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THE VARIANCE OF ONE-SIDED STOPPING RULES

BY DAVID SIEGMUND[1]

Stanford University

Let $x_1, x_2, \cdots$ be independent random variables with means $\mu_1, \mu_2, \cdots$ for which for some $0 < \mu < \infty$

$$(1) \qquad n^{-1} \sum_1^n \mu_k \rightarrow \mu \qquad (n \rightarrow \infty).$$

Let $s_n = \sum_1^n x_k$, and for each $c > 0$ define

$$(2) \qquad t = t(c) = \text{first } n \geqq 1 \text{ such that } s_n > c$$
$$= \infty \text{ if no such } n \text{ exists.}$$

It is easily inferred from the results and methods of [5] that if

$$(3) \qquad \sup_n n^{-1} \sum_1^n E(x_k - \mu_k)^- < \infty,$$

and if for each $\epsilon > 0$

$$(4) \qquad \lim_{n\rightarrow\infty} n^{-1} \sum_1^n \int_{\{x_k - \mu_k > \epsilon n\}} (x_k - \mu_k) = 0,$$

then $Et < \infty$ for each $c > 0$ and $Et \sim c\mu^{-1} (c \rightarrow \infty)$. Under more restrictive conditions on the distributions of the x's an asymptotic expression for the variance of t may be obtained. To be specific, if the x's are identically distributed, non-negative, and if $\sigma^2 = Ex_1^2 - \mu^2 < \infty$, then it has been shown by Feller [2] in the lattice and Smith [6] in the non-lattice case that

$$(5) \qquad \operatorname{Var} t \sim c\sigma^2\mu^{-3} \qquad (c \rightarrow \infty).$$

Recently, using combinational results of Spitzer [7], Heyde [4] has shown that (5) holds without the restriction to non-negative variables. The methods of Feller, Smith, and Heyde involve finding sufficiently detailed expansions of Et^2 and Et, from which (5) may be deduced. Smith and Heyde use Blackwell's Renewal Theorem.

In this note we generalize (5) to a large class of non-identically distributed x's. Our method involves Wald's lemma for squared sums [1] and the technique of Gundy and Siegmund [3] (see also [5]).

THEOREM. *Let $x_1, x_2, \cdots$ be independent random variables with means μ_1, $\mu_2, \cdots$ such that for some $0 < \mu < \infty$*

$$(6) \qquad \sum_1^n \mu_k - n\mu = o(n^{\frac{1}{2}}).$$

Let $\sigma_n^2 = Ex_n^2 - \mu_n^2$, $b_n^2 = \sum_1^n \sigma_k^2$ $(n = 1, 2, \cdots)$, and suppose that for some $0 < \sigma^2 < \infty$

$$(7) \qquad b_n^2 \sim n\sigma^2.$$

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Let t be defined by (2). *If for each $\epsilon > 0$*

$$\lim_{n\to\infty} n^{-1} \textstyle\sum_1^n \int_{\{x_k-\mu_k>\epsilon n^{\frac{1}{2}}\}} (x_k - \mu_k)^2 = 0, \tag{8}$$

then

$$Et = c\mu^{-1} + o(c^{\frac{1}{2}}) \tag{9}$$

and (5) *holds.*

We shall utilize the following lemmas.

LEMMA 1. *If* (1) *and* (7) *hold, then for any stopping time τ with finite expectation*

$$Es_\tau = E(\textstyle\sum_1^\tau \mu_k) \tag{10}$$

and

$$Eb_\tau^2 = E(s_\tau - \textstyle\sum_1^\tau \mu_k)^2. \tag{11}$$

PROOF. By Theorem 2 of [1], in order that (11) and

$$E(s_\tau - \textstyle\sum_1^\tau \mu_k) = 0 \tag{12}$$

hold it suffices that $Eb_\tau^2 < \infty$, which by (7) is implied by $E\tau < \infty$. Since (1) and $E\tau < \infty$ imply that $E|\sum_1^\tau \mu_k| < \infty$, (10) follows from (12).

LEMMA 2. *If* (6), (7), *and* (8) *hold, then for any non-decreasing family $\{\tau(r), r > 0\}$ of stopping times for which*

$$\infty > E\tau(r) \uparrow \infty \qquad \text{as} \quad r \to \infty,$$

we have

$$E(x_{\tau(r)}^+)^2 < \infty \qquad \text{for all} \quad r > 0$$

and

$$E(x_{\tau(r)}^+)^2 = o(E\tau(r)) \qquad (r \to \infty).$$

PROOF. For any $r > 0$, $E(x_\tau^+)^2 \leqq 2[E((x_\tau - \mu_\tau)^+)^2 + E|\mu_\tau|^2]$. From (6) it follows that $\mu_n = o(n^{\frac{1}{2}})$, and hence $E|\mu_\tau|^2 < \infty$ for all $r > 0$,

$$E|\mu_\tau|^2 = o(E\tau) \qquad (r \to \infty).$$

The remainder of the proof may be completed along the lines of the proof of Theorem 1 of Gundy and Siegmund [3].

