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THE REMAINDER IN THE CENTRAL LIMIT THEOREM FOR MIXING STOCHASTIC PROCESSES

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1. Introduction. Let $\langle x_n, n = 1, 2, \cdots \rangle$ be a sequence of independent random variables centered at expectations and uniformly bounded by 1 almost surely. It follows from the Berry-Esseen theorem ([2], p. 288) that

$$P(s_N^{-1} \sum_{n \leqq N} x_n < x) = \phi(x) + O(s_N^{-1}) \tag{1}$$

where we set $s_N^2 = \sum_{n \leqq N} E(x_n^2)$ and $\phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-t^2/2}\, dt$. The constant implied by O is numerical. As is well known, in general, the order of magnitude of the estimate of the remainder cannot be improved.

In a paper to appear shortly [3] I investigated the central limit problem for mixing sequences of random variables; in particular, necessary and sufficient conditions were given for the central limit theorem to hold. In the present paper a modest attempt is made to estimate the remainder for such mixing stochastic processes. Unfortunately I was unable to show the central limit theorem in the above strong form (1) but could prove only that the order of magnitude of the remainder does not exceed $s_N^{-\frac{1}{2}} \log^3 s_N$ in case the random variables are uniformly bounded almost surely and satisfy a certain additional condition. Since in this direction nothing appears to be known and since the proof of the above statement turned out to be not quite so simple as I first anticipated I felt that it might be worthwhile to supply the details. Moreover, in a subsequent paper [5] the results are used to prove the law of the iterated logarithm for mixing stochastic processes.

2. Preliminaries and statement of the theorems. Let $\langle x_n, n = 1, 2, \cdots \rangle$ be a sequence of random variables centered at expectations with $\sup_n E(x_n^2) \leqq 1$ and $s_N^2 = E(\sum_{n \leqq N} x_n)^2 \to \infty$. Denote by $\mathfrak{M}_{ab}$ the σ-algebra generated by the events $\{x_n < \alpha\}$, $1 \leqq a \leqq n \leqq b \leqq \infty$. We shall be concerned with the following two mixing conditions:

$$(\mathrm{I}^*) \qquad |P(AB) - P(A)P(B)| \leqq \psi(n)P(A)P(B)$$

for all $A \,\varepsilon\, \mathfrak{M}_{1t}$, $B \,\varepsilon\, \mathfrak{M}_{t+n,\infty}$ with $\psi(n) \downarrow 0$ as $n \to \infty$

$$(\mathrm{II}^*) \qquad \sup_t \sup_{B \varepsilon \mathfrak{M}_{t+n,\infty}} |P(B \mid \mathfrak{M}_{1t}) - P(B)| \leqq \varphi(n) \downarrow 0$$

with probability 1. (II*) is equivalent to (for a proof see [1])

(II*′) For any events $A \,\varepsilon\, \mathfrak{M}_{1t}$ and $B \,\varepsilon\, \mathfrak{M}_{t+n,\infty}$ we have

$$|P(AB) - P(A)P(B)| \leqq \varphi(n)P(A).$$

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THEOREM 1. *In addition to the standard hypotheses let* $\langle x_n \rangle$ *satisfy condition* (II*) *with* $\varphi(n) = e^{-\lambda n}$ *where* $\lambda > 0$ *is a constant. Moreover, suppose that the* x_n *are uniformly bounded by* 1 *almost surely and that*

$$\sum_{n=M+1}^{M+H} \| x_n \|_4 \ll E\left(\sum_{n=M+1}^{M+H} x_n\right)^2 \tag{2}$$

uniformly in $M = 0, 1, 2, \cdots$. *Then*

$$P\left(s_N^{-1} \sum_{n \leqq N} x_n < x\right) = \phi(x) + O(s_N^{-\frac{1}{2}} \log^3 s_N).$$

(Here we use the Vinogradov symbol $\ll$ to indicate an inequality containing some unspecified positive constant factor.)

