

ON BRANCHING PROCESSES IN RANDOM ENVIRONMENTS¹

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0. Summary. $\{\zeta_n\}$ is a sequence of iid "environmental" variables in an abstract space Θ . Each point $\zeta \in \Theta$ is associated with a pgf $\phi_\zeta(s)$. The branching process $\{Z_n\}$ is defined as a Markov chain such that $Z_0 = k$, a finite integer, and given Z_n and ζ_n , Z_{n+1} is distributed as the sum of Z_n iid random variables, each with pgf $\phi_{\zeta_n}(s)$. Set $\xi(\zeta_n) = \phi'_{\zeta_n}(1)$ and assume that $E|\log \xi(\zeta_n)| < \infty$. Then: (i) $P\{Z_n = 0\} \rightarrow 1$ if $E \log \xi(\zeta_n) \leq 0$; (ii) $q_k =_{\text{def}} \lim P\{Z_n = 0\} < 1$ if $E \log \xi(\zeta_n) > 0$ and $E|\log(1 - \phi_{\zeta_n}(0))| < \infty$. Furthermore $\{q_k\}$, $k = 1, 2, \dots$, forms a *moment sequence*.

1. Introduction and description of the process. In the preface to *The Theory of Branching Processes*, Harris (1963) defines a branching process as "a mathematical representation of the development of a population whose members reproduce and die, according to laws of chance. The objects may be of different types, depending on their age, energy, position, or other factors. However, they must not interfere with one another." This assumption, that different objects reproduce independently, unifies the mathematical theory and characterizes virtually all of the branching process models in the literature. While this assumption allows the definition to encompass a large number of models, it also limits the application of the models of branching processes, since the natural processes of multiplication are often affected by interaction among objects or other factors which introduce dependencies.

The model with which we shall be concerned in this paper may be described mathematically as follows. Let $\{\zeta_n\}$, $n = 0, 1, 2, \dots$, be an infinite sequence of independent and identically distributed "environmental variables" taking values in some space Θ . We suppose that associated with each point $\zeta \in \Theta$ is a probability generating function (pgf)

$$(1.1) \quad \phi_\zeta(s) = \sum_{j=0}^{\infty} p_j(\zeta) s^j, \quad 0 \leq s \leq 1,$$

and we further suppose that for each fixed s , $0 \leq s \leq 1$, $\{\phi_{\zeta_n}(s)\}$ is a sequence of independent and identically distributed random variables. Define a matrix (P_{ij}) with elements

$$(1.2) \quad P_{ij} = \text{coefficient of } s^j \text{ in } E[\phi_{\zeta_n}(s)]^i, \quad i, j = 0, 1, \dots$$

Because of our assumptions, P_{ij} is independent of n . Clearly $P_{ij} \geq 0$ for all i and j , and since

$$(1.3) \quad \sum_{j=0}^{\infty} P_{ij} s^j = E[\phi_{\zeta_n}(s)]^i$$

it follows, putting $s = 1$, that $\sum_{j=0}^{\infty} P_{ij} = 1$ for all i .

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We can define a temporally homogeneous Markov chain $\{Z_n\}$ on the non-negative integers by choosing initial probabilities

$$\begin{aligned} P(Z_0 = i) &= \delta_{i,k} = 1, \quad i = k \\ &= 0, \quad i \neq k, \end{aligned}$$

for some positive integer k , and defining

$$P(Z_0 = a_0, \dots, Z_n = a_n) = P(Z_0 = a_0)P_{a_0 a_1} \cdots P_{a_{n-1} a_n}.$$

If $P(Z_n = i) > 0$, then P_{ij} is the transition probability

$$P(Z_{n+1} = j | Z_n = i).$$

While all our results follow from the mathematical description of the model given above, we may interpret the process $\{Z_n\}$ as a branching process developing in an environment which changes stochastically in time and which affects the reproductive behavior of the population. For example, the development of an animal population is often affected by such environmental factors as weather conditions, food supply, and so forth. In the physical interpretation of the model, each point in the space Θ represents a possible state of the environment, and the assumptions about $\{\zeta_n\}$ imply that environmental states are independently sampled from one generation to the next.

As in the classical Galton-Watson process [cf. Harris (1963), Chapter I], we consider objects that can generate additional objects of the same kind. The initial set of objects, called the zeroth generation, has offspring that constitute the first generation; their offspring constitute the second generation, and so on. Since we are interested only in the sizes of the successive generations, and not the number of offspring of individual objects, we shall let Z_n , $n = 0, 1, \dots$, denote the size of the n th generation, and shall hereafter assume that $Z_0 = 1$ unless stated otherwise.

In the Galton-Watson process, it is assumed that the number of offspring of different objects are independent, identically distributed random variables with probability generating function $\phi(s)$, say. In our model, the probability generating function $\phi(s)$ is replaced by one of a family of probability generating functions $\{\phi_\zeta(s), \zeta \in \Theta\}$, depending on the "state" of the environment at the generation under consideration. Thus the probability $p_j(\zeta_n)$ in (1.1) is interpreted as the probability that an object existing in the n th generation has j offspring in the $(n + 1)$ st generation.

