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A CHARACTERIZATION OF THE UPPER AND LOWER CLASSES IN TERMS OF CONVERGENCE RATES[1]

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For a given sequence of independent random variables $\{X_n\}$ a monotonic increasing positive sequence $\{\varphi_n\}$ is said to be in the upper class $\mathfrak{U}$ if

$$P[S_n > n^{\frac{1}{2}}\varphi_n \text{ infinitely often}] = 0,$$

where $S_n = \sum_{k=1}^{n} X_k$.

Otherwise $\{\varphi_n\}$ is in the lower class $\mathfrak{L}$ and the above probability is zero. In 1946, Feller [3] characterized these sequences as follows:

Theorem (Feller). *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with*

$$EX_1 = 0, \qquad EX_1^2 = 1, \qquad \textit{and} \quad \int_{|t|>y} t^2\, dF = O((\lg\lg y)^{-1}).$$

Then a monotonic increasing sequence $\{\varphi_n\}$ is in the upper class if and only if

$$\sum_{n=1}^{\infty} \varphi_n e^{-\varphi_n^2/2} n^{-1} < \infty.$$

The main result here is a characterization of the upper class in terms of a prescribed convergence rate for the partial sums of the random variables. This result represents an improvement of work previously done in this area. In [1] Baum and Katz show the following:

Theorem [Baum and Katz]. *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with*

(1) $EX_1 = 0$,

(2) $EX_1^2 = 1$,

(3) $EX_1^2(\lg|X_1|)^{1+\delta} < \infty$ *for some* $\delta > 0$.[2] *Then a monotonic increasing sequence $\{\varphi_n\}$ is in the upper class for $\{X_n\}$ if and only if*

$$\sum_{n=1}^{\infty} \varphi_n^2 n^{-1} P[S_n > n^{\frac{1}{2}}\varphi_n] < \infty.$$

In [2] the author shows that the same conclusion may be drawn if hypothesis (3) is weakened to $EX_1^2 \lg|X_1| \lg\lg|X_1| < \infty$. Here we obtain a similar conclusion under a moment condition slightly stronger than Feller's O-condition.

Theorem. *Let $\{X_n\}$ be a sequence of independent identically distributed random variables with $EX_1 = 0$, $EX_1^2 = 1$, and $EX_1^2 \lg\lg|X_1| < \infty$. Then a positive monotonic increasing sequence is in the upper class for $\{X_n\}$ if and only if*

$$\sum_{n=3}^{\infty} (\lg\lg n) n^{-1} P[S_n > n^{\frac{1}{2}}\varphi_n] < \infty$$

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[2] Throughout this paper $\lg X$ denotes the function $\lg X = \log_e X$ for $X > 1$ and 0 otherwise.

or equivalently

$$\sum_{n=3}^{\infty} (\lg\lg n)n^{-1}P[|S_n| > n^{\frac{1}{2}}\varphi_n] < \infty.$$

The proof of the theorem is deferred until the following lemma is established.

LEMMA[3]. *Let* $\{\varphi_n\}$ *be a positive monotonic increasing sequence and* $K \geqq 0$. *Then*

$$\sum_{n=3}^{\infty} \varphi_n n^{-1} e^{-\varphi_n^2/2} < \infty$$

$$\Leftrightarrow \sum_{n=3}^{\infty} (\lg\lg n)(n\varphi_n)^{-1} \exp[(-\varphi_n^2/2)(1 + K/(\lg\lg n))] < \infty.$$

PROOF OF LEMMA. If $\varphi_n^2 \geqq (2+\epsilon)\lg\lg n$ for all n both of the series converge.

$$\sum \varphi_n n^{-1} e^{-\varphi_n^2/2} \leqq (2+\epsilon)^{\frac{1}{2}} \sum (\lg\lg n)^{\frac{1}{2}}/n(\lg n)^{1+\epsilon/2} < \infty$$

where the inequality is from the fact $xe^{-x^2/2}$ is monotone decreasing in x.

$$\sum (\lg\lg n)(n\varphi_n)^{-1} \exp[(-\varphi_n^2/2)(1 + K/(\lg\lg n)]$$
$$\leqq (2+\epsilon)^{-\frac{1}{2}} \sum (\lg\lg n)^{\frac{1}{2}} n^{-1} e^{-\varphi_n^2/2}$$
$$\leqq (2+\epsilon)^{-\frac{1}{2}} \sum (\lg\lg n)^{\frac{1}{2}}/n(\lg n)^{1+\epsilon/2} < \infty.$$