THEOREM 2. *If in Theorem* 1, *instead of* (II*), *condition* (I*) *is satisfied with* $\psi(n) = e^{-\lambda n}$ $(\lambda > 0)$ *condition* (2) *can be relaxed to*

$$\sum_{n=M+1}^{M+H} E|x_n| \ll E\left(\sum_{n=M+1}^{M+H} x_n\right)^2 \tag{3}$$

and the conclusion of Theorem 1 *remains valid.*

As will be clear from the proof of the theorems one could prove theorems of the above type assuming e.g. $\sum \varphi^{\frac{1}{2}}(n) < \infty$ and

$$\sum_{n=M+1}^{M+H} \|x_n\|_4 \ll \left(E\left(\sum_{n=M+1}^{M+H} x_n\right)^2\right)^\alpha, \qquad 1 \leqq \alpha \leqq 2. \tag{4}$$

However, this would affect the estimate of the error term in the central limit theorem. Similarly, if the assumption of the x_n being uniformly bounded is dropped, Theorems 1 and 2 would continue to hold with an error term $O(s_N^{-2/5})$. (see Theorem 4).

In a later application [4] we shall be confronted with a more general situation. Let $\langle x_{Nn}, n = 1, 2, \cdots, N; N = 1, 2, \cdots \rangle$ be a double sequence of random variables centered at expectations with

$$\sup_{n,N} E(x_{Nn}^2) \leqq 1 \qquad \text{and} \qquad s_N^2 = E\left(\sum_{n \leqq N} x_{Nn}\right)^2 \to \infty.$$

Denote by $\mathfrak{M}_{ab}^{(N)}$ the σ-algebra generated by the events $\{x_{Nn} < \alpha\}$, $1 \leqq a \leqq n \leqq b \leqq N$. We shall employ the following mixing conditions.

(I) For any events $A \,\varepsilon\, \mathfrak{M}_{1t}^{(N)}$ and $B \,\varepsilon\, \mathfrak{M}_{t+n,N}^{(N)}$ we have

$$|P(AB) - P(A)P(B)| \leqq \psi(n\gamma_N)P(A)P(B)$$

with $\psi(n) \downarrow 0$ as $n \to \infty$ and a constant $\gamma_N > 0$.

$$\text{(II)} \qquad \sup_t \sup_{B \varepsilon \mathfrak{M}_{t+n,N}^{(N)}} |P(B \mid \mathfrak{M}_{1t}^{(N)}) - P(B)| \leqq \varphi(n\gamma_N)$$

with probability 1 where $\varphi(n) \downarrow 0$ as $n \to \infty$ and $\gamma_N > 0$ is a constant. Again (II) is equivalent with

(II′) For any events $A \,\varepsilon\, \mathfrak{M}_{1t}^{(N)}$ and $B \,\varepsilon\, \mathfrak{M}_{t+n,N}^{(N)}$ we have

$$|P(AB) - P(A)P(B)| \leqq \varphi(n\gamma_N)P(A)$$

THEOREM 3. *Let* $\langle x_{Nn} \rangle$ *be as above with* $\sup_{n,N} \|x_{Nn}\|_\infty \leqq 1$. *Suppose that*

$$\sum_{n=M+1}^{M+H} \|x_n\|_4 \ll E\left(\sum_{n=M+1}^{M+H} x_n\right)^2 \tag{5}$$

uniformly in $M = 0, 1, 2, \cdots$ *and that the* $\langle x_{Nn} \rangle$ *satisfy condition* (II) *with*

$\varphi(n\gamma_N) = 1$ *for* $1 \leqq n \leqq 4 \log s_N$ *and* $\varphi(n\gamma_N) = s_N^2 e^{-\lambda n}$ *for* $4 \log s_N < n \leqq N$ *where* $0 < \lambda \leqq 1$ *is a constant. Then*

$$P(s_N^{-1} \textstyle\sum_{n \leqq N} x_{Nn} < x) = \phi(x) + O(s_N^{-\frac{1}{2}} \log^3 s_N).$$

