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# THE CANONICAL CORRELATION COEFFICIENTS OF BIVARIATE GAMMA DISTRIBUTIONS

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**1. Introduction.** Let $F(x, y)$ be a bivariate distribution function with marginal distribution functions $G(x)$ and $H(y)$. Lancaster [8] has studied the structure of bivariate distributions using orthogonal functions on the marginal distributions. Let $\{\zeta_i(x)\}$ and $\{\eta_j(y)\}$ be complete orthonormal sets of functions on $G(x)$ and $H(y)$ respectively such that $E(\zeta_i(x)\eta_j(y)) = \rho_j\delta_{ij}$, $1 \geqq \rho_1^2 \geqq \rho_2^2 \geqq \cdots$, where $\delta_{ij}$ is the Kronecker delta. $\{\zeta_i\}$, $\{\eta_j\}$ are called the canonical variables of $(X, Y)$, and $\{\rho_i\}$ the canonical correlation coefficients of $(X, Y)$. The sets $\{\zeta_i\}$, $\{\eta_j\}$ and $\{\rho_i\}$ determine the bivariate distribution function $F(x, y)$ uniquely given $G(x)$ and $H(y)$. $F(x, y)$ is said to be $\phi^2$-bounded with respect to its marginal distributions if $\phi^2 + 1 = \int \{dF(x, y)/dG(x)dH(y)\}^2 \, dG(x) \, dH(y) < \infty$, or equivalently $\sum_{n=1}^{\infty} \rho_n^2 = \phi^2 < \infty$. $\phi^2$-bounded distributions have a canonical expansion of the form $dF(x, y) = dG(x) \, dH(y)\{1 + \sum_{n=1}^{\infty} \rho_n \zeta_n^{(x)} \eta_n^{(y)}\}$, in mean square.

Sarmanov [10] has characterized the canonical correlation coefficients of $\phi^2$-bounded distributions, whose marginal distributions are normal and whose canonical variables are the Hermite-Chebyshev polynomials. The series expansion of a bivariate normal frequency function in Hermite-Chebyshev polynomials,

$$(2\pi)^{-1}(1 - \rho^2)^{-\frac{1}{2}} \exp\{-(x^2 - 2\rho xy + y^2)/2(1 - \rho^2)\}$$
$$= (2\pi)^{-1} \exp\{-(x^2 + y^2)/2\}\{1 + \textstyle\sum_{n=1}^{\infty} \rho^n H_n(x)H_n(y)\}$$

is used in this characterization, $\{H_n(x)\}$ being orthonormal on $(2\pi)^{-\frac{1}{2}} \exp \cdot\{-x^2/2\}$. There is a similar expansion in the Laguerre polynomials of a bivariate gamma frequency function derived by Kibble [5]. A multivariate extension of this frequency function has been derived by Krishnamoorthy and Parthasarathy [7] and some properties of this multivariate case discussed by Krishnaiah and Rao [6].

In this note, the canonical correlation coefficients of bivariate gamma distributions, with canonical variables the Laguerre polynomials, are considered, making use of the frequency function derived by Kibble [5]. A class of these distributions which are $\phi^2$-bounded is obtained, the general proof not depending on $\phi^2$-boundedness.

The connection with Bochner's work [1] on stochastic processes is shown and thus a class of stochastic processes associated with the Laguerre polynomials is constructed.

Moran [9] has obtained a minimum bound for the ordinary correlation coeffi-

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cient between two correlated gamma variables. For those bivariate distributions in this note, further restrictions on $\rho$ are obtained.

**2. Bivariate gamma distributions.** The Laguerre polynomials $\{L_n^a(x)\}$ are a system of polynomials, orthonormal and complete with respect to the gamma distribution whose frequency function is $w^a(x) = x^{a-1}e^{-x}/\Gamma(a)$, $x > 0$. The Laguerre polynomials have the expansion

$$L_n^a(x) = \binom{n+a-1}{n}^{-\frac{1}{2}} \sum_{m=0}^{n} \binom{n+a-1}{n-m} (-x)^m/m! \tag{2.1}$$

The notation is such that if $\{L*_n^a(x)\}$ is the set of Laguerre polynomials referred to in Erdélyi [4], then $L_n^a(x) = \binom{n+a-1}{n}^{-\frac{1}{2}} L*_n^{a-1}(x)$.