Thus, if the sequence $\{\varphi_n\}$ is truncated above at $[(2+\epsilon)\lg\lg n]^{\frac{1}{2}}$ the convergence of neither series is affected. Let

$$\begin{aligned}\varphi_n'^2 &= (2+\epsilon)\lg\lg n \qquad \text{if } \varphi_n^2 > (2+\epsilon)\lg\lg n\\ &= \varphi_n^2 \qquad \text{otherwise.}\end{aligned}$$

If $\varphi_n^2 \leqq (2-\epsilon)\lg\lg n$ infinitely often, both series diverge. To see this the truncated series φ_n' may be used.

$$\sum_{n=3}^{m} \varphi_n' n^{-1} e^{-\varphi_n'^2/2} \geqq e^{-\varphi_m'^2/2} \sum_{n=3}^{m} \varphi_1' n^{-1}$$
$$\geqq c(\lg m)/(\lg m)^{1-\epsilon/2} \to \infty$$

and

$$\sum_{n=3}^{m} (\lg\lg n)(n\varphi_n')^{-1} \exp[(-\varphi_n'^2/2)(1 + K/(\lg\lg n))]$$
$$\geqq \exp[(-\varphi_m'^2/2)(1+\epsilon/4)](2+\epsilon)^{-\frac{1}{2}} \sum_{n=3}^{m} (\lg\lg n)^{\frac{1}{2}} n^{-1}$$
$$\geqq c(\lg m)/(\lg m)^{1-\epsilon/4} \to \infty$$

Now consider the new sequence

$$\begin{aligned}\varphi_n''^2 &= (2-\epsilon)\lg\lg n \qquad \text{if } \varphi_n^2 < (2-\epsilon)\lg\lg n\\ &= \varphi_n^2 \qquad \text{if } (2-\epsilon)\lg\lg n \leqq \varphi_n^2 \leqq (2+\epsilon)\lg\lg n\\ &= (2+\epsilon)\lg\lg n \qquad \text{if } \varphi_n^2 > (2+\epsilon)\lg\lg n.\end{aligned}$$

Each of the series being considered will then converge for $\{\varphi_n\}$ if and only if it

[3] Minor modifications in the proof extend the lemma to all real K, but this is not needed for our purpose.

converges for $\{\varphi_n''\}$. It is now shown that the series converge and diverge together with $\{\varphi_n''\}$,

$$\begin{aligned}\sum (\lg\lg n)(n\varphi_n'')^{-1} &\exp[(-\varphi_n''^2/2)(1+K/(\lg\lg n))]\\ &\leqq (2-\epsilon)^{-\frac{1}{2}} \sum (\lg\lg n)^{\frac{1}{2}} n^{-1} e^{-\varphi_n''^2/2}\\ &\leqq \sum \varphi_n'' n^{-1} e^{-\varphi_n''^2/2}\\ &\leqq (2+\epsilon)^{\frac{1}{2}} e^{K(1+\epsilon/2)} \sum (\lg\lg n)^{\frac{1}{2}} n^{-1} \exp[-\varphi_n''^2/2 - K(1+\epsilon/2)]\\ &\leqq (2+\epsilon) e^{K(1+\epsilon/2)} \sum (\lg\lg n)(n\varphi_n'')^{-1} \exp[(-\varphi_n''^2/2)(1+K/(\lg\lg n))]\end{aligned}$$

and the lemma is proven.