THEOREM 4. *Suppose that in Theorem* 3, *instead of* $\sup_{n,N} \|x_{Nn}\|_\infty \leqq 1$ *we only assume* $\sup_{n,N} E(x_{Nn}^2) \leqq 1$ *then*

$$P(s_N^{-1} \textstyle\sum_{n \leqq N} x_{Nn} < x) = \phi(x) + O(s_N^{-2/5}).$$

We now state a result analogous to Theorem 3 assuming that, instead of (I*), condition (I) is satisfied.

THEOREM 5. *Let* $\langle x_{Nn} \rangle$ *be as in Theorem* 3 *but suppose that, instead of* (5), *condition* (3) *holds and that, instead of* (II), *condition* (I) *is satisfied with* $\psi(n\gamma_N) = 1$ *for* $1 \leqq n \leqq 4 \log s_N$ *and* $\psi(n\gamma_N) = s_N^2 e^{-\lambda n}$ *for* $4 \log s_N < n \leqq N$ *where* $0 < \lambda \leqq 1$ *is a constant. Then the conclusion of Theorem* 3 *remains valid.*

Similarly Theorem 4 *remains true if we replace condition* (II) *by* (I) *in the hypotheses.*

Again as in Theorems 1 and 2 there are several variations of the hypotheses possible with some of them affecting the estimate of the remainder. I shall prove Theorem 3 only and indicate the changes necessary for the proofs of the other ones. Finally two applications are given.

3. Some lemmas.

LEMMA 1. *Suppose that condition* (II) *is satisified and that* ξ *and* η *are random variables measurable over* $\mathfrak{M}_{1t}^{(N)}$ *and* $\mathfrak{M}_{t+n,N}^{(N)}$ *respectively. If*

$$E|\xi|^p < \infty \quad \text{and} \quad E|\eta|^q < \infty \quad \text{with } p, q > 1, p^{-1} + q^{-1} = 1$$

then

$$|E(\xi\eta) - E(\xi)E(\eta)| \leqq 2(\varphi(n\gamma_N))^{1/p} \|\xi\|_p \|\eta\|_q .$$

Moreover, if ξ *and* η *are both essentially bounded then*

$$|E(\xi\eta) - E(\xi)E(\eta)| \leqq 4\varphi(n\gamma_N) \|\xi\|_\infty \|\eta\|_\infty .$$

For a proof see [1].

LEMMA 2. *Under the hypotheses of Theorem* 3 *we can represent* X_N *in the form*

$$X_N \equiv \textstyle\sum_{n \leqq N} x_{Nn} = Y_N + Z_N \equiv \sum_{j \leqq l+1} y_{Nj} + \sum_{j \leqq l} z_{Nj}$$

subject to the following conditions:

$$\begin{aligned} y_{N1} &= x_{N1} + \cdots + x_{Nh_1}, & z_{N1} &= x_{Nh_1+1} + \cdots + x_{Nh_1+k}, \\ &\cdots & &\cdots \\ y_{Nl} &= x_{N\rho_l+1} + \cdots + x_{N\rho_l+h_l}, & z_{Nl} &= x_{N\rho_l+h_l+1} + \cdots + x_{N\rho_{l+1}}, \\ y_{Nl+1} &= x_{N\rho_{l+1}+1} + \cdots + x_{NN}, \end{aligned}$$

where we set $\rho_i = \min (N, \sum_{v<i} (h_v + k))$

(6) $\quad E(y_{Nj}^2) = s_N^\alpha + O(\log s_N), \qquad E(y_{Nl+1}^2) \leqq s_N^\alpha + O(\log s_N)$

uniformly in $1 \leqq j \leqq l$. *Here* $0 < \alpha < 2$ *denotes a constant to be chosen later. Moreover, there is a constant* $c > 0$ *such that for all* $1 \leqq j \leqq l$

