by Cheng [4] to the theory of non-linear transformation of random processes in deriving the closed form solutions of the output autocorrelation functions of half-wave smooth and hard limiters. Previously, McFadden [11] gave a series expansion for the orthant probability $M(\rho_{12}, \rho_{23}, \rho_{34})$ of a Markov process whose correlation matrix is

$$\begin{bmatrix} 1 & \rho_{12} & \rho_{12}\rho_{23} & \rho_{12}\rho_{23}\rho_{34} \\ \rho_{12} & 1 & \rho_{23} & \rho_{23}\rho_{34} \\ \rho_{12}\rho_{23} & \rho_{23} & 1 & \rho_{34} \\ \rho_{12}\rho_{23}\rho_{34} & \rho_{23}\rho_{34} & \rho_{34} & 1 \end{bmatrix}.$$

$\Phi_3(a, b)$ is also the orthant probability of a Markov process when

$$\rho_{12} = \rho_{34} = a, \qquad \rho_{23} = b;$$

i.e.

$$\Phi_3(a, b) = M(a, b, a). \tag{2.20}$$

Our method gives an integral expression for $M(\rho_{12}, \rho_{23}, \rho_{34})$ as:

$$\begin{aligned}M(\rho_{12}, \rho_{23}, \rho_{34}) &= \tfrac{1}{16} \\ &+ \{\arcsin(\rho_{12}) + \arcsin(\rho_{23}) + \arcsin(\rho_{34}) + \arcsin(\rho_{23}\rho_{34}) \\ &+ \arcsin(\rho_{12}\rho_{23}) + \arcsin(\rho_{12}\rho_{23}\rho_{34})\}/8\pi \\ &+ [\arcsin(\rho_{12})\arcsin(\rho_{23})]/4\pi^2 \\ &+ \tfrac{1}{4}\pi^{-2}\int_0^{\rho_{12}\rho_{23}\rho_{34}} (1 - \gamma^2)^{-\frac{1}{2}} \arcsin g_5(\gamma)\, d\gamma\end{aligned} \tag{2.21}$$

where

$$g_5(\gamma) = \{\gamma(1 - \rho_{12}^2)^{\frac{1}{2}}(1 - \rho_{34}^2)^{\frac{1}{2}}[\rho_{12}\rho_{34}(1 - \gamma^2/\rho_{12}^2)^{\frac{1}{2}}(1 - \gamma^2/\rho_{34}^2)^{\frac{1}{2}}]^{-1}\},$$

and where $|\rho_{12}| \leqq 1$, $|\rho_{23}| \leqq 1$, $|\rho_{34}| \leqq 1$. It is then easily verified that

$$\begin{aligned}M(-\rho_{12}, \rho_{23}, \rho_{34}) &= \tfrac{2}{16} \\ &+ \{\arcsin(\rho_{23}) + \arcsin(\rho_{34}) + \arcsin(\rho_{23}\rho_{34})\}/4\pi \\ &- M(\rho_{12}, \rho_{23}, \rho_{34}).\end{aligned} \tag{2.22}$$

From (2.20) and (2.22), closed form expression for $M(-a, b, a)$ can also be deduced.

In concluding, we remark that the method illustrated in this paper can also be combined with other methods, such as the transformation used by McFadden [11], to obtain closed form expressions for more orthant probabilities of four Gaussian variates.

3. Discussion. A method has been demonstrated and applied to obtain closed form expressions for the orthant probabilities of the quadrivariate normal distribution when the correlation matrices are of certain specific forms. Many of these orthant probabilities have practical applications. However, this method does not appear to yield closed form solution for the general orthant probability of four Gaussian variates. Since the general orthant probability is of considerable importance in many fields (see, for examples, [13], [14]), it appears desirable and challenging for mathematicians to continue the search for solution of this problem.

APPENDIX

To evaluate

$$I(a, b, k) = \int_0^{ab} (1 - x^2)^{-\frac{1}{2}} \arcsin g_6(x)\, dx$$

where

$$g_6(x) = \{x[1 - 2k^2/(1 + a)]\cdot[1 - 2k^2x^2/(1 + a)]^{-1}\},$$

or equivalently

$$I(a, b, k) = \int_0^{\arcsin(ab)} \arcsin g_7(\theta)\, d\theta$$

where

$$g_7(\theta) = \{\sin \theta[1 - 2k^2/(1 + a)]\cdot[1 - 2k^2 \sin^2 \theta/(1 + a)]^{-1}\}.$$

Put

(A1) $u(\theta, a, k) = \arcsin \{\sin \theta[1 - 2k^2/(1 + a)][1 - 2k^2 \sin^2 \theta/(1 + a)]^{-1}\},$

and define $u(\theta, a, k) = 0$ whenever $2k^2 = 1 + a$, then it can be verified that $u(\theta, a, k)$ is a well defined, real-valued function in the region where $|k| \leqq 1$, $|b| \leqq 1$ and $-3^{-1} \leqq a \leqq 1$. $u(\theta, a, k)$ may be complex valued outside this region. Moreover, the integrand and its partial derivative with respect to k are continuous in the same region, therefore differentiation under the integral sign is permissible and

$$(\partial/\partial k)I(a, b, k) = \int_0^{\arcsin(ab)} g_8(\theta)\, d\theta$$

where

$$g_8(\theta) = -4k \sin \theta \cos \theta/(1 + a) \cdot\{[1 - 4k^4 \sin^2 \theta/(1 + a)^2]^{\frac{1}{2}}[1 - 2k^2 \sin^2 \theta/(1 + a)]\}^{-1}.$$

