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A REMARK ON THE KOLMOGOROV-PETROVSKII CRITERION[1]

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Let $X(t, \omega)$ be any separable version of the standard Wiener process (Brownian motion) defined on a probability space $(\Omega, \mathcal{A}, P)$. Let ψ be any nonnegative function on $(0, \infty)$ such that $\lambda(t) = t^{-\frac{1}{2}}\psi(t)$ is monotone nondecreasing $(\uparrow)$. Define $T_\psi(\omega) = \sup\{t: X(t, \omega) \geqq \psi(t)\}$ and $\Lambda_\psi(\omega) = \lambda\{t: X(t, \omega) \geqq \psi(t)\}$ where λ is Lebesgue measure on $(0, \infty)$. The Kolmogorov-Petrovskii criterion (proved for coin tossing by Erdös) states that,

$$P[T_\psi < \infty] = 1 \tag{1}$$

if and only if,

$$Q(\psi) = \int_1^\infty t^{-3/2}\psi(t) \exp[-\tfrac{1}{2}\psi^2(t)t^{-1}]\, dt < \infty. \tag{2}$$

A beautiful treatment of these results is given in Strassen [3].

It is a trivial consequence of this criterion that if ψ is such that λ is $\uparrow$ and (2) holds then,

$$P[\Lambda_\psi < \infty] = 1. \tag{3}$$

The purpose of this note is to prove a partial converse of (3).

Theorem. *Suppose ψ is such that λ is $\uparrow$ and*

$$\sup_{t \geqq 1} t^{-1}\psi(t) < \infty. \tag{4}$$

If $Q(\psi) = \infty$, then

$$P[\Lambda_\psi = \infty] = 1. \tag{5}$$

We begin with a lemma which is well known.

Lemma. *Let $t_1 < t_2 < \cdots < t_n < \cdots$ where $t_n \uparrow \infty$ be a given sequence of numbers. Suppose $\mathcal{B}$ is the σ field generated by the variables $\{X(t_i, \cdot)\}$, $i \geqq 1$, and $\mathcal{B}_j$, $j \geqq 1$, is the σ field generated by the variables $\{X(t, \cdot)\}$, $t_{j-1} \leqq t < t_j$ where $t_0 = 0$. Then the σ fields $\mathcal{B}_1, \mathcal{B}_2, \cdots$, are conditionally independent given $\mathcal{B}$.*

Proof. This is, of course, a general fact about Markov processes. It evidently suffices to check the independence of events $A_1, \cdots, A_r$ where $A_j \varepsilon \mathcal{B}_j$ is a cylinder set based on i_j of the variables $\{X(t, \cdot)\}, t_{j-1} \leqq t < t_j$, where r, i_j and the variables chosen are arbitrary. Let $\mathcal{B}^{(n)}$ be the σ field generated by $X(t_1, \cdot)$, $\cdots, X(t_n, \cdot)$. Since $\mathcal{B}_n \uparrow \mathcal{B}$ by the martingale convergence theorem it suffices to show that $A_1, \cdots, A_r$ are conditionally independent given $\mathcal{B}^{(n)}$ for all n sufficiently large. Therefore we need only check that if $X_1, \cdots, X_N$ is

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a discrete parameter Markov process $(X_1, \cdots, X_{i_1})$, $(X_{i_1+1}, \cdots, X_{i_2})$, $\cdots$ $(X_{i_{r-1}}, \cdots, X_N)$ are conditionally independent given $X_{i_1}, \cdots, X_{i_{r-1}}, X_N$. It is easy to see that this follows from

$$(6) \quad P[X_{k+1} \varepsilon C_{k+1}, \cdots, X_{k+r} \varepsilon C_{k+r} \mid X_1, \cdots, X_k, X_{k+r+1}, \cdots, X_N] = P[X_{k+1} \epsilon C_{k+1}, \cdots, X_{k+r} \epsilon C_{k+r} \mid X_k, X_{k+r+1}].$$

If I_C if the indicator of a Borel set C, (6) follows from

$$(7) \quad E(P[X_{k+1} \varepsilon C_{k+1}, \cdots, X_{k+r} \varepsilon C_{k+r} \mid X_k, X_{k+r+1}] \cdot I_{C_1}(X_1) \cdots I_{C_k}(X_k) I_{C_{k+r+1}}(X_{k+r+1}) \cdots I_{C_N}(X_N)) = P[X_1 \varepsilon C_1, \cdots, X_N \varepsilon C_N].$$

But the left hand side of (7) equals

$$(8) \quad E(E(I_{C_{k+1}}(X_{k+1}) \cdots I_{C_{k+r}}(X_{k+r}) P[X_1 \varepsilon C_1, \cdots, X_k \varepsilon C_k, X_{k+r+1} \varepsilon C_{k+r+1}, \cdots, X_N \epsilon C_N \mid X_k, X_{k+r+1}] \mid X_k, X_{k+r+1})).$$