$$h_j \geqq c s_N^{\alpha} (\log s_N)^{-1}, \tag{7}$$

$$k = [8 \lambda^{-1} \log s_N], \tag{8}$$

$$l = s_N^{2-\alpha} + O(s_N^{2-2\alpha} \log s_N), \tag{9}$$

$$E(Y_n^2) = s_N^2 + O(s_N^{2-\alpha} \log s_N), \tag{10}$$

and finally

$$E(Z_N^2) \ll s_N^{2-\alpha} \log s_N . \tag{11}$$

Proof. In what follows we shall drop the index N in x_{Nn}, y_{Nj} and z_{Nj}. We choose h_1 to be the largest integer $h \leqslant N$ such that

$$E(\textstyle\sum_{n \leqq h} x_n)^2 \leqq s_N^{\alpha}.$$

Next we choose k as in (8) which determines the random variable z_1. In general, we define h_j inductively as the largest integer $h \leqslant N$ such that

$$E(\textstyle\sum_{n=\rho_j+1}^{\rho_j+h} x_n)^2 \leqq s_N^{\alpha}.$$

It might happen, of course that $h_{l+1} = 0$ or that even z_{Nl} consists of less than $[8 \lambda^{-1} \log s_N]$ terms. To prove (6) we remark that as a consequence of Lemma 1

$$|E(x_v x_\mu)| \leqq 2\varphi^{\frac{1}{2}}(n\gamma_N)E(x_v^2)E(x_\mu^2) \leqq 2\varphi^{\frac{1}{2}}(|v - \mu|\gamma_N). \tag{12}$$

(The reason why we do not make use of the boundedness of the x_{Nn} is that we want (12) to continue to hold under the hypothesis of Theorem 4). Hence

$$E(y_j^2) = E(\textstyle\sum_{n=\rho_j+1}^{\rho_j+h_j} x_n)^2 \leqq s_N^{\alpha} \leqq E(y_j + x_\rho)^2$$

with $\rho = \rho_j + h_j + 1$. But

$$E(y_j + x_\rho)^2 \leqq E(y_j^2) + 2.2 \textstyle\sum_{v<\rho} \varphi^{\frac{1}{2}}(\gamma_N(\rho - v)) + 1 = E(y_j^2) + O(\log s_N)$$

proving (6). On the other hand with $I_j = [\rho_j + 1, \rho_j + h_j + 1]$

$$s_N^{\alpha} \leqq E(y_j + x_\rho)^2 \leqq \textstyle\sum_{i \varepsilon I_j} E(x_i^2) + 4 \sum_{v<\rho \varepsilon I_j} \varphi^{\frac{1}{2}}(\gamma_N(\rho - v)) \leqq (h_j + 1) + O(h_j \log s_N)$$

which proves (7). The proofs of (9)–(11) are similar to those of formula (3.9) and corollary to Lemmas 4 and 5 in [3] so that we can deal with them briefly. In what follows a summation or a product of terms involving the y_j or the z_j is always extended over all possible values of j, e.g. $\sum y_j$ stands for $\sum_{j<l+1} y_j$. As in [3] we expand

$$\begin{aligned} s_N^2 &= E(\textstyle\sum y_j + \sum z_j)^2 \\ &= \textstyle\sum E(y_j^2) + \sum E(z_j^2) + 2\sum_{i<j} E(y_i y_j) + 2\sum_{i<j} E(z_i z_j) + \sum E(y_i z_j) \\ &= \textstyle\sum^{(1)} + \cdots + \sum^{(5)} \end{aligned}$$

and obtain

$$\sum^{(1)} = ls_N^{\alpha} + O(l \log s_N) + O(s_N^{\alpha}),$$
$$\sum^{(2)} \ll lk \ll l \log s_N,$$
$$\sum^{(3)} \leqq 8 \cdot s_N^{\frac{1}{2}\alpha} l \sum_{n \geqq 4\lambda^{-1} \log s_N} e^{-\frac{1}{2}\lambda n} \ll l,$$
$$\sum^{(4)} \ll l \quad \text{and} \quad \sum^{(5)} \ll l$$

which proves (9). (10) follows from $E(Y_N^2) = \sum^{(1)} + \sum^{(3)}$ and (11) from $E(Z_N^2) = \sum^{(2)} + \sum^{(4)}$.