Integrating with respect to k, we obtain

(A2) $$I(a, b, k) = \int_0^k \int_0^{\arcsin(ab)} g_9(\theta, z)\, d\theta\, dz + I(a, b, 0)$$

where

$g_9(\theta, z) =$

$$-4z \sin\theta \cos\theta/(1 + a)\{[1 - 4z^4 \sin^2\theta/(1 + a)^2]^{\frac{1}{2}}[1 - 2z^2 \sin^2\theta/(1 + a)]\}^{-1}.$$

In particular

(A3) $$I(a, b, 1) = \int_0^1 \int_0^{\arcsin(ab)} g_{10}(\theta, z)\, d\theta\, dz + I(a, b, 0)$$

where $g_{10}(\theta, z)$ is as in $(A2)$. Substituting

$$t = 2^{\frac{1}{2}} z \sin\theta/(1 + a)^{\frac{1}{2}},$$

$$s = 2z^2 \sin\theta/(1 + a),$$

the Jacobian of the transformation is

$$|(1 + a)^{\frac{1}{2}}/[2^{\frac{1}{2}} s(1 - t^4/s^2)^{\frac{1}{2}}]|,$$

whence

(A4) $$I(a, b, 1) = \int_0^{2ab(1+a)^{-1}} \int_{(1+a)^{\frac{1}{2}} s/2^{\frac{1}{2}}}^{(abs)^{\frac{1}{2}}} g_{11}(t, s)\, dt\, ds + I(a, b, 0)$$

where

$$g_{11}(t, s) = -2t(1 - t^2)^{-1}[s(1 - s^2)^{\frac{1}{2}}]^{-1};$$

thus

$$I(a, b, 1) = \int_0^{2ab(1+a)-1} g_{12}(s)\, ds + I(a, b, 0)$$

where

$$g_{12}(s) = \{\ln(1 - abs)[s(1 - s^2)^{\frac{1}{2}}]^{-1} - \ln\{1 - (1 + a)s^2/2\}[s(1 - s^2)^{\frac{1}{2}}]^{-1}\}.$$

The dilogarithm function [10] has been defined as:

(A5) $$Li_2[z] = -\int_0^z \ln(1 - v)/v\, dv,$$

where z may be real or complex. Following the notation of Lewin [10], the real part of the dilogarithm function is

(A6) $$Li_2[r, \theta] = \Re Li_2[\mathrm{Re}^{i\theta}], = -\tfrac{1}{2}\int_0^r \ln(1 - 2v\cos\theta + v^2)/v\, dv$$

where $\Re$ denotes the real part of the function followed. The properties, including series representations, of these functions have been discussed, and these functions have also been tabulated; see, for example, Lewin [10].

By means of the transformation

$$s = 2v/(1 + v^2),$$

we obtain

(A7) $\int_0^{2ab/(1+a)} \ln(1 - abs)[s(1 - s^2)^{\frac{1}{2}}]^{-1}\, ds$

$$= -2Li_2[f, \arccos(ab)] + \tfrac{1}{2}Li_2[-f^2],$$

(A8) $\int_0^{2ab/(1+a)} \ln[1 - (1 + a)s^2/2][s(1 - s^2)^{\frac{1}{2}}]^{-1}\, ds$

$$= -Li_2[f^2, \arccos(a)] + Li_2[-f^2],$$

where

(A9) $$f = f(a, b) = \{(1 + a) - [(1 + a)^2 - 4a^2b^2]^{\frac{1}{2}}\}/2ab,$$

with $f(0, b) = f(a, 0) = 0$. Substituting (A7) and (A8) into (A4), we get

(A10) $$I(a, b, 1) = [\arcsin(ab)]^2/2 - 2Li_2[f, \arccos(ab)] - \tfrac{1}{2}Li_2[-f^2] + Li_2[f^2, \arccos(a)].$$

From (A2), it is readily verified that

(A11) $$2Li_2[c, \arccos(b)] + \tfrac{1}{2}Li_2[-c^2] - Li_2[c^2] = [\arcsin(b)]^2/2,$$

where

(A12) $$c = c(b) = [1 - (1 - b^2)^{\frac{1}{2}}]/b,$$

with $c(0) = 0$.

This method has been applied by Cheng [4] to show that in the region $|a| \leqq 1$, $|b| \leqq 1$, and defining $g_{13}(x) = \{x(1 - 1/a^2)(1 - x^2/a^2)^{-1}\}$,

(A13) $$\int_0^{ba^2} (1 - x^2)^{-\frac{1}{2}} \arccos g_{13}(x)\, dx = \tfrac{1}{2}\pi \arcsin(a^2b) - \tfrac{1}{2}[\arcsin(a^2b)]^2 + \tfrac{1}{2}Li_2[-c^2] - Li_2[c^2, \arccos(2a^2 - 1)] + 2Li_2[c, \arccos(a^2b)].$$

In particular, for $a^2 = \frac{1}{2}$,

(A14) $$\int_0^{b/2} (1 - x^2)^{-\frac{1}{2}} \arccos\{-x/(1 - 2x^2)\}\, dx = \tfrac{1}{2}\pi \arcsin(b/2) - \tfrac{1}{2}[\arcsin(b/2)]^2 + \tfrac{1}{2}Li_2[-c^2] - Li_2[-c^4]/4 + 2Li_2[c, \arccos(b/2)].$$

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