Definition. Let $X$ and $Y$ be two gamma variates with parameters $a$ and $b$ respectively. $(X, Y)$ will be said to be distributed in gamma correlation it $E(L_n^a(X)L_m^b(Y)) = \delta_{nm}\rho_n$, $\{\rho_n\}$ being a sequence of constants. $(X, Y)$ then has canonical variables $\{L_n^a(X)\}$, $\{L_n^b(Y)\}$ and canonical correlation coefficients $\{\rho_n\}$.

Kibble [5] has obtained a bivariate gamma distribution with joint frequency function

$$w^a(x)w^a(y)\{1 + \sum_{n=1}^{\infty} z^n L_n^a(x)L_n^a(y)\} \qquad \text{for } 0 \leqq z < 1, \tag{2.2}$$

and moment generating function (mgf)

$$E(\exp(t_1X + t_2Y)) = (1 - t_1)^{-a}(1 - t_2)^{-a}(1 - zt_1t_2/(1 - t_1)(1 - t_2))^{-a} \tag{2.3}$$

when $a$ is half-integer. Recently, Vere-Jones [11] showed that the "symmetric" bivariate gamma distribution is infinitely divisible and so (2.3) is a moment generating function for any real $a > 0$. For $z = 1$ this is also a mgf of the random variable $(X, Y = X)$. A possible "mgf" for a bivariate gamma distribution is thus

$$(1 - t_1)^{-L}(1 - t_2)^{-M}(1 - zt_1t_2/(1 - t_1)(1 - t_2))^{-P}, \tag{2.4}$$

with $L > 0$, $M > 0$, $P > 0$ and $0 \leqq z \leqq 1$. However Kibble [5] has pointed out that the resulting "frequency function" is negative for some values of $(x, y)$ if $zP > \min(L, M)$. Krishnamoorthy and Parthasarathy [7] have considered a moment generating function of the form

$$\phi(t_1, \cdots, t_m) = |I - \Omega D|^{-p} \tag{2.5}$$

where $p$ is a half-integer, $I$ is an identity matrix, $D$ is diagonal matrix whose diagonal elements are $t_1, \cdots, t_m$ and $\Omega$ is the covariance matrix of the "accompanying" multivariate normal. The multivariate distribution with mgf (2.5) has gamma marginals and an expansion in the Laguerre polynomials under covergence conditions on $\Omega$.

Some properties of (2.5) have been discussed by Krishnaiah and Rao [6].

Although (2.4) is not in general a mgf, it is for certain values of $L$, $M$, $P$ and $z$.

Lemma 1. *(2.4) is a* mgf *if* $M = b$, $L = a$, $P = a$, $b \geqq a$ *and* $0 \leqq z \leqq 1$. *The resulting random variable* $(X, Y)$ *is in gamma correlation with canonical correlation coefficients* $\rho_n = \binom{n+a-1}{n}^{\frac{1}{2}}\binom{n+b-1}{n}^{-\frac{1}{2}} z^n$.

PROOF. Since (2.3) is a mgf for all $a > 0$ and $0 \leqq z \leqq 1$, so $(1 - t_1)^{-a} \cdot (1 - t_2)^{-(a+c)}(1 - zt_1t_2/(1 - t_1)(1 - t_2))^{-a}$ is a mgf for all $a, c > 0$ and $0 \leqq z \leqq 1$. Placing $b = a + c$ gives the first part of the lemma. The random variable $(X, Y)$ corresponding to this mgf is of the form $(X, Y = U + V)$, $(U, X)$ having the mgf (2.3), and $V$ being a gamma $(b - a)$ variable, independent of $U$ and $X$.

It is known that the Laguerre polynomials satisfy a "Runge-type" identity

$$\tilde{L}_n^b(U + V) = \sum_{t=0}^n \binom{n}{t} \tilde{L}_t^a(U) \tilde{L}_{n-t}^{b-a}(V), \tag{2.6}$$

where $U$ and $V$ are independent and $\tilde{L}_n^a(U)$ is $L_n^a(U)$ scaled to have a leading coefficient of unity; (see Eagleson [2]). The leading coefficient of $L_n^a(U)$ is $(-1)^n\binom{n+a-1}{n}^{-\frac{1}{2}}/n!$, so from this and (2.6) $E(L_n^a(X)L_m^b(Y)) = \binom{n+a-1}{n}^{\frac{1}{2}} \binom{n+b-1}{n}^{-\frac{1}{2}} z^n \delta_{nm}$. When $z = 1$ this corresponds to a distribution with "random elements in common" as discussed by Eagleson [2].