We set

$$g_N^3 = 8\, s_N^{-3} \sum E|y_j^3|.$$

LEMMA 3. *For large N we have uniformly in $0 \leqq t \leqq 2\, g_N^{-3}$*

$$|\prod E(\exp\,(ity_j s_N^{-1})) - e^{-t^2/2}| \leqq 2\, t^2(tg_N^3 + O(s_N^{-\alpha} \log s_N))e^{-t^2/8}.$$

The proof follows exactly Loève [2], p. 286. In case that $tg_N \geqq 1$ it is enough to show that for large N

$$|\prod E(\exp\,(ity_j s_N^{-1}))| \leqq \exp\,(-\tfrac{1}{8}t^2).$$

From Lemma 2 we infer that uniformly in $1 \leqq j \leqq l$

$$|E(\exp\,(ity_j s_N^{-1}))| \leqq 1 - \tfrac{1}{2}t^2(s_N^{\alpha-2} + O(s_N^{-2} \log s_N)) + \tfrac{1}{6}t^3 E|y_j^3| s_N^{-3}$$

and

$$|E(\exp\,(ity_{l+1} s_N^{-1}))| \leqq 1 + t^2 \cdot O(s_N^{\alpha-2} \log s_N) + \tfrac{1}{6}t^3 E|y_{l+1}^3| s_N^{-3}.$$

Using the estimate $1 + x \leqq e^x$ and taking the product over $1 \leqq j \leqq l + 1$ we get from Lemma 2 for large N

$$\prod |E(\exp\,(ity_j s_N^{-1}))| \leqq \exp\,(-\tfrac{1}{2}t^2(1 + o(1)) + \tfrac{1}{6}t^3 g_N^3) \leqq \exp\,(-\tfrac{1}{8}t^2).$$

If, on the other hand $tg_N < 1$ then for $1 \leqq j \leqq l + 1$

$$E(\exp\,(ity_j s_N^{-1})) = 1 - \tfrac{1}{2}t^2 E(y_j^2) s_N^{-2} + \tfrac{1}{6}\theta t^3 E|y_j^3| s_N^{-3} = 1 - r_j.$$

Here and below $|\theta| \leqq 1$ denotes a constant not always necessarily the same. Since $\|y_j\|_2 \leqq \|y_j\|_3 \leqq \frac{1}{2} g_N s_N$ we get $|r_j| < \frac{1}{2}$ and thus

$$\log E(\exp\,(ity_j s_N^{-1})) = -r_j + \theta r_j^2 = -\tfrac{1}{2}t^2 E(y_j^2) s_N^{-2} + \theta \tfrac{1}{1}\tfrac{1}{8} t^3 E|y_j^3| s_N^{-3}$$

using the identity $(a + b)^2 = a^2 + b(2a + b)$. Summing over $1 \leqq j \leqq l + 1$ we obtain from Lemma 2

$$\log \prod E(\exp\,(ity_j s_N^{-1})) = -\tfrac{1}{2}t^2(1 + O(s_N^{-\alpha} \log s_N) + (11\,\theta/144)t^3 g_N^3.$$

