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A STABLE LIMIT THEOREM FOR MARKOV CHAINS

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1. Introduction and summary. In [2] Doeblin obtained a central limit for discrete parameter Markov chains with discrete state space. In obtaining this theorem the principal tool is the Doeblin dissection of the sequence of partial sums of a functional of a Markov chain into a random sum of independent, identically distributed random variables. Indeed, given this dissection, the remainder of Doeblin's proof is, in essence, a proof of a random central limit theorem. Although Doeblin does not state a stable limit theorem for Markov chains, at the end of the paper he observes that the dissection should also be of use in obtaining such theorems, and comments on a possible method of obtaining theorems of this nature. In this paper three stable limit theorems for Markov chains together with the appropriate solidarity theorems are obtained depending on whether the index α of the limiting distribution is <1, $=1$, or >1. The principal tools used in obtaining these theorems are the Doeblin dissection, a well known result concerning the rate of growth of the sequence of norming coefficients of a random variable in the domain of attraction of a stable law, and a random stable limit theorem [8], [10].

2. Preliminary definitions and results. Let $\{x_n, n \geqq 0\}$ be a positive recurrent, irreducible Markov chain with stationary transition probabilities defined on a probability space $(\Omega, \mathfrak{F}, P)$ and having state space S. Let f be a function from S to $(-\infty, +\infty)$. Then the sequence of random variables $\{y_n, n \geqq 0\}$ where $y_n = f \circ x_n$ is termed a functional of the Markov chain $\{x_n\}$. For such a Markov chain we now define several associated random variables. Our definitions and notation will, in general, be those of [1, pages 81–99].

For $i \varepsilon S$, $t_r(i)$ is the random variable giving the time of rth entrance into the state i, $r = 1, 2, 3, \cdots$. We denote by $L(i, n)$ the random variable giving the number of entries into state i not later than time n. For functionals of a positive recurrent irreducible Markov chain we recall that the Doeblin dissection of $S_n = y_0 + \cdots + y_n$ on an arbitrary but fixed state i of the state space is given by:

$$S_n = Y'(i, n) + \textstyle\sum_{s=1}^{L(i,n)-1} Y_s(i) + Y''(i, n), \tag{1}$$

where

$$Y'(i, n) = \textstyle\sum_{r=0}^{t_1(i)-1} y_r ; \qquad Y''(i, n) = \sum_{t_{L(i,n)}(i)}^{n} Y_r , \tag{2}$$

$$Y_s(i) = \textstyle\sum_{t_s(i)}^{t_{s+1}(i)-1} y_j . \tag{3}$$

Our interest in this dissection stems from the following lemma which is contained in the theorems and discussion of [1, pages 83–89].

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LEMMA 1. *Let* $\{y_n, n \geqq 0\}$ *be a functional of a positive recurrent irreducible Markov chain* $\{x_n, n \geqq 0\}$. *Let* $\{Y'(i, n), n \geqq 1\}$, $\{Y''(i, n), n \geqq 1\}$, $\{Y_r(i), r \geqq 1\}$ *be the sequences of random variables yielded by the dissection of* S_n *on* i. *Then the sequences of random variables* $\{Y'(i, n), n \geqq 1\}$, $\{Y''(i, n), n \geqq 1\}$ *are bounded in probability and the random variables of the sequence* $\{Y_r(i), r \geqq 1\}$ *are independent and identically distributed.*

The following random stable limit theorem [8] and the Doeblin dissection are perhaps the most important tools in obtaining stable limit theorems for Markov chains. We observe that the random stable limit theorem of [8] is contained in Theorem 5.2 of [10] for $\alpha \neq 1$. For if $\alpha > 1$ and the sequence of random variables $\{Y_j\}$ is assumed centered, we may let $A_n = 0$, while if $\alpha < 1$, we need not center to obtain a limit distribution by Theorem 2 of [3, p. 546]. In either of these cases the cited theorem of Wittenberg applies. Thus it is only in the case $\alpha = 1$ that we shall need a random stable limit theorem which can handle centering constants.

