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# A NOTE ON SEQUENCES OF CONTINUOUS PARAMETER MARKOV CHAINS

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We will consider continuous parameter Markov chains which for convenience, all have the same state space $S = \{1, 2, 3, \cdots\}$. We will assume that the transition probabilities

$$p_{ij}(t) = P\{X(t) = j \mid X(0) = i\}$$

satisfy the usual conditions

$$(1) \qquad p_{ij}(t) \geqq 0,$$

$$(2) \qquad \textstyle\sum_{j=1}^{\infty} p_{ij}(t) \leqq 1,$$

$$(3) \qquad \textstyle\sum_{k=1}^{\infty} p_{ik}(s)p_{kj}(t) = p_{ij}(s+t),$$

and

$$(4) \qquad \lim_{t\to 0} p_{ij}(t) = \delta_{ij}.$$

These conditions imply the continuous differentiability of $p_{ij}(t)$ (see Chung [1]), and we define

$$(5) \qquad q_{ij} = p'_{ij}(0).$$

The $q_{ij}$ satisfy

$$(6) \qquad 0 \leqq q_{ij} < \infty \qquad \text{for} \quad i \neq j$$

and

$$(7) \qquad \textstyle\sum_{j=1}^{\infty} q_{ij} \leqq 0.$$

In addition we will assume

$$(8) \qquad q_i \equiv -q_{ii} < \infty.$$

For a given matrix $Q = ((q_{ij}))$, whose elements satisfy (6)–(8), there exists at least one matrix of transition probabilities $P(t) = ((p_{ij}(t)))$ satisfying (1)–(5). Any such matrix is called a $Q$-transition matrix, and a Markov chain with these transition probabilities is called a $Q$-process.

We are interested in the behavior of a sequence of transition matrices $P_n(t) = ((p_{ij}^n(t)))$ with corresponding matrices $Q^n = ((q_{ij}^n))$ satisfying (1)–(8) under the assumptions that

$$(9) \qquad \lim_{n\to\infty} q_{ij}^n = q_{ij} \qquad \text{exists for all } i \text{ and } j,$$

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and

$$q_i = -q_{ii} < \infty \qquad \text{for all} \quad i. \tag{10}$$

Let $X_n(t)$ denote a Markov chain with transition matrix $P_n(t)$ and set

$$_Kp_{ij}^n(t) = P\{X_n(t) = j, X_n(s) \leqq K, 0 \leqq s \leqq t \mid X_n(0) = i\}, \quad 1 \leqq i,j \leqq K. \tag{11}$$

It can be shown (see Chung [1, II.11, II.17]) that the $_Kp_{ij}^n(t)$ satisfy conditions (1)–(4) and that for $1 \leqq i, j \leqq K$

$$\exp\{q_{ii}^n t\} \leqq {_Kp_{ii}^n}(t) \leqq p_{ii}^n(t), \tag{12}$$

$$\int_0^t q_{ij}^n \exp\{q_{ii}^n s + q_{jj}^n(t-s)\}\, ds = {_Kp_{ij}^n}(t) \leqq p_{ij}^n(t), \qquad i \neq j. \tag{13}$$

(we note that the first term in (13) is just $P\{X_n(t) = j, X_n(s)$ has only one jump in $0 < s < t \mid X_n(0) = i\}$), and hence

$$_Kp_{ij}^{n\prime}(0) = q_{ij}^n. \tag{14}$$

Letting $Q_K^n$ denote the $K \times K$ matrix $((q_{ij}^n))$, $1 \leqq i, j \leqq K$, it follows that

$$((_Kp_{ij}^n(t))) = \exp\{tQ_K^n\} = \sum_{l=0}^\infty (t^l/l!)[Q_K^n]^l. \tag{15}$$

For finite dimensional matrices, $\lim_{n\to\infty} Q_K^n = Q_K$ implies the exponentials converge also, so that $\lim_{n\to\infty} {_Kp_{ij}^n}(t) = {_Kp_{ij}}(t)$. Since $p_{ij}^n(t) > {_Kp_{ij}^n}(t)$ for all $K$, we have

$$\begin{aligned} \liminf_{n\to\infty} p_{ij}^n(t) \geqq \lim_{K\to\infty} {_Kp_{ij}}(t) &= \bar{p}_{ij}(t) \\ &= P\{X(t) = j, \sup_{0\leqq s\leqq t} X(s) < \infty \mid X(0) = i\}, \end{aligned} \tag{16}$$

for any $Q$-process $X(t)$. We observe that $((\bar{p}_{ij}(t)))$ is just the transition matrix for the minimal $Q$-process. The following theorem is immediate.

THEOREM 1. *For each $n$, let $((p_{ij}^n(t)))$ satisfy* (1)–(4) *and suppose*

$$\lim_{n\to\infty} p_{ij}^{n\prime}(0) = q_{ij} \quad \text{for all} \quad i,j \quad \text{and} \quad -q_{ii} < \infty \quad \text{for all} \quad i. \tag{17}$$

*Let $\bar{p}_{ij}(t)$ denote the transition probabilities for the minimal $((q_{ij}))$-process. Then*

$$\lim_{n\to\infty} \sum_{j=1}^\infty p_{ij}^n(t) = \sum_{j=1}^\infty \bar{p}_{ij}(t) \quad \text{for all} \quad i, t, \tag{18}$$

*implies*

$$\lim_{n\to\infty} p_{ij}^n(t) = \bar{p}_{ij}(t), \quad \text{for all} \quad i, j, t. \tag{19}$$

*In particular* (18) *must hold if*

$$\sum_{j=1}^\infty \bar{p}_{ij}(t) = 1. \tag{20}$$

REMARK. The equicontinuity of the $p_{ij}^n(t)$ implies that the convergence in (19) is uniform in compact $t$-intervals. Condition (20) implies $((\bar{p}_{ij}(t)))$ is the unique $Q$-transition matrix. If condition (20) does not hold, then there are infinitely many $Q$-transition matrices, and clearly there exist sequences of transition matrices satisfying (17) that do not satisfy (18) and (19).