LEMMA 3. *If* (1) *and* (7) *hold, then*

$$Et \sim c\mu^{-1} \qquad (c \to \infty). \tag{13}$$

PROOF. By the result mentioned in the first paragraph of this note, it suffices to verify (3) and (4). For any $k = 1, 2, \cdots$

$$E|x_k - \mu_k| \leqq E(1 + |x_k - \mu_k|)^2 \leqq 2E(1 + (x_k - \mu_k)^2) = 2(1 + \sigma_k^2),$$

which in conjunction with (7) proves (3); (4) follows from (7) and the observation that for any $\epsilon > 0, n = 1, 2, \cdots, k = 1, \cdots, n,$

$$\int_{\{x_k-\mu_k>\epsilon n\}} (x_k - \mu_k) \leqq (\epsilon n)^{-1}\sigma_k^2.$$

PROOF OF THE THEOREM. For ease of exposition we shall henceforth assume that $\mu_n \equiv \mu$, $\sigma_n \equiv \sigma$. By Lemma 1, for all $c > 0$

$$\mu Et = Es_t = c + E(s_t - c). \tag{14}$$

(By Lemma 3 $Et < \infty$ for all c.) From Lemmas 2 and 3 it follows that

$$[E(s_t - c)]^2 \leqq E(s_t - c)^2 \leqq Ex_t^2 = o(Et) = o(c), \tag{15}$$

which together with (14) establishes (9). From Lemma 1 we obtain

$$\begin{aligned} &\mu^{-1}\sigma^2[c + E(s_t - c)] \\ &\quad = \mu^{-1}\sigma^2 Es_t = \sigma^2 Et = E(s_t - \mu t)^2 = E(s_t - c + c - \mu t)^2 \\ &\quad = E(s_t - c)^2 + 2\mu E(s_t - c)(c\mu^{-1} - t) + \mu^2 E(t - c\mu^{-1})^2, \end{aligned} \tag{16}$$

so

$$\begin{aligned} \mu^2 E(t - c\mu^{-1})^2 = \sigma^2\mu^{-1}c &+ 2\mu E(s_t - c)(t - c\mu^{-1}) \\ &+ \sigma^2\mu^{-1}E(s_t - c) - E(s_t - c)^2. \end{aligned} \tag{17}$$

By (15) and the Cauchy-Schwarz inequality

$$|E(s_t - c)(t - c\mu^{-1})| \leqq [E(s_t - c)^2 E(t - c\mu^{-1})^2]^{\frac{1}{2}} = o(c^{\frac{1}{2}})[E(t - c\mu^{-1})^2]^{\frac{1}{2}}. \tag{18}$$

From (15), (17), and (18) we obtain

$$\mu^2 E(t - c\mu^{-1})^2 \leqq \sigma^2\mu^{-1}c + o(c^{\frac{1}{2}})[E(t - c\mu^{-1})^2]^{\frac{1}{2}} + o(c),$$

and it follows that

$$E(t - c\mu^{-1})^2 = O(c). \tag{19}$$

Hence by (15), (17), (18), and (19) we obtain

$$E(t - c\mu^{-1})^2 = \sigma^2\mu^{-3}c + o(c). \tag{20}$$

But by (15)

$$\begin{aligned} \operatorname{Var} t &= E(t - c\mu^{-1})^2 - [E(t - c\mu^{-1})]^2 \\ &= E(t - c\mu^{-1})^2 - [\mu^{-1}E(s_t - c)]^2 \\ &= E(t - c\mu^{-1})^2 + o(c), \end{aligned}$$

which together with (20) implies (5). (Note that in expanding $E(s_t - \mu t)^2$ in formula (16) we have tacitly assumed that $Et^2 < \infty$ and $Es_t^2 < \infty$. That $Es_t^2 < \infty$ follows from Lemma 2. To show that $Et^2 < \infty$, let $\tau = \min(t, n)$ $(n = 1, 2, \cdots)$. Then by reasoning very similar to that employed above it may be inferred that

$$E\tau^2 \leqq \text{const.}\ (E\tau + (E\tau^2)^{\frac{1}{2}}(E\tau)^{\frac{1}{2}}),$$

from which it follows that

$$Et^2 = \lim_{n\to\infty} E\tau^2 < \infty.)$$

REMARK. It is easy to deduce Lemma 3 directly without reference to the results of [5]. The essential ingredients are already present in formulas (14) and (15) (minus the $o(c)$ term in (15), which is a consequence of Lemma 3). Whereas this approach makes our proof self-contained, it was deemed of some value to point out that the present assumptions actually imply those of [5].

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