THEOREM 1. *Let* $\{Y_j, j \geqq 1\}$ *be independent, identically distributed random variables for which there exist sequences of constants* $\{A_n\}$, $\{B_n\}$, $B_n > 0$ *such that the limit distribution* F *of* $\sum_1^n (Y_j - A_n)/B_n$ *exists and is non-degenerate (hence stable of index* α, $0 < \alpha \leqq 2$). *Let* $\{R(n), n \geqq 1\}$ *be a sequence of positive integer valued random variables for which there exists a positive constant* π *such that* plim $R(n)/n = \pi$. *Let* $n^* = [n\pi]$. ($[x]$ *denotes the greater integer in* x). *Then the limit in distribution of* $\sum_1^{R(n)} (Y_j - A_{n^*})/B_{n^*}$ *exists and is* F.

The next theorem is a randomized form of a theorem of Marcinkiewitz [9, page 243] from which it is easily obtained. As usual plim denotes limit in probability while Plim denotes limit with probability one.

THEOREM 2. *Let* $\{Y_j, j \geqq 1\}$ *be independent and identically distributed random variables having finite* rth *moment,* $1 < r < 2$. *Let* $\{R(n), n \geqq 1\}$ *be a sequence of positive integer valued random variables such that* plim $R(n)/n = \pi > 0$. *Then* plim $\sum_1^{R(n)} (Y_j - E[Y_j])/(R(n))^{1/r} = 0$.

In order to obtain a stable limit theorem for Markov chains covering the case $\alpha = 1$, we shall need the following lemma comparing the relative rates of growth of the centering and norming constants for stable distributions of index $\alpha = 1$. The author is indebted to the referee for pointing out that it follows easily from equations (5.14) and (5.17) of [3, Chapter 17].

LEMMA 2. *Let* $\{X_j, j \geqq 1\}$ *be independent, identically distributed random variables for which there exist sequences of constants* $\{A_n\}$, $\{B_n\}$, $B_n > 0$ *such that the limit in distribution of* $\sum_1^n (X_j - A_n)/B_n$ *is stable of index* 1. *Then for any* $\beta > 0$, $\lim |A_n|/B_n^{\beta} = 0$.

3. Principal theorems. In our stable limit theorems for Markov chains we obtain sequences of norming and centering constants which are independent of the state used in the dissection for $\alpha \neq 1$, while for $\alpha = 1$ the norming constants do not depend on the state used in the dissection. In order to obtain norming constants independent of the state used in the dissection, the characterization given in [7] of norming constants in terms of dispersion is used. (Recall that for

$0 \leqq \gamma \leqq 1$, arbitrary but fixed, the dispersion of a random variable X for the probability γ, denoted $D(\gamma, X)$, is the infimum of lengths of the closed intervals $[a, b]$ such that $P\{a \leqq X \leqq b\} \geqq \gamma$). The precise stable limit theorem for Markov chains that is obtained will vary according as the index α of the limiting distribution is < 1, $= 1$, or > 1. We begin by considering the case $\alpha < 1$ which is simplest.

THEOREM 3. *Let $\{y_n, n \geqq 0\}$ be a functional of a positive recurrent, irreducible Markov chain $\{x_n, n \geqq 0\}$. Assume that for some state i in the state space, the common distribution F_i of $Y_r(i)$ (in the dissection of $S_n = y_0 + \cdots + y_n$ on i) belongs to the domain of attraction of a stable law G of index α, $0 < \alpha < 1$. Then the limit distribution of $S_n/D(\gamma, S_n)$ for $0 < \gamma < 1$, arbitrary but fixed, exists and is of the same type as G.*

PROOF. Since F_i is in the domain of attraction of a stable distribution of index $0 < \alpha < 1$, there exists [3, page 546] a sequence of constants $\{B_n, n > 0\}$, $B_n > 0$ such that the limit distribution G of $\sum_{r=1}^{n} Y_r(i)/B_n$ exists and is stable of index α. (Observe that $\{B_n\}$ may depend on the state i used in the dissection.) By Corollary 1 of [1, p. 93] we know Plim $\{L(i, n) - 1\}/n = \pi_i$, where $\pi_i > 0$ is the stationary probability for the state i. Hence it follows from Theorem 1 that if $n^* = [n\pi_i]$, the limit in distribution of $\sum_{r=1}^{L(i,n)-1} Y_r(i)/B_{n^*}$ is also G. Since $\{Y'(i, n)\}$, $\{Y''(i, n)\}$ are, as mentioned earlier, bounded in probability, it follows that the limit distribution of S_n/B_{n^*} is G. However, by Corollary 1 of Theorem 4 of [7], it follows that the limit distribution of $S_n/D(\gamma, S_n)$ for $0 < \gamma < 1$, arbitrary but fixed, is $G(xD(\gamma, G))$. Thus the theorem.