The lemma follows now from the fact that $e^a = 1 + \theta a e^{\theta a}$ with

$$a = t^2 \cdot O(s_N^{-\alpha} \log s_N) + (11/144).$$

LEMMA 4. *Under the hypotheses of Theorem* 3 *we have uniformly for* $0 \leqq t \leqq 1$

$$E(\exp(itY_N s_N^{-1})) - \prod E(\exp(ity_j s_N^{-1})) \ll t^2.$$

PROOF. From Lemma 2 we get

$$|E(\exp(itY_N s_N^{-1})) - \prod E(\exp(ity_j s_N^{-1}))| \leqq \tfrac{1}{2}t^2(1 + o(1)) + \tfrac{1}{2}t^2(1 + o(1)) \ll t^2,$$

since

$$\prod (1 - \tfrac{1}{2}t^2\theta_j E(y_j^2)s_N^{-2}) = 1 - \tfrac{1}{2}t^2 \sum \theta_j E(y_j^2)s_N^{-2} \prod_{k>j} (1 - \tfrac{1}{2}t^2\theta_k E(y_k^2)s_N^{-2}) = 1 - \tfrac{1}{2}t^2 \sum \theta_j E(y_j^2)s_N^{-2}.$$

The following lemma says that Bernstein's inequality holds for the random variables z_j $(1 \leqq j \leqq l)$. Again we follow Loève [2], p. 254.

LEMMA 5. *We have*

$$P\{|Z_N| \geqq 16\,\lambda^{-1}s_N^{1-\alpha/2}\log^3 s_N\} \ll s_N^{-1}.$$

PROOF. Since $\|z_j\|_\infty \leqq 8\lambda^{-1}\log s_N = C_N$ we have for $1 \leqq j \leqq l$ and $0 < t < C_N^{-1}$

$$E(\exp(tz_j)) < 1 + (t^2E(z_j^2)/2)(1 + tC_N/3 + t^2C_N^2/3.4 + \cdots) < 1 + (t^2C_N^2/2)(1 + tC_N/2) < \exp(t^2C_N^2).$$

Applying Lemma 1 l times we obtain

$$|E(\exp(tZ_N)) - \prod E(\exp(tz_j))| \leqq 4\,l \prod E(\exp(tz_j)) \cdot \varphi(\gamma_N h_j).$$

Hence it follows from (7) and (9) that

$$E(\exp(tZ_N)) \ll \prod E(\exp(tz_j)) \ll \exp(t^2C_N^2 s_N^{2-\alpha}).$$

Therefore setting $t = \frac{1}{8}\lambda s_N^{-1+\alpha/2}\log s_N$

$$P\{|Z_N| \geqq 16\,\lambda^{-1}s_N^{1-\alpha/2}\log^3 s_N\} \ll \exp(-ts_N^{1-\alpha/2}\log^3 s_N + t^2C_N^2 s_N^{2-\alpha}) \ll s_N^{-1}.$$

4. The normal law for Y_N/s_N. In order to apply what in [2], p. 285, is called the basic inequality we have to estimate

$$\int_0^T |E(\exp(itY_N s_N^{-1})) - e^{-t^2/2}|t^{-1}\,dt \leqq \int_0^T |\prod E(\exp(ity_j s_N^{-1})) - e^{-t^2/2}|t^{-1}\,dt + \left(\int_0^{t_0} + \int_{t_0}^1 + \int_1^T\right)(|E(\exp(itY_N s_N^{-1})) - \prod E(\exp(ity_j s_N^{-1}))|t^{-1}\,dt$$

where we set $t_0 = s_N^{-1}$ and $T = \min(2\,g_N^{-3}, s_N^2)$. Call these integrals $I_1, \cdots, I_4$ respectively. By means of Lemma 3

$$I_1 \ll g_N^3 \int_0^\infty t^2e^{-t^2/8}\,dt + s_N^{-\alpha}\log s_N \cdot \int_0^\infty te^{-t^2/8}\,dt \ll g_N^3 + s_N^{-\alpha}\log s_N.$$

As a consequence of Lemma 4

$$I_2 \ll \int_0^{t_0} |\prod E(\exp(ity_j s_N^{-1})) - E(\exp(itY_N s_N^{-1}))|t^{-1}\,dt \ll \int_0^{t_0} t\,dt \ll s_N^{-2}.$$