More specifically, we assume that the environment passes through a sequence of "states" governed by the process $\{\zeta_n\}$. Given ζ_n , the number of offspring of different objects in the n th generation are independent, identically distributed random variables with probability generating function $\phi_{\zeta_n}(s)$.

As $s \uparrow 1$, for arbitrary fixed $\zeta \in \Theta$,

$$[(1 - \phi_\zeta(s))/(1 - s)]$$

increases to a limit $\xi(\zeta)$, say, which is the mean family size for families born in

environment ζ . It is not difficult to show that $\{\xi(\zeta_n)\}$ is a sequence of independent and identically distributed random variables. We make the important assumption, however, that

$$P\{\xi(\zeta_n) < \infty\} = 1.$$

To avoid triviality we shall introduce two further assumptions, namely that, for every n ,

$$A(i): P\{p_0(\zeta_n) < 1\} = 1$$

$$A(ii): P\{p_0(\zeta_n) + p_1(\zeta_n) < 1\} > 0.$$

In the theory of the classical Galton-Watson branching process the determination of conditions for almost certain extinction is of paramount importance. The primary purpose of the present paper is to study the same fundamental question of extinction, but for the more general branching process in random environments, as outlined above. If such a process will become extinct, with probability one, we find it convenient to say the process is *mortal*; if the probability of ultimate extinction is strictly less than one, we shall say the process is *immortal*. Our main result, Theorem 3.1, shows: (a) the process is mortal if $E \log \xi(\zeta_n) \leq 0$; (b) the process is immortal if $E \log \xi(\zeta_n) > 0$ and if *additionally* $E|\log \{1 - \phi_{\zeta_n}(0)\}| < \infty$. Since a preliminary account of the present research appeared (Wilkinson, 1967) it has been shown (Smith, 1968) that the conditions (b) are *necessary* for immortality of the process. Thus the latter paper and the present one completely settle, at an acceptable level of generality, the matter of necessary and sufficient conditions for almost certain extinction of a branching process in random environments. The striking feature of conditions (b) for immortality is the involvement of the probability of no offspring to a particular parent; this feature of conditions (b) can be seen in Smith (1968) to be concerned with preventing "catastrophes" in which almost the entire population dies out in a single generation.

It might be pointed out that the elegant functional equations that play such a vital role in the theory of the classical Galton-Watson process, and many published generalizations thereof, do not arise in the present study. Instead a certain dual Markov process $\{X_n\}$ emerges, taking values on the unit interval. It transpires that this X_n -process converges in distribution, there being some limiting ergodic distribution function $G(x)$ on $[0, 1]$. This df $G(x)$ has an intriguing and intimate connection with the Z_n -process. If we write q_k for the probability of ultimate extinction of the Z_n -process when, initially, $Z_0 = k$, then Theorem 4.4 shows that

$$(1.4) \quad q_k = \int_0^1 x^k dG(x).$$

In other words *the extinction probabilities* q_k form a moment sequence. Important use will be made of this fact in a further paper (Wilkinson, 1969) concerned with actually calculating the probabilities $\{q_k\}$.

The present theory includes the classical theory as a special case; in that earlier theory the ergodic df $G(x)$ reduces to a degenerate distribution placing unit probability at some point q , $0 \leq q \leq 1$. As is well known in this case $q_k = q^k$, in agreement with (1.4). Furthermore, it now appears that the present theory can be generalized, with little difficulty, to cover the situation where the environmental variables $\{\zeta_n\}$ have Markovian dependence rather than being independent. This generalization will be published in the near future.

2. Some preliminary results concerning Z_n . In this section we shall prove a number of theorems about Z_n which are basic to our derivation of a.s. extinction conditions.

THEOREM 2.1. *If $\Pi_n(s)$ is the pgf of Z_n then*

$$(2.1) \quad \Pi_n(s) = E\Pi_{n-1}(\phi_{\zeta_{n-1}}(s)), \quad n = 1, 2, \dots$$

PROOF. Clearly,

$$\begin{aligned} \Pi_n(s) &= E(s^{Z_n}) \\ &= \sum_{i=0}^{\infty} P(Z_{n-1} = i)E(s^{Z_n} | Z_{n-1} = i) \\ &= \sum_{i=0}^{\infty} P(Z_{n-1} = i)E[\phi_{\zeta_{n-1}}(s)]^i \end{aligned}$$

from (1.3). By bounded convergence, we obtain

$$\Pi_n(s) = E\{\sum_{i=0}^{\infty} P(Z_{n-1} = i)[\phi_{\zeta_{n-1}}(s)]^i\} = E\Pi_{n-1}(\phi_{\zeta_{n-1}}(s)),$$

and the proof is complete.

By repeated application of Theorem 2.1, we obtain the representation

$$(2.2) \quad \Pi_n(s) = E\phi_{\zeta_0}(\phi_{\zeta_1}(\dots \phi_{\zeta_{n-1}}(s) \dots)).$$

THEOREM 2.2. *If $m = E\xi(\zeta_n) < \infty$ and if $Z_0 = 1$ then*

$$(2.3) \quad EZ_n = m^n, \quad n = 0, 1, \dots$$

PROOF. For convenience, let us write $\Pi_n'(1)$, for example, for the left-hand derivative of $\Pi_n(s)$ at $s = 1$. If $m < \infty$, we can deduce from (2.1) by monotone convergence that

$$\Pi_n'(1) = \Pi_{n-1}'(1)E\xi(\zeta_{n-1}) = m \Pi_{n-1}'(1).$$

Thus (2.3) follows by an obvious inductive argument.