We now consider the case $\alpha = 1$, which is, as might be expected, somewhat more difficult. In this case the choice of the centering constants may depend on the state used in the dissection. Before stating the theorem we recall that the rth return time for the state i is given by $P_r(i) = t_{r+1}(i) - t_r(i)$. As we are assuming that the Markov chain is positive recurrent, it follows that $E[p_r(i)]$ exists for all states i of the state space.

THEOREM 4. *Let $\{y_n, n \geqq 0\}$ be a functional of a positive recurrent, irreducible Markov chain $\{x_n, n \geqq 0\}$. Assume that for some state i in the state space the (common) distribution F_i of $Y_r(i)$ (in the dissection of $S_n = y_0 + \cdots + y_n$ on i) belongs to the domain of attraction of a stable law G of index 1. Further, assume that for some $\beta > 0$, $E[|p_r(i)|^{1+\beta}]$ exists. Then there exists a sequence of constants $\{A_n\}$ such that for $0 < \gamma < 1$, arbitrary but fixed, the limit distribution of $(S_n - A_n)/D(\gamma, S_n)$ exists and is of the same type as G.*

PROOF. Without loss of generality we assume $1 + \beta < 2$. Since F_i belongs to the domain of attraction of a stable law of index 1, we know there exist sequences of constants $\{M_n\}$, $\{B_n\}$, $B_n > 0$ such that the limit distribution G of $\sum_1^n (Y_r(i) - M_n)/B_n$ exists and is stable of index 1. Letting $n^* = [n\pi_i]$ where π_i is the stationary probability for the state i, it follows from Theorem 1 that the limit distribution of

$$\sum_{r=1}^{L(i,n)-1} (Y_r(i) - M_{n^*})/B_{n^*} \tag{4}$$

is also G. Using our assumption that for some $\beta > 0$ the $(1 + \beta)$th moment of $p_r(i)$ exists, we prove that

$$\text{(5)} \qquad \operatorname{plim} \{ \textstyle\sum_{r=1}^{L(i,n)-1} (p_r(i) - 1/\pi_i)\pi_i M_{n^*}/B_{n^*} \} = 0.$$

To do this, let us first recall that by equation (5.14) of [3, Chapter 17], $B_n = nZ(n)$ where Z is a slowly varying function. Hence, if $\gamma = 1/(1 + \beta)$ the expression in parentheses in the preceding equation can be written as the product of

$$\text{(a)}\ \textstyle\sum_{r=1}^{L(i,n)-1} (p_r(i) - 1/\pi_i)/(L(i, n) - 1)^{\gamma}; \qquad \text{(b)}\ (L(i, n) - 1)^{\gamma}/n^{*\gamma};$$

$$\text{(c)}\ M_{n^*}/(n^* Z(n^*))^{(1-\gamma)/2}; \qquad \text{(d)}\ \pi_i/\{(n^*)^{(1-\gamma)/2}(Z(n^*))^{(1+\gamma)/2}\}.$$

However, the limit in probability of expression (a) is zero by Theorem 2. The limit of (b) is one by the definition of n^* and Corollary 1 of [1, page 93]. By Lemma 2 we see that the limit of (c) is zero. Finally, since Z a slowly varying function implies $(Z)^{(1+\gamma)/2}$ is slowly varying, it follows from [5, page 45] that the limit of (d) is zero. Hence we have obtained (5). Since the limit in distribution of (4) is G, it follows from (5) and a well known lemma (see, e.g., [3, page 247]) that the limit in distribution of $\sum_{r=1}^{L(i,n)-1} (Y_r(i) - p_r(i)\pi_i M_{n^*})/B_{n^*}$ is also G. Now let us observe that by Theorem 2 of [1, page 82], the sequence of random variables $\{n - t_{L(i,n)}(i) + t_1(i);\ n \geqq 0\}$ is bounded in probability. Thus it follows from Lemma 2 that $\operatorname{plim}\{(n - t_{L(i,n)}(i) + t_1(i))\pi_i M_{n^*}/B_{n^*}\} = 0$. But this and the preceding observation imply that the limit in distribution of $\{\sum_{r=1}^{L(i,n)-1} Y_r(i) - n\pi_i M_{n^*}\}/B_{n^*}$ is G since $\sum_{r=1}^{L(i,n)-1} p_r(i) = t_{L(i,n)}(i) - t_1(i)$. Combining this with the fact that $\{Y'(i, n)\}$, $\{Y''(i, n)\}$ are bounded in probability, it follows that the limit in distribution of $(S_n - A_{n^*})/B_{n^*}$, where $A_{n^*} = n\pi_i M_{n^*}$ is G. But just as in the preceding theorems, we then see that for $0 < \gamma < 1$, arbitrary but fixed, the limit in distribution of $(S_n - A_{n^*})/D(\gamma, S_n)$ is $G(x\, D(\gamma, G))$. Thus the theorem.