For the estimate of I_3 we apply Lemma 1 l times to obtain

(13) $$|E(\exp(itY_N s_N^{-1})) - \prod E(\exp(ity_j s_N^{-1}))| \leqq 4\, l\varphi(\gamma_N k)$$

and hence by Lemma 2

$$I_3 = \int_{t_0}^{1} |\cdot| t^{-1}\, dt \ll t_0^{-1} l\varphi(\gamma_N k) \ll s_N^{-3},$$
$$I_4 = \int_1^T |\cdot| t^{-1}\, dt \ll \log T \cdot l\varphi(\gamma_N k) \ll s_N^{-3}.$$

Adding all the estimates we obtain with $T = \min(2\, g_N^{-3}, s_N^2)$ and $0 < \alpha < 2$

$$\int_0^T |E(\exp(itY_N s_N^{-1})) - e^{-t^2/2}| t^{-1}\, dt \ll g_N^3 + s_N^{-\alpha} \log s_N.$$

From the basic inequality in [2], p. 285, we conclude that

(14) $$P(Y_N s_N^{-1} < x) - \phi(x) \ll g_N^3 + s_N^{-\alpha} \log s_N.$$

5. Proof of Theorem 3. We now apply Lemma 10([3])—or rather a slightly modified version of it—with $H = h_j$, $A_1 = A_2 = 40\, \lambda^{-1} \log s_N$. Then $P(H) = P(h_j) \ll s_N^{\alpha}$ and from the hypotheses of Theorem 3 $E(y_j^4) \ll s_N^{2\alpha}$. Hence $E|y_j^3| \leqq \|y_j\|_4^3 \ll s_N^{3\alpha/2}$ and from Lemma 2 we get

(15) $$g_N^3 \ll s_N^{\alpha/2-1}.$$

From Lemma 5 we obtain setting $\epsilon_N = 16\, \lambda^{-1} s_N^{-\alpha/2} \log^3 s_N$

$$\begin{aligned} P[X_N s_N^{-1} < x) &= P(((Y_N + Z_N)s_N^{-1} < x) \cap (|Z_N| s_N^{-1} \geqq \epsilon_N)) \\ &\quad + P(((Y_N + Z_N)s_N^{-1} < x) \cap (|Z_N| s_N^{-1} < \epsilon_N)) \\ &= P(Y_N s_N^{-1} < x + \epsilon_N) + O(s_N^{-1}) \end{aligned}$$

which, if we set $\alpha = 1$, together with (14) and (15) implies the result.

In the proof of Theorem 4 we cannot apply Lemma 5 since the x_n are no longer assumed to be essentially bounded. We shall replace it by Chebyshev's inequality and get

$$P(|Z_N| \geqq \epsilon_N (E|Z_N^2|)^{\frac{1}{2}}) \leqq \epsilon_N^{-2}.$$

Setting $\epsilon_N = s_N^{\alpha/6} \log^{1/6} s_N$ and using (11) we obtain

(16) $$P(|Z_N s_N^{-1}| \geqq s_N^{-\alpha/3} \log^{-1/3} s_N) \leqq s_N^{-\alpha/3} \log^{-1/3} s_N.$$

If we set $\alpha = 6/5$ Theorem 4 follows from (14)–(16).

For the proof of Theorem 2 we apply Lemma 9 [3] instead of the modified version of Lemma 10 [3]. Theorem 1 is a corollary of Theorem 3.

6. Some applications. In this section we shall sketch the proofs of two special cases that were already considered in [3].

THEOREM 6. *Let $\langle E_n, n = 1, 2, \cdots \rangle$ be a sequence of events with $P(E_n) \to 0$. Let $\mathfrak{M}_{ab}$ be the σ-algebra generated by the E_n $(a \leqq n \leqq b)$ and suppose that* (i) *is satisfied with $\psi(n) = e^{-\lambda n}$ $(\lambda > 0)$. If*

$$\phi(N) = \sum_{n \leqq N} P(E_n) \to \infty$$

then

(17) $$P(s_N^{-1}(\sum_{n\leqq N}\varphi_n - \phi(N)) < x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-t^2/2}\,dt + O(s_N^{-\frac{1}{2}}\log^3 s_N)$$

where φ_n *is the indicator of* E_n *and* $s_N^2 = E(\sum_{n\leqq N}\varphi_n - \phi(N))^2$.