THEOREM 2.3. *For every positive integer N ,*

$$P(0 < Z_n < N) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

PROOF. Let $z > 0$, be an integer. If $P(\phi_{\zeta_n}(0) > 0) > 0$, then

$$P(Z_{n+1} = 0 | Z_n = z) > 0.$$

Hence

$$P(Z_{n+t} = z \text{ for some } t = 1, 2, \dots | Z_n = z) \leq 1 - P(Z_{n+1} = 0 | Z_n = z) < 1,$$

so the state z of the Markov chain $\{Z_n\}$ is transient.

If $P(\phi_{i_n}(0) > 0) = 0$, then by A (ii)

$$P(Z_{n+1} > z \mid Z_n = z) > 0.$$

But if $\phi_{i_n}(0)$ almost surely vanishes then $\{Z_n\}$ must be nondecreasing, almost surely. Hence

$$P(Z_{n+t} = z \text{ for some } t = 1, 2, \dots \mid Z_n = z) \leq 1 - P(Z_{n+1} > z \mid Z_n = z) < 1.$$

Therefore, once again, z is transient. A consequence of the transience of every "state" $z > 0$ is that

$$\lim_{n \rightarrow \infty} P(Z_n = z) = 0.$$

The theorem now follows.

THEOREM 2.4. *There is a constant c , $0 \leq c \leq 1$, and for every $s \in [0, 1)$*

$$\Pi_n(s) \rightarrow c, \quad \text{as } n \rightarrow \infty.$$

PROOF. Clearly $\Pi_n(0)$ is a nondecreasing function of n , and so tends to a limit c , say ($0 \leq c \leq 1$), as n tends to infinity.

If N is a positive integer,

$$\Pi_n(s) = P(Z_n = 0) + \sum_{i=1}^N P(Z_n = i)s^i + \sum_{i=N+1}^{\infty} P(Z_n = i)s^i.$$

Given s , $0 < s < 1$, and an arbitrarily small $\epsilon > 0$, choose N sufficiently large that $s^{N+1}/(1-s) < \epsilon/2$. Then

$$\sum_{i=N+1}^{\infty} P(Z_n = i)s^i < \epsilon/2.$$

By Theorem 2.3, we can choose n sufficiently large that

$$\sum_{i=1}^N P(Z_n = i) < \epsilon/2.$$

Thus, for n sufficiently large,

$$\Pi_n(s) \leq \Pi_n(0) + \epsilon.$$

Hence, since ϵ is arbitrary,

$$\limsup_{n \rightarrow \infty} \Pi_n(s) \leq c.$$

But $\Pi_n(0) \leq \Pi_n(s)$, so $c \leq \liminf_{n \rightarrow \infty} \Pi_n(s)$ and the proof is complete.

Extinction is the event that the random sequence $\{Z_n\}$ consists of zeros for all but a finite number of values of n . Since $P(Z_{n+k} = 0 \mid Z_n = 0) = 1$ for $k = 1, 2, \dots$, extinction is equivalently the event that $Z_n = 0$ for some $n = 1, 2, \dots$. It follows that if q is the probability of extinction, then

$$(2.4) \quad q = \lim_{n \rightarrow \infty} \Pi_n(0).$$

The following theorem shows that $m \leq 1$ is a sufficient condition for extinction with probability one; this condition is, however, not necessary.

THEOREM 2.5. *If $m \leq 1$, then $q = 1$.*

PROOF. Since $EZ_n = m^n$, it follows that for $m \leq 1$ and N an arbitrary positive integer,

$$P(Z_n \geq N) \leq (1/N)EZ_n \leq 1/N.$$

Given $\epsilon > 0$, we can choose N sufficiently large that

$$(2.5) \quad P(Z_n \geq N) < \epsilon/2$$

for all n , and then choose n sufficiently large that

$$(2.6) \quad P(0 < Z_n < N) < \epsilon/2,$$

(Theorem 2.3). It follows from (2.5) and (2.6) that for n sufficiently large,

$$P(Z_n = 0) > 1 - \epsilon.$$

Hence

$$\lim_{n \rightarrow \infty} \Pi_n(0) \geq 1 - \epsilon.$$

The theorem follows from the arbitrary smallness of ϵ .

3. Conditions for almost certain extinction. Consider the discrete-parameter Markov process $\{X_n\}$ on the unit interval, defined as follows: for arbitrary but fixed $s_0 \in [0, 1)$, let $X_0 = s_0$, and define

$$X_{n+1} = \phi_{\zeta_n}(X_n), \quad n = 0, 1, \dots$$

The stochastic process $\{X_n\}$ will be called the *dual process* associated with the branching process $\{Z_n\}$.