Finally we consider the case $1 < \alpha < 2$. Both the statement of the theorem and its proof are similar to the improvement by Kendall [6] of Doeblin's central limit theorem for Markov chains. In this theorem we eliminate the necessity of a requirement on the moment of $p_r(i)$ by requiring instead that $E[Y_r(i)]$ exist and by shifting our attention to the sequence of random variables $Z_r(i) = Y_r(i) - mp_r(i)$, where $m = E[Y_r(i)]/E[p_r(i)]$. (Recall that as we are restricting our attention to positive recurrent, irreducible Markov chains we know $E[p_r(i)]$ exists. Further, by the corollary of [1, page 115], we know that the value of m is independent of the state i.) In addition, since the random variables $\{Y_r(i), r \geqq 1\}$ are independent and identically distributed and the random variables $\{p_r(i), r \geqq 1\}$ are independent and identically distributed, it follows that the random variables $\{Z_r(i), r \geqq 1\}$ are independent and identically distributed.

THEOREM 5. *Let $\{y_n, n \geqq 0\}$ be a functional of a positive recurrent, irreducible Markov chain $\{x_n, n \geqq 0\}$. Let $\{Y_r(i), r \geqq 1\}$ be the sequence of independent, identically distributed random variables yielded in the dissection of S_n on i, and assume $E[y_r(i)]$ exists. Let $Z_r(i) = Y_r(i) - mp_r(i)$, where $m = E[Y_r(i)]/E[p_r(i)]$. If the* *common distribution F_i of $Z_r(i)$ belongs to the domain of attraction of a stable law G of index α, $1 < \alpha < 2$, then for $0 < \gamma < 1$, arbitrary but fixed, the limit in distribution of $(S_n - nm)/D(\gamma, S_n)$ exists and is of the same type as G.*

PROOF. Since $\sum_{r=1}^{L(i,n)-1} p_r(i) = t_{L(i,n)}(i) - t_1(i)$, we may write:

$$(6) \quad S_n - nm = Y'(i, n) + \sum_{r=1}^{L(i,n)-1} (Y_r(i) - mp_r(i)) + Y''(i, n) + (n - t_{L(i,n)}(i) + t_1(i))m.$$

It follows from the definition of $Z_r(i)$ that $E[Z_r(i)] = 0$. Hence the assumption that F_i belongs to the domain of attraction of a stable law G of index $1 < \alpha < 2$ implies the existence of a sequence of constants $\{B_n\}$, $B_n > 0$ such that the limit in distribution of $\sum_{r=1}^{n} Z_r(i)/B_n$ exists and is of the same type as G. As in the preceding theorems, it follows that if $n^* = [n\pi_i]$, the limit in distribution of:

$$\sum_{r=1}^{L(i,n)} Z_r(i)/B_{n^*} = \sum_{r=1}^{L(i,n)} (Y_r(i) - mp_r(i))/B_{n^*}$$

exists and is also G. But as the first, third and fourth terms on the right hand side of (6) are bounded in probability, this implies that the limit in distribution of $(S_n - nm)/B_{n^*}$ exists and is G. Hence, as in the earlier theorems, for $0 < \gamma < 1$, arbitrary but fixed, the limit distribution of $(S_n - nm)/D(\gamma, S_n)$ exists and is $G(xD(\gamma, G))$.

From the preceding theorem we can obtain the following corollary, the statement of which corresponds to that of Theorem 4.