The case when the marginal distributions of $(X, Y)$ are identical is considered first.

THEOREM 1. *A sequence $\{\rho_n\}$ is a sequence of canonical correlation coefficients of a distribution in gamma correlation with identical marginals if and only if $\rho_n = \int_0^1 t^n \, d\mu(t)$, $\mu(t)$ being a distribution function on* [0, 1].

PROOF. *Sufficiency*. Any mixture of the distribution function corresponding to (2.3) with respect to a distribution function $\mu(z)$, defined on [0, 1] will again be a distribution function. The canonical variables will be $\{L_n^a(X)\}$, $\{L_n^a(Y)\}$ and canonical correlation coefficients $\rho_n = \int_0^1 t^n \, d\mu(t)$.

*Necessity*. Let $\{\rho_n\}$ be a sequence of canonical correlation coefficients. $E(L_n^a(Y) \mid X) = \rho_n L_n^a(X)$. Now $y^n = (-1)^n n! \binom{n+a-1}{n}^{\frac{1}{2}} L_n^a(y) + a_2 L_{n-1}^a(y) + \cdots + a_n$, where the $a_j$ are easily determined constants.

$$\begin{aligned} E(Y^n \mid X) &= (-1)^n n! \binom{n+a-1}{n}^{\frac{1}{2}} \rho_n L_n^a(X) + b_2 L_{n-1}^a(X) + \cdots + b_n \\ &= \rho_n X^n + c_2 X^{n-1} + \cdots + c_n \end{aligned} \tag{2.7}$$

where the $b_j$ and $c_j$ are constants. From (2.7) $\rho_n = \lim_{x\to\infty} E\{(Y/X)^n \mid X\}$. These are the moments of a positive distribution, and since $\rho_n^2 \leqq 1$, the distribution is concentrated on [0, 1].

Sarmanov [10] has characterized the canonical correlation coefficients of a bivariate normal distribution, and the above proof is essentially due to him. However for bivariate gamma distributions the theorem can be extended to the nonsymmetric case.

THEOREM 2. *A sequence $\{\rho_n\}$ is a sequence of canonical correlation coefficients of a random variable $(X, Y)$ distributed in gamma correlation with marginals gamma $a$ and gamma $b$ respectively if and only if*

$$\rho_n = \binom{n+a-1}{n}^{\frac{1}{2}} \binom{n+b-1}{n}^{-\frac{1}{2}} \int_0^1 t^n \, d\mu(t), \tag{2.8}$$

*for $b \geqq a$, where $\mu(t)$ is a distribution function concentrated on* [0, 1].

PROOF. *Sufficiency*. From Lemma 1 (2.4) is a mgf if $M = b$, $L = a$ and $P = a$, $b \geqq a$ and $0 \leqq z \leqq 1$. A mixture of the corresponding distribution function with respect to a distribution function $\mu(z)$ concentrated on [0, 1] is again a distribution function. The correlation coefficients of this distribution are then mixtures of the correlation coefficients in Lemma 1; that is, of $\binom{n+a-1}{n}^{\frac{1}{2}}\binom{n+b-1}{n}^{-\frac{1}{2}}z^n$.

*Necessity.* Let $\{\rho_n\}$ be a sequence of canonical correlation coefficients. Then $E(L_n^b(Y) \mid X) = \rho_n L_n^a(X)$. Now $y^n = (-1)^n n! \binom{n+b-1}{n}^{\frac{1}{2}} L_n^b(y) + a_2 L_{n-1}^b(y) + \cdots + a_n$, where the $a_j$ are easily determined constants. Thus

$$E(Y^n \mid X) = (-1)^n n! \binom{n+b-1}{n}^{\frac{1}{2}} \rho_n L_n^a(X) + b_2 L_{n-1}^a(X) + \cdots + b_n$$
$$= \binom{n+b-1}{n}^{\frac{1}{2}} \binom{n+b-1}{n}^{-\frac{1}{2}} \rho_n X^n + c_2 X^{n-1} + \cdots + c_n,$$

where the $b_j$ and $c_j$ are constants. So

$$\rho_n = \binom{n+a-1}{n}^{\frac{1}{2}} \binom{n+b-1}{n}^{-\frac{1}{2}} \lim_{x\to\infty} E\{(Y/X)^n \mid X\}.$$