Now,

$$(9) \quad P[X_1 \varepsilon C_1, \cdots, X_k \varepsilon C_k, X_{k+r+1} \varepsilon C_{k+r+1}, \cdots, X_N \varepsilon C_N \mid X_k, \cdots, X_{k+r+1}]$$
$$= E\{[I_{C_1}(X_1) \cdots I_{C_k}(X_k) E(I_{C_{k+r+1}}(X_{k+r+1}) \cdots I_{C_N}(X_N) \mid X_1, \cdots, X_{k+r+1})] \mid X_k, \cdots, X_{k+r+1}\}$$
$$= E(I_{C_1}(X_1) \cdots I_{C_k}(X_k) \mid X_k) E(I_{C_{k+r+1}}(X_{k+r+1}) \cdots I_{C_N}(X_N) \mid X_{k+r+1})$$
$$= P[X_1 \varepsilon C_1, \cdots, X_k \varepsilon C_k, X_{k+r+1} \varepsilon C_{k+r+1}, \cdots, X_N \varepsilon C_N \mid X_k, X_{k+r+1}]$$

by the Markov property (future and past; see Loève [2], p. 351–2).

Substituting the left hand side of (9) in (8) we see that

$$(10) \quad E(I_{C_{k+1}}(X_{k+1}) \cdots I_{C_{k+r}}(X_{k+r}) P[X_1 \varepsilon C_1, \cdots, X_k \varepsilon C_k, X_{k+r+1} \varepsilon C_{k+r+1}, \cdots, X_N \varepsilon C_N \mid X_k, X_{k+r+1}] \mid X_k, X_{k+r+1}) = E(I_{C_1}(X_1) \cdots I_{C_N}(X_N) \mid X_k, \cdots, X_{k+r+1}),$$

and (7) follows. □

Returning to the theorem let $t_i = i$. We need to show that with the given assumptions,

$$(11) \qquad P^{(\mathcal{B})}(\Lambda_\psi = \infty) = 1 \qquad \text{a.s.}$$

(where the superscript $(\mathcal{B})$ is used to indicate conditional probability).

Write

$$(12) \qquad \Lambda_\psi = \textstyle\sum_{i=1}^{\infty} \Lambda_\psi^{(i)},$$

where

$$(13)\qquad \Lambda_\psi^{(i)} = \lambda[t\colon X(t,\cdot) \geqq \psi(t), (i-1) \leqq t < i].$$

Given $\mathcal{B}$ by our lemma the $\Lambda_\psi^{(i)}$ are independent. Since they are also bounded and nonnegative, the Kolmogorov three series theorem ([2], p. 237) states that (11) is equivalent to

$$(14)\qquad \sum_{i=1}^\infty E^{\mathcal{B}}(\Lambda_\psi^{(i)}) = \infty \qquad \text{a.s.},$$

which by (6) and Fubini's theorem reduces to

$$(15)\qquad \sum_{i=1}^\infty \int_{(i-1)}^i P[X(t) \geqq \psi(t) \mid X((i-1),\cdot), X(i,\cdot)]\, dt = \infty \qquad \text{a.s.}$$

Of course, the right hand side of (15) equals ∞

$$(16)\qquad \sum_{i=1}^\infty \int_{(i-1)}^i \bar{\Phi}([\psi(t) - (i-t)X(i-1,\cdot) - (t-i+1)X(i,\cdot)][(t-i+1)(i-t)]^{-\frac{1}{2}})\, dt,$$

where

$$(17)\qquad \bar{\Phi}(s) = (2\pi)^{-\frac{1}{2}}\int_s^\infty \exp -\tfrac{1}{2}t^2\, dt.$$

Consider

$$(18)\qquad H_i(t) = [(i-t)(t-i+1)]^{-\frac{1}{2}}[t^{\frac{1}{2}} - (i-t)(i-1)^{\frac{1}{2}} - (t-i+1)i^{\frac{1}{2}}] \quad \text{for} \quad (i-1) \leqq t < i.$$

We claim

$$(19)\qquad H_i(t) \leqq \tfrac{1}{4}i^{-\frac{1}{2}}(i-1)^{-1}.$$

To see this write

$$(20)\qquad \begin{aligned} H_i(t) &= [(i-t)(t-i+1)]^{-\frac{1}{2}}\{(i-t)(i-1)^{\frac{1}{2}} \\ &\quad \cdot\{(1+(t-i+1)(i-1)^{-1})^{\frac{1}{2}} - 1\} - (t-i+1)i^{\frac{1}{2}} \\ &\quad \cdot\{1-(1-(i-t)i^{-1})^{\frac{1}{2}}\}\} \\ &\leqq \tfrac{1}{2}[(i-t)(t-i+1)]^{\frac{1}{2}}\{(i-1)^{-\frac{1}{2}} - i^{-\frac{1}{2}}\} \end{aligned}$$

by using $(1+x)^{\frac{1}{2}} \leqq 1 + \frac{1}{2}x$ for $x \geqq -1$. The same inequality yields (19).