For the expected location of the dual process after n steps, we obtain straightforwardly

$$(3.1) \quad E(X_n | X_0 = s_0) = E\phi_{\zeta_{n-1}}(\phi_{\zeta_{n-2}}(\dots \phi_{\zeta_0}(s_0) \dots)).$$

Since the random variables $\zeta_0, \zeta_1, \dots, \zeta_n$ are independent and identically distributed, we may renumber them without affecting the value of the right-hand side of (3.1). Hence, replacing ζ_r by ζ_{n-1-r} , $r = 0, 1, \dots, n - 1$, we have

$$E(X_n | X_0 = s_0) = E\phi_{\zeta_0}(\phi_{\zeta_1}(\dots \phi_{\zeta_{n-1}}(s_0) \dots)).$$

On comparing this expression with (2.2), we discover that

$$(3.2) \quad E(X_n | X_0 = s_0) = \Pi_n(s_0).$$

Let U_1, U_2, \dots be a sequence of independent, identically distributed random variables such that $E|U_1| < \infty$.

Define a sequence of random variables $W_n, n = 0, 1, \dots$, as follows: $W_0 = 0$, and

$$\begin{aligned} W_{n+1} &= W_n + U_{n+1}, & \text{if } W_n + U_{n+1} > 0, \\ &= 0, & \text{if } W_n + U_{n+1} \leq 0, \end{aligned} \quad n = 0, 1, \dots$$

A sequence $\{W_n\}$ of nonnegative random variables defined in this manner will be called a *Lindley process*. The following result is due to Lindley (1952), who introduced such processes in connection with a waiting-time problem in queueing theory.

THEOREM A. *Suppose $E|U_1|$ is finite and U_1 is not zero with probability one. Then a necessary and sufficient condition that the distribution function $P(W_n \leq x)$ tends to a nondefective limit distribution function as $n \rightarrow \infty$ is that $EU_1 < 0$. If $EU_1 \geq 0$, $P(W_n \leq x)$ tends to zero for any $x \geq 0$.*

For typographical simplicity we shall henceforth write η_n for $\phi_{\zeta_n}(0)$. Note that, in view of our assumptions, $\{\eta_n\}$ is a sequence of independent and identically distributed random variables.

THEOREM 3.1. *Suppose that $E|\log \xi(\zeta_n)| < \infty$.*

(a) *If $E \log \xi(\zeta_n) \leq 0$, then $P(Z_n > 0) \rightarrow 0$ as $n \rightarrow \infty$, i.e. the branching process is mortal.*

(b) *If $E \log \xi(\zeta_n) > 0$ and if, additionally,*

$$E|\log (1 - \eta_n)| < \infty,$$

then $P(Z_n > 0)$ tends to some strictly positive limit as $n \rightarrow \infty$, i.e. the branching process is immortal.

PROOF. (a) $E \log \xi(\zeta_n) \leq 0$. If $\xi(\zeta_n) = 1$ with probability one, then $m = 1$, and by Theorem 2.5, $\Pi_n(0) \rightarrow 1$ as $n \rightarrow \infty$. Thus we can henceforth assume that $P(\xi(\zeta_n) = 1) < 1$.

Let $M_1 = \{\zeta: \xi(\zeta) \leq 1\}$ and $M_2 = \{\zeta: \xi(\zeta) > 1\}$. Define piecewise linear function ψ_ζ , $\zeta \in \Theta$, on the unit interval by

$$\psi_\zeta(s) = (1 - \xi(\zeta)) + \xi(\zeta)s, \quad \zeta \in M_1,$$

(3.3) and

$$\begin{aligned} \psi_\zeta(s) &= 0, & s < 1 - 1/\xi(\zeta), \\ &= (1 - \xi(\zeta)) + \xi(\zeta)s, & s \geq 1 - 1/\xi(\zeta), \end{aligned} \quad \zeta \in M_2.$$

Plainly $\psi_\zeta(1) = 1$ and $\psi'_\zeta(1) = \phi'_\zeta(1)$. In addition, $\psi_\zeta(s) \leq \phi_\zeta(s)$ for all $s \in [0, 1]$.

Let us now define on the unit interval a process $\{Y_n\}$, $n = 0, 1, 2, \dots$, based on the functions ψ_ζ as follows:

$$\begin{aligned} Y_0 &= 0 \\ Y_{n+1} &= \psi_{\zeta_n}(Y_n), \quad n = 0, 1, \dots \end{aligned}$$

It is not hard to see that

$$\begin{aligned} E(Y_n | Y_0 = 0) &= E\psi_{\zeta_{n-1}}(\psi_{\zeta_{n-2}}(\dots \psi_{\zeta_0}(0) \dots)) \\ &\leq E\psi_{\zeta_{n-1}}(\psi_{\zeta_{n-2}}(\dots \phi_{\zeta_0}(0) \dots)) \\ &\leq \dots \\ &\leq E\phi_{\zeta_{n-1}}(\phi_{\zeta_{n-2}}(\dots \phi_{\zeta_0}(0) \dots)) \\ &= E(X_n | X_0 = 0). \end{aligned}$$

If we write $s = 1 - e^{-x}$, $x \geq 0$, the equations (3.3) become

$$\begin{aligned} \psi_{\zeta}(1 - e^{-x}) &= 1 - e^0, & x - \log \xi(\zeta) < 0, \\ &= 1 - e^{-(x - \log \xi(\zeta))}, & x - \log \xi(\zeta) \geq 0. \end{aligned}$$