Set

$$v_k = \lim_{x\to\infty} E\{(Y/X)^k \mid X\} \text{ and } v_k^* = \lim_{x\to\infty} E\{(X/Y)^k \mid Y\}.$$

These are both moments of positive distributions. From a similar argument to the above $\rho_n = \binom{n+a-1}{n}^{-\frac{1}{2}}\binom{n+b-1}{n}^{\frac{1}{2}} v_n^*$. So $v_n^* = \binom{n+a-1}{n}\binom{n+b-1}{n}^{-1} v_n$. Now $\binom{n+a-1}{n}\binom{n+b-1}{n}^{-1} = \Gamma(n+a)\Gamma(b)/\{\Gamma(a)\Gamma(n+b)\}$ is the $n$th moment of the distribution with frequency function $\Gamma(b)/\{\Gamma(a)\Gamma(b-a)\}x^{a-1}(1-x)^{(b-a)-1}$, $0 < x < 1$ and so is bounded by unity. Since also $\rho_n^2 \leqq 1$, $v_n^*$ is the moment of a distribution defined on [0, 1].

Assume $v_n$ is the moment of a distribution on $[0, h]$, $h > 1$. Then $v_n^* \sim n^{-(b-a)} v_n \Gamma(b)/(a)$, $v_n = \int_0^h t^n \, d\partial(t) \geqq (1+m)^n \int_{1+m}^h d\partial(t)$, choosing $1 < 1 + m < h$. This implies $v_n^* \to \infty$ as $n \to \infty$, which is a contradiction; i.e., $v_n$ is the moment of a distribution on [0, 1], and the necessity is proved.

Theorem 2 expresses the distribution function of a random variable $(X, Y)$ in gamma correlation as a mixture belonging to a convex set, the extreme points of which are the distribution functions corresponding to the mgf $(1-t_1)^{-a} \cdot (1-t_2)^{-b}(1 - zt_1t_2/(1-t_1)(1-t_2))^{-a}$, $0 \leqq z \leqq 1$.

**3. $\phi^2$-boundedness.** $(X, Y)$ is $\phi^2$-bounded if $\sum_{n=1}^{\infty} \rho_n^2 < \infty$. This implies that if $F(x, y)$ is the distribution function of $(X, Y)$, then

(3.1) $\quad dF(x,y) = w^a(x)w^b(y)\,dx\,dy\{1 + \sum_{n=1}^{\infty} \rho_n L_n^a(x) L_n^b(y)\}$ in mean square.

Some simple observations give the following two sufficient conditions for $\phi^2$-boundedness. (i) If $\mu(t)$ in Theorem 2 is defined on $[0, 1-\epsilon]$, $\epsilon > 0$, then the distribution is $\phi^2$-bounded. (ii) If $(X, Y)$ is in gamma correlation with $b - a > 1$, the distribution is always $\phi^2$-bounded. Another sufficient condition for $\phi^2$-boundedness is given in the next theorem.

THEOREM 3. *Let $T$ be the random variable associated with $\mu(t)$ in Theorem 2. Set $v_n = \int_0^1 t^n \, d\mu(t)$. $\sum_{n=1}^{\infty} \rho_n^2 < \infty$ is equivalent to $\sum_{n=1}^{\infty} n^{-(b-a)} v_n^2 < \infty$, and for these conditions to hold it is sufficient that there exists an $\alpha > 0$ such that*

