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A SHORT PROOF OF A KNOWN LIMIT THEOREM FOR SUM OF INDEPENDENT RANDOM VARIABLES WITH INFINITE EXPECTATIONS[1]

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The following theorem is proved by Feller ([1]) with slightly more general hypotheses. He proves it using Kronecker's theorem and a special case of the three series theorem. We shall prove it using an elementary application of the law of large numbers.

THEOREM. *Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed random variables with common distribution function V. Let $S_n = X_1 + \cdots + X_n$. Let $0 = a_0 < a_1 < \cdots$ be a convex sequence of numbers. Assume that $\int |x| \, dV(x) = \infty$. Then,*

$$P\{|S_n| > a_n \textit{ infinitely often}\} = 0 \textit{ or } 1$$

according as

$$\textstyle\sum_{n=1}^{\infty} \int_{|x|>a_n} dV(x) < \infty \textit{ or } = \infty.$$

PROOF. Assume first that

$$\textstyle\sum_{n=1}^{\infty} \int_{|x|>a_n} dV(x) = \infty.$$

Since $2a_n \leqq a_{2n}$ (which follows from the convexity of $\{a_n\}$), we conclude that

$$\textstyle\sum_{n=1}^{\infty} \int_{|x|>2a_n} dV(x) = \infty.$$

Hence

$$P\{|X_n| > 2a_n \text{ infinitely often}\} = 1$$

which implies the desired conclusion.

For the other half of the proof we can, and do, assume, with no loss of generality, that $X_n \geqq 0$ for all n. Of course, we assume that

$$\textstyle\sum_{n=1}^{\infty} \int_{a_n}^{\infty} dV(x) < \infty.$$

For fixed k we define a new sequence:

$$b_n = nk^{-1}a_k, \quad n = 0, 1, \cdots, k;$$

$$b_n = a_n, \qquad n = k+1, \cdots.$$

The sequence $0 = b_0 < b_1 < \cdots$ is convex. Let $b(x)$ be defined for all $x \geqq 0$ such that b is strictly increasing and convex, and such that $b(n) = b_n$ if n is a non-negative integer.

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We now define

$$Y_n = b^{-1}(X_n), \qquad n = 1, 2, \cdots .$$

Since $\int x\, dV(x) = \infty$, it follows that $n = o(a_n)$ as $n \to \infty$. Hence the fixed integer k can be chosen so that $E\{Y_n\} < 1$. By the strong law of large numbers it follows that

$$P\{Y_1 + \cdots + Y_n > n \text{ infinitely often}\} = 0.$$

The proof is complete once one notices that

$$S_n \leqq b(Y_1 + \cdots + Y_n).$$

REFERENCE

[1] FELLER, W. (1946). A limit theorem for random variables with infinite moments. *Amer. J. Math.* **68** 257–262.