At this point we introduce a further process:

$$W_n = -\log (1 - Y_n), \quad n = 0, 1, \dots$$

It follows that

$$P(W_n \leq x) = P(Y_n \leq 1 - e^{-x}), \quad x \geq 0, \quad n = 0, 1, \dots$$

Let us write U_1, U_2, \dots for the sequence of independent, identically distributed random variables given by

$$U_n = -\log \xi(\zeta_{n-1}), \quad n = 1, 2, \dots$$

Then

$$\begin{aligned} Y_{n+1} &= \psi_{\zeta_n}(Y_n) \\ &= \psi_{\zeta_n}(1 - e^{-W_n}) \\ &= \begin{cases} 0, & W_n + U_{n+1} < 0, \\ 1 - e^{-(W_n + U_{n+1})}, & W_n + U_{n+1} \geq 0. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} W_{n+1} &= -\log (1 - Y_{n+1}) \\ &= \begin{cases} 0, & W_n + U_{n+1} < 0, \\ W_n + U_{n+1}, & W_n + U_{n+1} \geq 0. \end{cases} \end{aligned}$$

It appears, therefore, that $\{W_n\}$ is a Lindley process. Since

$$EU_n = E(-\log \xi(\zeta_{n-1})) \geq 0,$$

it follows from Theorem A (Lindley) that

$$P(W_n \leq x) \rightarrow 0 \quad \text{for all } x \geq 0, \quad \text{as } n \rightarrow \infty.$$

Hence

$$P(Y_n \leq 1 - e^{-x} \mid Y_0 = 0) \rightarrow 0 \quad \text{for all } x \geq 0, \quad \text{as } n \rightarrow \infty;$$

i.e. $Y_n \rightarrow 1$ in probability, given $Y_0 = 0$. By bounded convergence, $E(Y_n \mid Y_0 = 0) \rightarrow 1$, and thus $E(X_n \mid X_0 = 0) \rightarrow 1$. Finally, by (3.2), $\Pi_n(0) \rightarrow 1$ as $n \rightarrow \infty$; that is, $P(Z_n > 0) \rightarrow 0$ so the process is mortal.

(b) $E \log \xi(\zeta_n) > 0$ and $E|\log (1 - \eta_n)| < \infty$. Write $a = E \log \xi(\zeta_n)$. Since $E|\log (1 - \eta_n)| < \infty$, we can choose c_0' sufficiently close to unity that

$$(3.4) \quad \left| \int_{\eta_n > c_0} \log (1 - \eta_n) dP \right| < a/4$$

for all $c_0 \geq c_0'$. Next choose $c_0 \geq c_0'$ so large that

$$\int_{\eta_n \leq c_0} \log \xi(\zeta_n) dP > 3a/4.$$

Then we can find a large M' such that

$$(3.5) \quad \int_{\eta_n \leq c_0, \xi(\zeta_n) \leq M} \log \xi(\zeta_n) dP > a/2$$

for all $M \geq M'$. Finally, fix $M \geq M'$ so large that

$$(3.6) \quad \log M > |\log(1 - c_0)|$$

and

$$(3.7) \quad \int_{\xi(\zeta_n) > M} \log \xi(\zeta_n) dP < a/4.$$

Since

$$\int_{\xi(\zeta_n) > M} \log \xi(\zeta_n) dP \geq (\log M)P\{\xi(\zeta_n) > M\}$$

it follows from (3.7) that

$$P\{\xi(\zeta_n) > M\} < a/(4 \log M).$$

Hence

$$(3.8) \quad |\int_{\xi(\zeta_n) > M, \eta_n \leq c_0} \log(1 - \eta_n) dP| \leq a|\log(1 - c_0)|/(4 \log M) < a/4,$$

by (3.6). Let us define

$$\begin{aligned} \beta_n(s) &= 1 - \eta_n, && \text{if } \eta_n > c_0, \\ &= (1 - \phi_{\zeta_n}(s))/(1 - s), && \text{if } \eta_n \leq c_0 \text{ and } \xi(\zeta_n) \leq M, \\ &= 1 - \eta_n, && \text{if } \eta_n \leq c_0 \text{ and } \xi(\zeta_n) > M. \end{aligned}$$

Thus, for each fixed s , $0 \leq s < 1$, $\beta_n(s)$ is a random variable. We wish to show that for s sufficiently close to 1, $E \log \beta_n(s) > 0$. To this end we note that, by bounded convergence,

$$\lim_{s \uparrow 1} \int_{\eta_n \leq c_0, \xi(\zeta_n) \leq M} \log[(1 - \phi_{\zeta_n}(s))/(1 - s)] dP = \int_{\eta_n \leq c_0, \xi(\zeta_n) \leq M} [\log \xi(\zeta_n)] dP.$$

Hence

$$\begin{aligned} \lim_{s \uparrow 1} E \log \beta_n(s) &= \int_{\eta_n > c_0} \log(1 - \eta_n) dP + \int_{\eta_n \leq c_0, \xi(\zeta_n) \leq M} [\log \xi(\zeta_n)] dP \\ &\quad + \int_{\eta_n \leq c_0, \xi(\zeta_n) > M} \log(1 - \eta_n) dP > -(a/4) + (a/2) - (a/4) = 0, \end{aligned}$$

by inequalities (3.4), (3.5) and (3.8). Therefore there exists t such that $E \log \beta_n(t) > 0$.