$$P(T > 1 - t)/t^{\frac{1}{2}(a-b+1)+\alpha} \to 0 \text{ as } t \to 0.$$

PROOF. $v_n = \int_0^{1-\delta} t^n \, d\mu(t) + \int_{1-\delta}^{1} t^n \, d\mu(t) \leqq (1-\delta)^n + P(T > 1-\delta)$ for all $\delta > 0$. Consider a sequence $\delta_n = n^{-\beta}$, choosing $\frac{1}{2}/(\frac{1}{2}+\alpha) < \beta < 1$. Given any $\epsilon > 0$ there exists an $N_0(\epsilon)$ such that for $n \geqq N_0(\epsilon)$ $P(T > 1-\delta_n) \leqq \epsilon n^{-\gamma}$, $\gamma = \frac{1}{2}(a-b)\beta + \beta(\frac{1}{2}+\alpha)$, from (3.2). So $\sum_{N_0}^{\infty} n^{-(b-a)} P(T > 1-\delta_n)^2 \leqq \epsilon \sum_{N_0}^{\infty} n^{-\partial}$, $\partial = 2\beta(\frac{1}{2}+\alpha) + (b-a)(1-\beta) > 1$. The sum on the right converges since $\partial > 1$. Also $(1 - 1/n^{\beta})^n \leqq \exp(-n^{1-\beta})$, and so $\sum_{N_0}^{\infty} n^{-(b-a)}(1 - 1/n^{\beta})^{2n}$ converges by the integral comparison test since, $\int_{x_0}^{\infty} x^j \exp(-2x^m) \, dx$ converges for fixed $j$ with $m > 0$ and $x_0 > 0$. $\sum_{N_0}^{\infty} P(T > 1-\delta_n)(1-\delta_n)^n n^{-(b-a)}$ converges by a similar integral comparison test. i.e.

$$\sum_{n=1}^{\infty} n^{-(b-a)} v_n^2 < \infty. \tag{3.3}$$

Since $\binom{n+a-1}{n}\binom{n+b-1}{n}^{-1} \sim n^{-(b-a)} \Gamma(b)/\Gamma(a)$ (3.3) is equivalent to $\sum_{n=1}^{\infty} \rho_n^2 < \infty$.

REMARK. In view of Féjer's asymptotic formula

$$\binom{n+a-1}{n}^{\frac{1}{2}} L_n^a(x) = \pi^{-\frac{1}{2}} e^{x/2} x^{-a/2+\frac{1}{4}} n^{a/2-\frac{3}{4}} \cos[2(nx)^{\frac{1}{2}} - a/2 + \pi/4] + o(n^{a/2-5/4})$$

for fixed $x > 0$, (see Erdélyi [4 p. 199])

$$\rho_n |L_n^a(x) L_n^b(y)| \sim v_n n^{-(b-a)/2-3/2} f_n(x, y),$$

where $v_n = \int_0^1 t^n \, d\mu(t)$ and hence $f_n(x, y)$ is bounded with respect to $n$ for fixed $x, y > 0$. This implies the expansion (3.1) is pointwise convergent for fixed $x, y > 0$.

## 4. Positive definite sequences.

DEFINITION. Let $\{Q_n(X)\}$ be a complete orthogonal system on a distribution $\mu(x)$. $\{t_n\}$ is a positive definite sequence with respect to $\{Q_n(X)\}$ if whenever $\sum_{n=0}^{M} a_n Q_n(X) \geqq 0$ for all $X$, any $M$, then $\sum_{n=0}^{M} a_n t_n Q_n(X) \geqq 0$ for all $X$ any $M$. Bochner [1] characterizes those sequences $\{t_n\}$ which are positive definite with respect to the ultraspherical polynomials $\{P_n(X)\}$, orthogonal on $d\mu(x) = (1-x^2)^{\gamma-\frac{1}{2}} dx / \int_{-1}^{1} (1-y^2)^{-\frac{1}{2}} dy$, and normalized so that $P_n(1) = 1$. He proves that a sequence $\{t_n\}$ is positive definite with respect to $\{P_n(X)\}$ if and only if $t_n = \int_{-1}^{1} P_n(x) \, d\mu^*(x)$, where $\mu^*(x)$ is a positive, bounded, monotonely increasing function in $-1 \leqq x \leqq 1$.

Eagleson [3] has shown that when a distribution function has a finite number of points of increase that there is a 1–1 correspondence between positive definite sequences and canonical correlation coefficients. Using the concept of positive definite sequences he characterizes the canonical correlation coefficients of bivariate binomial distributions with identical marginal distributions. In the case of bivariate gamma distributions with identical marginal distributions sequences of canonical correlation coefficients, $\{\rho_n\}$, are also positive definite sequences. To see this if $\sum_{n=0}^{M} a_n L_n^a(X) \geqq 0$, then $E(\sum_{n=0}^{M} a_n L_n(X) \mid Y) = \sum_{n=0}^{M} a_n \rho_n L_n^a(Y) \geqq 0$. Conversely if $\{t_n\}$ is a positive definite sequence, which is a canonical correlation sequence, then $t_n L_n^a(y) = \int_0^{\infty} L_n^a(x) \, d\psi(x \mid y)$, $\psi(x \mid y)$ being the distribution function of $X$ given $Y$. Setting $y = 0$ $t_n = \binom{n+a-1}{n}^{-\frac{1}{2}} \int_0^{\infty} L_n^a(x) \cdot$