COROLLARY. *Let $\{y_n, n \geqq 0\}$ be a functional of a positive recurrent, irreducible Markov chain $\{x_n, n \geqq 0\}$. Assume that for some state i in the state space the distribution F_i of $Y_r(i)$ belongs to the domain of attraction of a stable law G of index α, $1 < \alpha < 2$. Further assume that for some $\beta > 0$, $E[|p_r(i)|^{\alpha+\beta}]$ exists. Then there exists a constant m such that for $0 < \gamma < 1$, arbitrary but fixed, the limit in distribution of $(S_n - nm)/D(\gamma, S_n)$ exists and is of the same type as G.*

PROOF. We show that the hypotheses of the corollary imply those of the theorem. As it is obvious that $E[Y_r(i)]$ exists, it suffices to show that the common distribution F_i of $Z_r(i) = Y_r(i) - mp_r(i)$ belongs to the domain of attraction of a stable distribution of the same type as G. If $M_i = E[Y_r(i)]$, we know there exists a sequence of constants $\{B_n\}$, $B_n > 0$ such that the limit distribution G of $\sum_1^n (Y_r(i) - M_i)/B_n$ exists and is stable of index α. Next we show

$$(7) \qquad \operatorname{plim} \{\sum_{r=1}^{n} (p_r(i) - 1/\pi_i)\pi_i M_i/B_n\} = 0.$$

Let $m = \pi_i M_i$ and recall that as before m is independent of the state i. If we verify (7) it will then follow that the limit distribution of $\sum_{r=1}^{n} (Y_r(i) - mp_r(i))/B_n$ is G, which implies that the common distribution F_i of $Z_r(i)$ belongs to the domain of attraction of a stable distribution of the same type as G. The corollary will then follow from the theorem. To obtain (7) let us first observe that by equation (5.14) of [3, page 545] $B_n = n^{1/\alpha}Z(n)$, where Z is a slowly varying function and α is the index of the stable law G. Let $k = 1/(\alpha + \beta)$. Then the expression in parentheses in (7) can be rewritten: $\{\sum_{r=1}^{n} (p_r(i) - 1/\pi_i)n^k\} \cdot \{\pi_i M_i/(n^{1/\alpha - k}Z(n))\}$. The limit of the first term above is zero almost surely by a theorem of Marcinkiewitz [9, page 243]. Since $|\pi_i M_i| = |m| < +\infty$, and Z is slowly varying, the limit of the second term is zero by a result in [5, page 45]. Thus we have verified (7) and the corollary follows.

4. Solidarity theorems. The results of the preceding section have been stated for a given state i of the state space. We now prove that under the conditions used respectively in obtaining Theorems 3–5, if $Y_r(i)$ belongs to the domain of attraction of a stable law G of index α, then $Y_r(j)$ also lies in the domain of attraction of G. In proving this, the following lemma will be of use.

LEMMA 3. *Let $\{y_r, r \geqq 0\}$ be a functional of a positive recurrent, irreducible Markov chain $\{x_r, r \geqq 0\}$. Let $t(i, n, j)$ be the random variable giving the time of first entrance into state i following the nth entrance into state j. Let*

$$N(i, n, j) = \sum_{r=0}^{t(i,n,j)} \delta_i(x_j) - 1,$$

where δ_i is the Kronecker delta function. Let $\{B_n\}$ be a sequence of positive constants tending to plus infinity. Then

$$\text{plim } \{|\textstyle\sum_{t_1(j)}^{t_n(j)} y_r - \sum_{t_1(i)}^{t(i,n,j)} y_r|/B_n\} = 0; \tag{8}$$

$$\text{plim } \{|\textstyle\sum_{t_1(i)}^{t(i,n,j)-1} y_r - \sum_{1}^{N(i,n,j)} Y_r(i)|/B_n\} = 0; \tag{9}$$

$$\text{plim } N(i, n, j)/n = m_j/m_i . \tag{10}$$

PROOF. To obtain (8), let $\epsilon > 0$ be given. Then

$$\begin{aligned} P\{|\textstyle\sum_{t_1(j)}^{t_n(j)} y_r - \sum_{t_1(i)}^{t(i,n,j)} y_r| > \epsilon B_n\} \leqq\ & P\{\textstyle\sum_0^{t_1(i)} |y_r| > \epsilon B_n/3\} \\ &+ P\{\textstyle\sum_0^{t_1(j)} |y_r| > \epsilon B_n/3\} \\ &+ P\{\max_{1 \leqq s \leqq k} |\textstyle\sum_{t_n(j)+1}^{t_n(j)+s} y_r| > \epsilon B_n/3\} \\ &+ P\{t(i, n, j) - t_n(j) > k\}. \end{aligned}$$

However, the first two terms on the right hand side tend to zero with n, and for fixed k, the third term on the right hand side also tends to zero with n. Finally, since $t_n(j)$ is an optional random variable, it follows that $P\{t(i, n, j) - t_n(j) = k\} = f_{ij}^{(k)}$, which is independent of n and thus the fourth term on the right hand side also tends to 0 with k.