If, in particular,

(18) $$\sum_{n\leqq N} P^2(E_n) \ll \phi^{3/4}(N)$$

then

(19) $$P((\phi(N))^{-\frac{1}{2}}(\sum_{n\leqq N}\varphi_n - \phi(N)) < x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-t^2/2}\,dt + O(\phi^{-\frac{1}{4}}(N)\log^3\phi(N)).$$

It follows from the proof of Theorem 8 [3] that the hypotheses of Theorem 2 are satisfied and hence the first part of the theorem follows. We also know from the very same proof that $s_N^2 = \phi(N)(1 + o(1))$. In order to show (19) we have to improve upon this last relation and then to show that the order of magnitude of the error term does not increase if we replace s_N by $(\phi(N))^{\frac{1}{2}}$ on the left hand side of (17).

In the proof of Theorem 8 [3] we set $M = 0$ and obtain

$$s_N^2 = \phi(N) - \sum_{n\leqq N} P^2(E_n) + \theta\sum_{m<n\leqq N} P(E_m)P(E_n)e^{-\lambda(n-m)}$$

with $|\theta| \leqq 1$. Substituting the inequality $P(E_m)P(E_n) \leqq P^2(E_m) + P^2(E_n)$ in the double sum we conclude from (18) that $s_N^2 - \phi(N) \ll \sum_{n\leqq N} P^2(E_n) \ll \phi^{3/4}(n)$. Hence as a consequence of (17)

$$\begin{aligned} P((\phi(N))^{-\frac{1}{2}}&(\sum_{n\leqq N}\varphi_n - \phi(N)) < x) \\ &= P(s_N^{-1}(\sum_{n\leqq N}\varphi_n - \phi(N)) < x(1 + \phi^{-\frac{1}{4}}(N))) \\ &= (2\pi)^{\frac{1}{2}}\int_{-\infty}^{x} e^{-t^2/2}\,dt + O(\phi^{-\frac{1}{4}}(N)\log^3\phi(n)) \end{aligned}$$

which is (19). We observe that condition (18) is certainly satisfied if, for example, $P(E_n) \ll n^{-1/5}$. In fact, we then have setting for a moment $a_n = P(E_n)$ and $A_N = \sum_{n\leqq N} P(E_n)$

$$\sum_{n\leqq N} a_n^2 = \sum_{n\leqq A_N^{5/4}} a_n^2 + \sum_{n>A_N^{5/4}}^{N} a_n^2 \ll \sum_{n\leqq A_N^{5/4}} n^{-2/5} + \sum_{n>A_N^{5/4}}^{N} a_n\cdot A_N^{-\frac{1}{4}} \ll A_N^{3/4}.$$

The second applications deals with stationary processes.

THEOREM 7. *Suppose that* $\langle x_n, n = 1, 2, \cdots\rangle$ *is a weak sense stationary process with* $E(x_n) = 0$ *and* $\sup_n E(x_n^4) \leqq 1$ *satisfying condition* (II*) *with* $\varphi(n) = e^{-\lambda n}(\lambda > 0)$. *Then*

$$\sigma^2 = E(x_1^2) + 2\sum_{v=1}^{\infty} E(x_1 x_{v+1})$$

exists. Moreover, if $\sigma \neq 0$

$$P(\sigma^{-1}N^{-\frac{1}{2}}\sum_{n\leqq N} x_n < x) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{x} e^{-t^2/2}\,dt + O(N^{-\frac{1}{4}}\log^3 N).$$

The theorem follows immediately from Theorem 1 and the proof of Theorem 9 [3].

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