We shall describe the construction of certain functions $\psi_{\zeta}(s)$ which dominate the functions $\phi_{\zeta}(s)$ and are piecewise linear. For each $\zeta \in \Theta$ we set

$$\begin{aligned} \xi'(\zeta) &= (1 - \phi_{\zeta}(t))/(1 - t), && \text{if } \phi_{\zeta}(0) \leq c_0 \text{ and } \phi_{\zeta}'(1) \leq M, \\ &= 1 - \phi_{\zeta}(0), && \text{otherwise.} \end{aligned}$$

(a) If $\phi_{\zeta}(0) > c_0$ or if $\phi_{\zeta}(0) \leq c_0$ and $\phi_{\zeta}'(1) > M$, let

$$\psi_{\zeta}(s) = \phi_{\zeta}(0) + [1 - \phi_{\zeta}(0)]s = [1 - \xi'(\zeta)] + [\xi'(\zeta)]s.$$

Since $\phi_\zeta(s)$ is convex for $0 \leq s \leq 1$ we must have $\phi_\zeta(s) \leq \psi_\zeta(s)$ for all s in this range.

(b) If $\phi_\zeta(0) \leq c_0$, $\phi_\zeta'(1) \leq M$, and $\phi_\zeta(t) \geq t$, let

$$\begin{aligned} \psi_\zeta(s) &= \phi_\zeta(t), & 0 \leq s < t, \\ &= \phi_\zeta(t) + [\xi'(\zeta)](s - t), & t \leq s \leq 1. \end{aligned}$$

The fact that $\psi_\zeta(s)$ dominates $\phi_\zeta(s)$ in the range $t \leq s \leq 1$, follows from the observation that (since $\phi_\zeta(s)$ is convex)

$$p\phi_\zeta(t) + q\phi_\zeta(1) \geq \phi_\zeta(pt + q)$$

with, in particular, $p = (1 - s)/(1 - t)$ and $q = 1 - p$. This yields, since $\phi_\zeta(1) = 1$, that

$$[(1 - s)\phi_\zeta(t) + (s - t)]/(1 - t) \geq \phi_\zeta(s)$$

from which the required result follows by our definition of $\xi'(\zeta)$.

We note that our definition of $\psi_\zeta(s)$ in this case is equivalent to the following:

$$\begin{aligned} \psi_\zeta(s) &= \phi_\zeta(t), & 0 \leq s < t, \\ &= [1 - \xi'(\zeta)] + [\xi'(\zeta)]s, & t \leq s \leq 1. \end{aligned}$$

(c) If $\phi_\zeta(0) \leq c_0$, $\phi_\zeta'(1) \leq M$, and $\phi_\zeta(t) < t$, let us write

$$\gamma = [\xi'(\zeta) - (1 - t)]/[\xi'(\zeta)]$$

and define

$$\begin{aligned} \psi_\zeta(s) &= t, & 0 \leq s < \gamma, \\ &= [1 - \xi'(\zeta)] + [\xi'(\zeta)]s, & \gamma \leq s \leq 1. \end{aligned}$$

The arguments of case (b) must apply to the present case. It follows quickly from the assumption $\phi_\zeta(t) < t$ that $\gamma > t$, and so from (b) we have

$$\phi_\zeta(\gamma) \leq [1 - \xi'(\zeta)] + [\xi'(\zeta)]\gamma = t,$$

by the definition of γ . Thus, for $s < \gamma$, $\phi_\zeta(s) < \phi_\zeta(\gamma) \leq t = \psi_\zeta(s)$. On the other hand, for $\gamma \leq s \leq 1$, the fact that $\phi_\zeta(s) \leq \psi_\zeta(s)$ is immediate from (b).

At this point we wish to introduce a certain sequence of independent and identically distributed random variables $\{W_n\}$. It is clearer, however, if we lead to these by means of some intermediate sequences. To begin with, let $x \geq 0$ and write

$$\alpha = |\log(1 - t)|.$$

(a) If $\phi_\zeta(0) > c_0$ or if $\phi_\zeta(0) \leq c_0$ and $\phi_\zeta'(1) > M$ we have

$$\begin{aligned} \psi_\zeta(1 - e^{-x}) &= 1 - \exp - \{x - \log[1 - \phi_\zeta(0)]\} \\ &= 1 - \exp - [x - \log \xi'(\zeta)]. \end{aligned}$$

(b) If $\phi_{\zeta}(0) \leq c_0$, $\phi_{\zeta}'(1) \leq M$, and $\phi_{\zeta}(t) \geq t$ we have

$$\begin{aligned} \psi_{\zeta}(1 - e^{-x}) &= \phi_{\zeta}(t), & x < \alpha, \\ &= 1 - \exp - [x - \log \xi'(\zeta)], & x \geq \alpha. \end{aligned}$$

(c) If $\phi_{\zeta}(0) \leq c_0$, $\phi_{\zeta}'(1) \leq M$, and $\phi_{\zeta}(t) < t$, we have

$$\begin{aligned} \psi_{\zeta}(1 - e^{-x}) &= 1 - e^{-\alpha}, & x < \alpha + \log \xi'(\zeta), \\ &= 1 - \exp - [x - \log \xi'(\zeta)], & x \geq \alpha + \log \xi'(\zeta). \end{aligned}$$