$d\partial(x)$, where $\partial(x)$ is the distribution function $\partial(x \mid y = 0)$. The next theorem gives the class of distributions $\partial(x)$ when $\rho_n < \delta < 1$.

THEOREM 4. *$\{\rho_n\}$, with $\rho_n < \delta < 1$ is a sequence of canonical correlation coefficients with respect to a random variable in gamma correlation if and only if* $\rho_n = \binom{n+b-1}{n}^{-\frac{1}{2}} \int_0^\infty L_n^a(x)\, d\partial(x)$, *where* $d\partial(x) = x^{a-1} \int_\delta^1 t^{-a} \exp(-x/t)\, d\partial(t)\, dx/\Gamma(a)$, *$\mu(1 - t)$ being a distribution function on* $[0, \delta]$.

PROOF.

$$(1 - z)^{-a} x^{a-1} \exp(-x/(1 - z))/\Gamma(a) = w^a(x)\{1 + \textstyle\sum_{n=1}^\infty \binom{n+a-1}{n}^{\frac{1}{2}} z^n L_n^a(x)\},\ 0 \leqq z < 1,$$

for which see Erdélyi [4 p. 189]. Set $f(x; z)$ equal to the right hand side of this equation. Then $z^n = \binom{n+a-1}{n}^{-\frac{1}{2}} \int_0^\infty L_n^a(x) f(x; z)\, dx$. From Theorem 2 and the condition $\rho_n < \delta < 1$ it is necessary and sufficient that $\rho_n = \binom{n+a-1}{n}^{\frac{1}{2}} \binom{n+b-1}{n}^{-\frac{1}{2}} \cdot \int_0^\delta z^n\, d\eta(z)$, $\eta(z)$ being a distribution function. So $\rho_n = \binom{n+b-1}{n}^{-\frac{1}{2}} \int_0^\infty L_n^a(x) \cdot d\partial(x)$, setting $\partial(x) = \int_0^\delta f(x; z)\, d\eta(z)$. Placing $\mu(t) = \eta(1 - t)$ gives the theorem.

Moran [9] has obtained bounds for the ordinary correlation coefficient of two correlated gamma variables. He proves that if the gamma variables have parameters $a$ and $b$, and $y = A(x)$ is defined by $\int_0^x w^a(u)\, du + \int_0^y w^b(v)\, dv = 1$ then $\rho_{\min} = \{\int_0^\infty uA(u) w^a(u)\, du - ab\}/(ab)^{\frac{1}{2}}$. When $a = b = 1$ then $\rho_{\min} = 2 - \pi^2/6 = -0.64493$. If it is assumed that the random variable $(X, Y)$ is in gamma correlation, then further restrictions are placed on the correlation coefficient. In fact the following result holds.

COROLLARY. *Let $\rho$ be the ordinary correlation coefficient of $(X, Y)$, distributed in gamma correlation, and $v$ be the mean of some distribution on* $[0, 1]$. *Then* $\rho = v(a/b)^{\frac{1}{2}}$ *and* $0 \leqq \rho \leqq (a/b)^{\frac{1}{2}}$.

PROOF. $L_1^a(X) = (a - X)/a^{\frac{1}{2}} = -(X - E(X))/(\operatorname{var} X)^{\frac{1}{2}}$ and so $\rho = \rho_1 = v(a/b)^{\frac{1}{2}}$.

**5. Stochastic processes.** Bochner [1] has constructed a homogeneous stochastic process associated with the ultraspherical polynomials using the concept of positive definite sequences. Using the same methods, with sequences of canonical correlation coefficients replacing positive definite sequences, a class of stochastic processes associated with the Laguerre polynomials may be constructed.