Proof of Theorem. The hypothesis of Theorem 1 in convergence rates for the law of the iterated logarithm [2] are satisfied and

$$\sum (\lg\lg n) n^{-1} \sup_x |P[S_n n^{-\frac{1}{2}} < x] - \Phi(x/\sigma_n)| < \infty$$

with Φ the standard normal distribution function and

$$\sigma_n^2 = \int_{[|x|<n^{\frac{1}{2}}]} x^2\, dF - \left(\int_{[|x|<n^{\frac{1}{2}}]} x\, dF\right)^2$$

where F is the distribution function for X_n. If $\{\varphi_n\}$ is bounded

$$\sum \varphi_n n^{-1} e^{-\varphi_n^2/2} = \infty \quad \text{and} \quad \sum (\lg\lg n) n^{-1} P[S_n > n^{\frac{1}{2}}\varphi_n] = \infty$$

by the central limit theorem. Thus assume $\varphi_n \uparrow \infty$. Now by the above

$$(*) \qquad \sum (\lg\lg n) n^{-1} |P[S_n/n^{\frac{1}{2}} > \varphi_n] - (1 - \Phi(\varphi_n/\sigma_n))| < \infty.$$

With $\varphi_n \uparrow \infty$ and $\sigma_n \uparrow 1$ the tail approximation for the normal distribution may be applied. That is

$$1 - \Phi(\varphi_n/\sigma_n) \sim \sigma_n (2\pi)^{-\frac{1}{2}} \varphi_n^{-1} e^{-\varphi_n^2/2\sigma_n^2}.$$

Now with $EX_1^2 \lg\lg |X_1| < \infty$ one obtains

$$\lg\lg n^{\frac{1}{2}}(1-\sigma_n^2) = \lg\lg n^{\frac{1}{2}} \int_{[|x|>n^{\frac{1}{2}}]} x^2\, dF \leqq \int_{[|x|>n^{\frac{1}{2}}]} x^2 \lg\lg x\, dF \to 0.$$

That is $1 - \sigma_n^2 = o((\lg\lg n)^{-1})$. Thus Feller's 0-condition is satisfied and for any $\epsilon > 0$ $\sigma_n^2 \geqq 1 - \epsilon/(\lg\lg n)$ for all n greater than some N_ϵ.
Then

$$\begin{aligned} c\varphi_n^{-1} e^{-\varphi_n^2/2} \geqq 1 - \Phi(\varphi_n/\sigma_n) &\geqq c\varphi_n^{-1} e^{-\varphi_n^2/2\sigma_n^2}\\ &\geqq c\varphi_n^{-1} \exp[(-\varphi_n^2/2)(1-\epsilon/(\lg\lg n))]\\ &\geqq c\varphi_n^{-1} \exp[(-\varphi_n^2/2)(1+2\epsilon/(\lg\lg n))].\end{aligned}$$

Consider the series

$$\sum (\lg\lg n) n^{-1} \varphi_n^{-1} e^{-\varphi_n^2/2}$$

$$\text{and} \quad \sum (\lg\lg n) n^{-1} \varphi_n^{-1} \exp[(-\varphi_n^2/2)(1+2\epsilon/(\lg\lg n))].$$

According to the lemma these converge if and only if $\{\varphi_n\}$ is in the upper class. Thus $\sum (\lg \lg n)n^{-1}(1 - \Phi(\varphi_n/\sigma_n)) < \infty$ if and only if $\{\varphi_n\} \varepsilon \mathfrak{U}$. By $(*)$ however, the convergence or divergence of the above forces the same for $\sum (\lg \lg n)n^{-1}P[S_n > \varphi_n n^{\frac{1}{2}}]$, and the result is established.

To emphasize the analogy with the Borel zero-one law the theorem may be restated as: For sequences of random variables $\{X_n\}$ and reals $\{\varphi_n\}$ satisfying the above hypothesis then $P[S_n > \varphi_n n^{\frac{1}{2}}$ infinitely often$] = 0$ or 1 according as

$$\sum (\lg \lg n)n^{-1}P[S_n > \varphi_n n^{\frac{1}{2}}] < \infty \quad \text{or} \quad = \infty.$$

REFERENCES

[1] BAUM, L. E. and KATZ, MELVIN (1965). Convergence rates in the law of large numbers. *Trans. Amer. Math. Soc.* **120** 108–123.

[2] DAVIS, J. A. (1968). Convergence rates for the law of the iterated logarithm. *Ann. Math. Statist.* **39** 1479–1485.

[3] FELLER, W. (1946). The law of the iterated logarithm for independent identically distributed random variables. *Ann. Math.* **47** 631–638.

[4] FRIEDMAN, KATZ, M., and KOOPMANS, (1966). Convergence rates for the central limit theorem. *Proc. Nat. Acad. Sci. U. S. A.* **56** 1062–1065.