Define a Markov process $\{Y_n\}$, $n = 0, 1, \dots$, on the unit interval as follows:

$$\begin{aligned} Y_0 &= 1 - e^{-\alpha}, \\ Y_{n+1} &= \psi_{\zeta_n}(Y_n), \quad n = 0, 1, \dots \end{aligned}$$

Because the ψ -functions dominate the corresponding ϕ -functions it is easily seen that

$$(3.9) \quad E\{X_n | X_0 = 1 - e^{-\alpha}\} \leq E\{Y_n | Y_0 = 1 - e^{-\alpha}\}.$$

Furthermore, $Y_n \geq 1 - e^{-\alpha}$ as is easily seen by an inductive argument.

Next define a Markov process $\{W_n\}$, $n = 0, 1, \dots$, by the relation

$$W_n = -\alpha - \log(1 - Y_n), \quad n = 0, 1, \dots,$$

and let $\{U_n\}$, $n = 1, 2, \dots$, be the sequence of independent and identically distributed random variables:

$$U_n = -\log \xi'(\zeta_{n-1}), \quad n = 1, 2, \dots$$

Since $Y_n \geq 1 - e^{-\alpha}$, it follows immediately that $W_n \geq 0$ ($W_0 = 0$). It is also important to note that $U_n \geq 0$ in cases (a) and (b), the reasoning being as follows:

(a) In this case $\xi'(\zeta_{n-1}) = 1 - \phi_{\zeta_{n-1}}(0)$ so that, necessarily, $-\log \xi'(\zeta_{n-1}) \geq 0$.

(b) In this case $\phi_{\zeta_n}(t) \geq t$ and

$$\xi'(\zeta_{n-1}) = (1 - \phi_{\zeta_{n-1}}(t))/(1 - t)$$

so that, again, $-\log \xi'(\zeta_{n-1}) \geq 0$.

Now, in view of our definitions, we have

$$Y_{n+1} = \psi_{\zeta_n}(1 - e^{-(W_n + \alpha)}).$$

If we consider the cases (a), (b), (c) separately, we can then show that

$$\begin{aligned} \psi_{\zeta_n}(1 - e^{-(W_n + \alpha)}) &= 1 - e^{-\alpha}, & W_n + U_{n+1} < 0, \\ &= 1 - \exp(-(W_n + U_{n+1} + \alpha)), & W_n + U_{n+1} \geq 0. \end{aligned}$$

Hence we find that

$$\begin{aligned} Y_{n+1} &= 1 - e^{-\alpha}, & W_n + U_{n+1} < 0, \\ &= 1 - \exp(-(W_n + U_{n+1} + \alpha)), & W_n + U_{n+1} \geq 0. \end{aligned}$$

Therefore

$$\begin{aligned} W_{n+1} &= 0, & W_n + U_{n+1} &< 0, \\ &= W_n + U_{n+1}, & W_n + U_{n+1} &\geq 0. \end{aligned}$$

We have thus shown $\{W_n\}$ to be a Lindley process with $W_0 = 0$. But $EU_n = E(-\log \xi'(\zeta_{n-1}))$, and

$$\begin{aligned} E \log \xi'(\zeta_n) &= \int_{\eta_n > c_0} \log(1 - \eta_n) dP + \int_{\eta_n \leq c_0, \xi(\zeta_n) \leq M} \log[(1 - \phi_{\zeta_n}(t))/(1 - t)] dP \\ &\quad + \int_{\eta_n \leq c_0, \xi(\zeta_n) > M} \log(1 - \eta_n) dP \\ &= E \log \beta_n(t). \end{aligned}$$

We have already shown that $E \log \beta_n(t) > 0$, so that $EU_n < 0$. By Theorem A it follows that the W_n -process has a non-defective limiting distribution, i.e.

$$(3.10) \quad \lim_{x \rightarrow \infty} \lim_{n \rightarrow \infty} P\{W_n \leq x\} = 1.$$

Thus, given any $y \in (0, 1)$, we can find $x_0(y)$ and $n_0(y)$ such that

$$P\{W_n \leq x_0\} > y, \quad \text{all } n \geq n_0.$$

This implies that

$$P\{Y_n \leq 1 - e^{-(x_0+\alpha)} \mid Y_0 = 1 - e^{-\alpha}\} > y, \quad \text{all } n \geq n_0,$$

and hence

$$E\{Y_n \mid Y_0 = 1 - e^{-\alpha}\} \leq 1 - ye^{-(x_0+\alpha)}, \quad \text{all } n \geq n_0.$$

Since the right-hand side of this inequality is independent of n , we infer that

$$\limsup_{n \rightarrow \infty} E\{Y_n \mid Y_0 = 1 - e^{-\alpha}\} < 1.$$

From (3.9) and (3.2) we can conclude that

$$\lim_{n \rightarrow \infty} \Pi_n(1 - e^{-\alpha}) < 1.$$

It follows, from Theorem 2.4, that for all $s \in [0, 1)$,

$$\lim_{n \rightarrow \infty} \Pi_n(s) < 1.$$

In particular, taking $s = 0$, we find that the probability of ultimate extinction is strictly less than one. This concludes the proof of Theorem 3.1.