DEFINITION. A sequence of functions $\{c_n(t)\}$ $0 \leqq t < \infty$, $n = 0, 1, 2, \cdots$ is a homogeneous stochastic process if

(i) for each $t$, $\{c_n(t)\}$ is a sequence of canonical correlation coefficients of a symmetric distribution in gamma correlation.

(ii) $c_n(u + v) = c_n(u) c_n(v)$

(iii) $c_0(t) = 1$

(iv) $c_n(0) = 1$

and $c_n(t)$ is continuous.

THEOREM 5. *$\{c_n(t)\}$ is a homogeneous stochastic process if and only if*

$$(5.1)\quad c_0(t) = 1,\ c_n(t) = \exp(-t \textstyle\int_0^1 \{(1 - y^n)/(1 - y)\}\, dG(y)),\ n \geqq 1,$$

*where $G(y)$ is a positive increasing function of $y$ in* $[0, 1]$ *such that* $\int_0^1 dG(y) < \infty$.

PROOF. *Sufficiency.* $y^n$ is a canonical correlation sequence for fixed $0 \leqq y \leqq 1$. By the closure properties of canonical correlation sequences $\exp(-t)\sum_{r=0}^{\infty} y^{nr}t^r/r! = \exp\{-t(1 - y^n)\}$ is a sequence of canonical correlations for $t \geqq 0$. Replacing $t$ by $t\,dG(y_j)/(1 - y_j)$, $j = 0, 1, \cdots m$ and $0 \leqq y_j < 1$, and using closure properties again; 1, $\exp\{-t\{\sum_{j=0}^{m}(1 - y_j^n)/(1 - y_j)\}\,dG(y_j)\}$ is a sequence of canonical correlation coefficients. $\exp\{-t\int_0^{1-0}\{(1 - y^n)/(1 - y)\}\,dG(y)\}$ is the limit of such approximating sums. If $G(y)$ has an atom at one, with a measure of $p$, then this atom contributes $np$ in the integral $\int_0^1\{(1 - y^n)/(1 - y)\}\,dG(y)$. 1, $\exp(-tnp)$ is a canonical correlation sequence and thus so is (5.1).

*Necessity.* Since $\{c_n(t)\}$ is a canonical correlation sequence by Theorem 1 $c_n(t) = \int_0^1 y^n\,d\mu(y, t)$, $\mu(y, t)$ being a distribution function on [0, 1] for all $t \geqq 0$. Properties (i) to (iv) imply that $c_n(t) = \exp(-t\gamma_n)$, $\gamma_0 = 0$. Set $G(y, t) = \int_0^y (1 - z)\,d\mu(z, t)/t$. For all $t > 0$, $G(y, t)$ is a positive increasing function of $y$. $\int_0^1\{(1 - y^n)/(1 - y)\}\,dG(y, t) = (1 - \exp(-t\gamma_n))/t$, $n \geqq 1$, so setting $n = 1$, $G(y, t)$ is seen to be bounded uniformly by a constant depending only on $\delta$ for $t \leqq \delta$. $\gamma_n = \lim_{t\to 0+}\int_0^1\{(1 - y^n)/(1 - y)\}\,dG(y, t)$, $n \geqq 1$. Using Helly's first theorem it is possible to choose a sequence $t_r \to 0+$ such that $G(y, t_r)$ converges weakly to a nondecreasing function $G(y)$. Since the interval is finite and $(1 - y^n)/(1 - y)$ is continuous in [0, 1], Helly's second theorem implies that $\gamma_n = \int_0^1\{(1 - y^n)/(1 - y)\}\,dG(y)$, $G(y)$ having the desired properties.

## 6. Examples.

6.1. For the particular case of identical marginal distributions and $\rho_n = \int_0^1 t^n\,dt = 1/(n + 1)$, a closed form for the frequency function $f(x, y)$ of the corresponding random variable $(X, Y)$ is known.

$$f(x, y) = w^a(x)w^a(y)\{1 + \sum_{n=1}^{\infty}(n + 1)^{-1}L_n^a(x)L_n^a(y)\}$$
$$= \{\Gamma(a, \max(x, y)) - \Gamma(a, x)\Gamma(a, y)/\Gamma(a)\}/\Gamma(a)$$

where $\Gamma(a, x) = \int_x^{\infty} t^{a-1}\exp(-t)\,dt$. (See Erdélyi [4 p. 215]).