Observe that (9) follows directly from the definition of $Y_r(i)$. To obtain (10), observe that it follows from the strong law of large numbers that

$$n^{-1}t_n(j) = n^{-1}\{\textstyle\sum_{r=1}^{n} p_r(j) + 1\} \to E[p_r(j)] = m_j .$$

Next, if we let $f = \delta_i(\cdot)$, it follows from Theorem 2 of [1, page 92] that Plim $\sum_1^n f(x_s)/n = \pi_i$. But this implies that Plim $\sum_1^{t_n(j)} f(x_s)/[nm_j] = \pi_i$. Finally, since $t(i, n, j) - t_n(j)$ is bounded in probability, uniformly in n, this last expression implies plim $n^{-1}N(i, n, j) =$ plim $n^{-1}\{\sum_1^{t(i,n,j)} f(x_s) - 1\} = \pi_i m_j = m_j/m_i$, yielding the lemma.

We state a solidarity theorem only for the case $\alpha = 1$, as the necessary modifications to cover the case $\alpha \neq 1$ under the assumptions of Theorem 3 or Theorem 5 or its corollary will be evident.

THEOREM 6. *Under the conditions of Theorem 4, if $Y(i)$ is in the domain of attraction of a stable law G of index 1, then for any other state j, $Y(j)$ also lies in the domain of attraction of G.*

PROOF. Since $Y(i)$ is in the domain of attraction of G, it follows that there exist sequences of constants $\{M_n(i)\}$, $\{B_n(i)\}$ such that the limit in distribution of $\sum_{r=1}^{n} (Y_r(i) - M_n(i))/B_n(i)$ exists and is G. Let $n^* = [nm_j/m_i]$. It then follows from the random stable limit theorem and the preceding lemma that the limit in distribution of $\sum_{r=1}^{N(i,n,j)} (Y_r(i) - M_{n^*}(i))/B_{n^*}(i)$ is also G. By using an argument patterned after that used in obtaining (5), the preceding is seen to imply that the limit in distribution of $\sum_{r=1}^{N(i,n,j)} (Y_r(i) - p_r(i)\pi_i m_{n^*}(i))/B_{n^*}(i)$ is G. However, from the preceding lemma, this is seen to imply that the limit in distribution of $\sum_{t_1(j)}^{t_n(j)} (y_r - \pi_i m_{n^*}(i))/B_{n^*}(i)$ is G. But by using once more a modification of the argument which yielded (5), the preceding statement is seen to imply that the limit in distribution of

$$\sum_1^{n-1} (Y_r(j) - \pi_i M_{n^*}(i)/\pi_j)/B_{n^*}(i)$$

is G, yielding the theorem.

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REFERENCES

[1] CHUNG, K. L. (1967). *Markov Chains with Stationary Transition Probabilities*, (2nd ed.). Springer, Berlin.

[2] DOEBLIN, W. (1938). Sur deux problems de M. Kolmogorov concernant les chaines denombrables. *Bull. Soc. Math. France* **66** 210–220.

[3] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications*, **2**. Wiley, New York.

[4] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables.* (Translated by K. L. Chung) Addison-Wesley, Reading.

[5] KARAMATA, J. (1933). Sur un mode de croissance reguliere. *Bull. Soc. Math. France* **61** 45–62.

[6] KENDALL, D. G. (1957). A note on Doeblin's central limit theorem, *Proc. Amer. Math. Soc.* **8** 1037–1039.

[7] KIMBLETON, S. R. (1968). A characterization of certain sequences of norming constants, *Ann. Math. Statist.* **39** 394–397.

[8] KIMBLETON, S. R. A random limit theorem for triangular arrays. To appear.

[9] LOÈVE, M., (1963). *Probability Theory.* (3rd ed.). Van Nostrand, Princeton.

[10] WITTENBERG, H. (1964). Limiting distributions of random sums of random variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 7–19.