4. The ergodic distribution. Let us write, for $k = 1, 2, \dots, 0 \leq s \leq 1$,

$$\Pi_n^{(k)}(s) = E\{s^{Z_n} \mid Z_0 = k\}.$$

For simplicity, we continue to write $\Pi_n^{(1)}(s)$ as $\Pi_n(s)$.

THEOREM 4.1. For $k = 1, 2, \dots$, the pgf $\Pi_n^{(k)}(s)$ is given recursively by

$$\begin{cases} \Pi_0^{(k)}(s) = s^k \\ \Pi_n^{(k)}(s) = E \Pi_{n-1}^{(k)}(\phi_{\zeta_{n-1}}(s)). \end{cases}$$

Since the proof of this theorem is so similar to that of Theorem 2.1, we omit it. Similarly we omit the proof of the following, analogous to Theorem 2.4.

THEOREM 4.2. *If for $k = 1, 2, \dots$, q_k is the probability of ultimate extinction conditional upon $Z_0 = k$, then*

$$\Pi_n^{(k)}(s) \rightarrow q_k \leq 1 \quad \text{as } n \rightarrow \infty,$$

for all $s \in [0, 1)$.

At this point we shall establish a relationship between $\Pi_n^{(k)}(s)$ and the dual process $\{X_n\}$. Evidently, for $s_0 \in [0, 1)$,

$$\begin{aligned} E\{X_n^k \mid X_0 = s_0\} &= E[\phi_{r_{n-1}}(\phi_{r_{n-2}}(\dots \phi_{r_0}(s_0) \dots))]^k \\ &= E[\phi_{r_0}(\phi_{r_1}(\dots \phi_{r_{n-1}}(s_0) \dots))]^k. \end{aligned}$$

However, Theorem 4.1 shows that

$$\Pi_n^{(k)}(s_0) = E[\phi_{r_0}(\phi_{r_1}(\dots \phi_{r_{n-1}}(s_0) \dots))]^k,$$

and hence that

$$\Pi_n^{(k)}(s_0) = E\{X_n^k \mid X_0 = s_0\}.$$

From this equation and Theorem 4.2 we have:

THEOREM 4.3. *As $n \rightarrow \infty$, for any $s_0 \in [0, 1)$,*

$$E\{X_n^k \mid X_0 = s_0\} \rightarrow q_k,$$

where q_k is the probability of ultimate extinction when $Z_0 = k$.

Because $0 \leq X_n \leq 1$ for all n , it follows from Theorem 4.3 that there exists a random variable X such that X_n tends to X , as $n \rightarrow \infty$, in distribution; and $q_k = EX^k$. If, for $0 \leq x \leq 1$, we write $G(x) = P\{X \leq x\}$ then we shall refer to $G(x)$ as the ergodic distribution (function) of the dual process $\{X_n\}$. Theorem 4.3 shows that $G(x)$ is independent of X_0 , provided $0 \leq X_0 < 1$.

THEOREM 4.4. *To any branching process $\{Z_n\}$ in a random environment there corresponds on the unit interval a dual Markov process $\{X_n\}$ possessing a limiting ergodic distribution G . The k th moment of G is the probability q_k that the branching process becomes extinct, given that $Z_0 = k$. Furthermore, if conditions (a) of Theorem 3.1 hold, $G(1-) = 0$ and so $q_k = 1$ for all $k = 1, 2, \dots$. On the other hand, if conditions (b) of Theorem 3.1 hold, $G(1-) = 1$ and*

$$q_k = \int_0^1 x^k dG(x) \downarrow 0, \quad \text{as } k \rightarrow \infty.$$

PROOF. (i) Under conditions (a) we have from Theorem 3.1 that

$$q_1 = \int_0^1 x dG(x) = 1.$$

Necessarily, therefore, $G(1-) = 0$ and

$$q_k = \int_0^1 x^k dG(x) = 1, \quad \text{all } k.$$

(ii) Under conditions (b) we have, in terms of processes already defined,

$$P\{Y_n \leq z \mid Y_0 = s_0\} \leq P\{X_n \leq z \mid X_0 = s_0\}, \quad 0 \leq s_0 < 1.$$

Hence,

$$\begin{aligned} \lim_{x \rightarrow \infty} \lim_{n \rightarrow \infty} P\{X_n \leq 1 - e^{-(x+\alpha)} \mid X_0 = 1 - e^{-\alpha}\} \\ \geq \lim_{x \rightarrow \infty} \lim_{n \rightarrow \infty} P\{W_n \leq x \mid W_0 = 0\} = 1, \quad \text{by (3.10).} \end{aligned}$$

Given a small $\epsilon > 0$, we can therefore choose x_0 sufficiently large that

$$\lim_{n \rightarrow \infty} P\{X_n \leq 1 - e^{-(x_0+\alpha)} \mid X_0 = 1 - e^{-\alpha}\} > 1 - \epsilon.$$

This establishes that

$$G(1 - e^{-(x_0+\alpha)}) > 1 - \epsilon$$

and hence that $G(1-) = 1$ as claimed. From this result it is easily seen that

$$q_k = \int_0^1 x^k dG(x) \quad .$$

decreases to zero as $k \rightarrow \infty$.

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