Now let us explore the possibility of a converse to the above theorem. Suppose, for each $n$, $((p_{ij}^n(t)))$ satisfies conditions (1)–(4) and

(21) $$\lim_{n\to\infty} p_{ij}^n(t) = p_{ij}(t) \qquad \text{exists for all} \quad i, j, t.$$

In order to eliminate obvious degenerate cases, assume that

(22) $$\lim_{t\to 0} p_{ij}(t) = \delta_{ij}.$$

Since

$$|p_{ij}^n(t+h) - p_{ij}^n(t)| \leqq 1 - p_{ii}^n(h),$$

we have

(23) $$|p_{ij}(t+h) - p_{ij}(t)| \leqq 1 - p_{ii}(h)$$

and hence the $p_{ij}(t)$ are continuous.

Fatou's lemma implies

(24) $$p_{ij}(t+s) \geqq \sum_{k=1}^{\infty} p_{ik}(t)p_{kj}(s),$$

and without further assumptions strict inequality is possible. The existence of derivatives

$$\lim_{t\to 0} (1 - p_{ii}(t))t^{-1} = q_i \leqq \infty$$

and

$$\lim_{t\to 0} p_{ij}(t)t^{-1} = q_{ij} < \infty$$

with

$$\sum_{j\neq i} q_{ij} \leqq -q_{ii},$$

follows by essentially the same proofs as in the case when equality holds in (24). We cannot, however, conclude that

$$\lim_{n\to\infty} q_{ij}^n = q_{ij},$$

as can be seen easily by considering the sequence of transition matrices corresponding to $Q^n$ given by

$$\begin{aligned} q_{ij}^n &= -1, && i = j = 1, \\ &= 1, && i = 1, \quad j = n, \\ &= -n, && i = j = n, \\ &= n, && i = n, \quad j = 1, \\ &= 0, && \text{otherwise.} \end{aligned}$$

Remark. The Markov chains in the sequence need not all have the same state space in order to obtain interesting results. Suppose $X_n(t)$ is a chain with state space $S_n$ and that there exist a countable set $S$ and 1-1 maps $Y_n$ from $S_n$ into $S$

such that for every $i \varepsilon S$ there exists $N(i)$ for which $n > N(i)$ implies $i$ is in the range of $Y_n$.

For $n > \max(N(i), N(i))$ let $i(n) = Y_n^{-1}(i)$ and $j(n) = Y_n^{-1}(j)$. If $\lim_{n\to\infty} q^n_{i(n)j(n)} = q_{ij}$ exists for all $i, j \varepsilon S$, and $-q_{ii} < \infty$ for all $i \varepsilon S$, then

$$\lim_{r\to\infty} \textstyle\sum_{j\epsilon S_n} p^n_{i(n)j(n)}(t) = \sum_{j\epsilon S} \bar{p}_{ij}(t) \quad \text{all} \quad i \varepsilon S, \qquad t \geqq 0,$$

implies

$$\lim_{n\to\infty} p_{i(n)j(n)}(t) = \bar{p}_{ij}(t).$$

For example, consider the following theorem originally proved by Stratton and Tucker [2]:

THEOREM 2. *Let $X_n(t)$ be a branching process with offspring distribution*

$$P\{k \quad \textit{offspring}\} = p_k^n$$

*and exponential lifetime distribution with parameter $\lambda^n$. Suppose*

$$\lim_{n\to\infty} n\lambda^n = \nu < \infty \tag{25}$$

*and*

$$\lim_{n\to\infty} p_k^n = p_k \qquad \textit{with} \quad \textstyle\sum_k p_k = 1. \tag{26}$$

*Then, setting $\nu_k = p_k\nu$,*

$$\lim_{n\to\infty} p^n_{i+n,j+n}(t) = \bar{p}_{ij}(t) \qquad \textit{for all integers} \quad i, j,$$

*where $\bar{p}_{ij}(t)$ are the transition probabilities of a spatially homogeneous process with*

$$\bar{p}'_{ij}(0) = \nu_{j-i+1}, \qquad i \neq j,$$

*and*

$$-\bar{p}'_{ii}(0) = \nu - \nu_1 .$$

PROOF. Letting $Y_n(i) = i - n$ and hence $Y_n^{-1}(i) = i + n$, we note that

$$q^n_{i+n,j+n} = (i + n)\lambda^n p^n_{j-i+1} \qquad i, j \geqq -n, \qquad i \neq j,$$

and

$$q^n_{i+n,i+n} = -(i + n)\lambda^n(1 - p_1^n) \qquad i \geqq -n.$$

Conditions (24) and (25) imply

$$\lim_{n\to\infty} q^n_{i+n,j+n} = \nu_{j-i+1} \qquad i \neq j,$$

and

$$\lim_{n\to\infty} q^n_{i+n,i+n} = \nu_1 - \nu.$$

Clearly the minimal process for

$$q_{ij} = \nu_{j-i+1} \qquad i \neq j,$$

and

$$q_{ii} = \nu_1 - \nu$$

satisfies

$$\textstyle\sum_j \bar{p}_{ij}(t) = 1,$$

and the theorem follows.

## REFERENCES

[1] Chung, K. L. (1967). *Markov Chains with Stationary Transition Probabilities* (2nd ed.). Springer-Verlag, New York.

[2] Stratton, Howard H., Jr., and Howard G. Tucker. (1964). Limit distributions of a branching stochastic process. *Ann. Math. Statist.* **35** 557–565.