Now suppose

$$(21)\qquad \lambda(t) = \lambda_i \quad \text{for} \quad (i-1) \leqq t < i.$$

Let $A = \{\omega\colon X((i-1),\omega) \geqq \lambda_i(i-1)^{\frac{1}{2}}$ and $X(i,\omega) \geqq \lambda_i i^{\frac{1}{2}}$ for infinitely many indices $i\}$. For $\omega \in A$, and this λ by (14)–(16) and (19), $\sum_{i=1}^\infty E^{\mathcal{B}}(\Lambda_\psi^{(i)}) = \infty$, if

$$(22)\qquad \liminf_i \bar{\Phi}(\tfrac{1}{4}\lambda_i i^{-\frac{1}{2}}(i-1)^{-1}) > 0.$$

But (22) follows from assumption (4). Therefore, for a λ satisfying the assumptions of the theorem and of the form (23) we need only check that $P(A) = 1$. Now, by the Kolmogorov-Petrovskii criterion (in the form given by (Strassen [3], Corollary 4.5)),

$$P[X((i-1), \cdot) \geqq \lambda_i (i-1)^{\frac{1}{2}} \text{ infinitely often}] = 1. \tag{23}$$

Let t_1 be the first index $(i-1)$ such that $X((i-1), \cdot) \geqq \lambda_i(i-1)^{\frac{1}{2}}$, t_2 be the second such index, etc. By (23) $\{t_n\}$ is a sequence of finite stopping times such that $t_n \uparrow \infty$.

Let $Z_i = X(i, \cdot) - X((i-1), \cdot)$. Evidently, $P(A) = 1$ if

$$P[Z_{t_n+1} \geqq \lambda_{t_n+1}\{(t_n+1)^{\frac{1}{2}} - t_n^{\frac{1}{2}}\} \text{ infinitely often}] = 1. \tag{24}$$

Define the σ fields $\mathfrak{F}_{t_n}$ in the usual way as the set of all events $A \varepsilon \mathfrak{A} \ni A \cap [t_n \leqq k] \varepsilon \mathfrak{B}(Z_1, \cdots, Z_k)$ for all k where $\mathfrak{B}(Z_1, \cdots, Z_k)$ is the σ field induced by $Z_1, \cdots, Z_k$. Clearly $\mathfrak{F}_{t_1} \subset \mathfrak{F}_{t_2} \subset \cdots$ and $Z_{t_1+1}, \cdots, Z_{t_{n-1}+1}$ as well as $t_1, \cdots, t_n$ are measurable $\mathfrak{F}_{t_n}$. We may therefore apply the P. Lévy $0-1$ law ([1], p. 398) to conclude that (24) holds if and only if,

$$\sum_{n=1}^{\infty} P^{\mathfrak{F}_{t_n}}[Z_{t_n+1} \geqq \lambda_{t_n+1}\{(t_n+1)^{\frac{1}{2}} - t_n^{\frac{1}{2}}\}] = \infty \quad \text{a.s.} \tag{25}$$

By a theorem of Doob ([1], Theorem 5.2, p. 145) Z_{t_n+1} is independent of $\mathfrak{F}_{t_n}$ and is distributed as Z_1. We conclude that (25) is equivalent to

$$\sum_{n=1}^{\infty} \bar{\Phi}[\lambda_{t_n+1}((t_n+1)^{\frac{1}{2}} - t_n^{\frac{1}{2}})] = \infty \quad \text{a.s.} \tag{26}$$

But this readily follows from assumption (4) and the theorem is proved for functions ψ such that λ satisfies (21).

To obtain the general case we need only note that for any ψ satisfying the assumptions of the theorem there exists $\psi^* \geqq \psi$ for t sufficiently large such that λ^* corresponding to ψ^* satisfies the assumptions of the theorem and (21). If $\lambda(t) \leqq 1$ for all t this is obvious. Otherwise, if $\lambda(a) \geqq 1$, $a \geqq 2$, $\lambda(t) \exp -\frac{1}{2}\lambda^2(t)$ is monotone decreasing for $t \geqq a$ and hence,

$$\sum_{n=a}^{\infty} \lambda(n)\, [\exp -\tfrac{1}{2}\lambda^2(n)] \log(1 + n^{-1}) \geqq \int_a^{\infty} \lambda(t) \exp(-\tfrac{1}{2}\lambda^2(t)) t^{-1}\, dt = \infty. \tag{27}$$

But

$$\begin{aligned} \sum_{n=a}^{\infty} \lambda(n)\, [\exp -\tfrac{1}{2}\lambda^2(n)] \log(1+n^{-1}) &\leqq \{\textstyle\sum_{n=a}^{\infty} \lambda(n)\, [\exp -\tfrac{1}{2}\lambda^2(n)] \log(1 + (n-1)^{-1})\} \\ &= \int_{a-1}^{\infty} \lambda^*(t) \exp(-\tfrac{1}{2}[\lambda^*(t)]^2) t^{-1}\, dt \end{aligned} \tag{28}$$

where $\lambda^*(t) = \lambda(n)$ for $(n-1) \leqq t < n$, $n \geqq 2$. This λ^* function will evidently do and the theorem is proved.

We do not know whether condition (4) may be dispensed with altogether. Evidently, we only used the fact that $\sup_n n^{-1}\psi(n) < \infty$ in order to conclude that (26) holds. Furthermore the choice of the natural numbers as "conditioning times" is arbitrary. Any arithmetic progression would have done.

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