6.2. Eagleson [2] gives the following example of a bivariate gamma distribution whose canonical variables are the Laguerre polynomials. Let $W_1$, $W_2$ and $W_3$ be independent gamma variables with parameters $a$, $b$ and $c$ respectively. Set $X = W_1 + W_2$, $Y = W_2 + W_3$. The canonical correlations are given by

$$\rho_r = [(\Gamma(a + b)\Gamma(b + c))/(\Gamma(a + b + r)\Gamma(b + c + r))]^{\frac{1}{2}}\Gamma(b + r)/\Gamma(b).$$

This is in the form of the sequences $\{\rho_r\}$ in Theorem 2 as

$$\rho_r = \binom{r+a+b-1}{r}^{\frac{1}{2}}\binom{r+b+c-1}{r}^{-\frac{1}{2}}\int_0^1 t^r\,d\mu(t)$$

where $d\mu(t) = \Gamma(a + b)t^{b-1}(1 - t)^{a-1}\,dt/[\Gamma(a)\Gamma(b)]$ if it is supposed that $c \geqq a$.

6.3. Let $\{H_n(X)\}$ be the Hermite-Chebyshev polynomials, orthonormal and complete on the normal frequency function $\phi(x) = \exp(-\frac{1}{2}x^2)/(2\pi)^{\frac{1}{2}}$, $-\infty < x < \infty$. Sarmanov [10] has shown that $\{c_n\}$ is a sequence of canonical correlation coefficients of a $\phi^2$-bounded distribution in generalized normal correlation if and only if $c_n$ is the $n$th moment of a distribution on the interior of $[-1, 1]$. Consider $(U, V)$ distributed in this way. Then $(X, Y)$, $X = \frac{1}{2}U^2$, $V = \frac{1}{2}V^2$, is distributed in gamma correlation with canonical correlations $\rho_n = c_{2n}$, and symmetric marginals gamma $\frac{1}{2}$. $E(H_{2k}(U)H_{2n}(V)) = \delta_{kn}c_{2n}$, and since $H_{2k}(U) = (-)^k L_k^{\frac{1}{2}}(X)$, (see Erdélyi [4 p. 193]), $E(L_k^{\frac{1}{2}}(X)L_n^{\frac{1}{2}}(Y)) = \delta_{kn}c_{2n}$ and $c_{2n}$ is the $n$th moment of a distribution on $[0, 1]$ as in Theorem 2.

6.4. If $(X_1, Y_1)$ and $(X_2, Y_2)$ are independent and both distributed in gamma correlation, then so is $(X_1 + X_2, Y_1 + Y_2)$. The Runge-type identity $\tilde{L}^{a_1+a_2}(X_1 + X_2) = \sum_{t=0}^{n} \binom{n}{t}\tilde{L}_t^{a_1}(X_1)\tilde{L}_{n-t}^{a_2}(X_2)$, and a similar identity for $Y_1$, $Y_2$ with notation as in Lemma 1 gives this immediately.

6.5. The expression (2.4) is a mgf if and only if $P \leqq \min(L, M)$ and $0 \leqq z \leqq 1$. Suppose $L \leqq M$. The sufficiency is clear from a slight extension of lemma 1. If (2.4) is a mgf it is necessary that the canonical correlations be of the form of those in Theorem 2. Since $\rho_n = \binom{n+P-1}{n}\binom{n+L-1}{n}^{-\frac{1}{2}}\binom{n+M-1}{n}^{-\frac{1}{2}}z^n$, it is necessary that $\Gamma(n + P)\Gamma(L)z^n/[\Gamma(P)\Gamma(n + L)]$ be a moment of a distribution on $[0, 1]$, $u_n$, say. Now $u_n^2/(u_{n+1}u_{n-1}) = (n + P - 1)(n + L)/((n + P)(n + L - 1))$ and this can only be less than or equal to unity if $P \leqq L$.

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## REFERENCES

[1] Bochner, S. (1954). Positive zonal functions on spheres. *Proc. Nat. Acad. Sci., U.S.A.* **40** 1141–1147.

[2] Eagleson, G. K. (1964). Polynomial expansions of bivariate distributions. *Ann. Math. Statist.* **35** 1208–1215.

[3] Eagleson, G. K. (1969). A characterization theorem for positive definite sequences on the Krawtchouk polynomials. *Austral. J. Statist.* **11** 29–38.

[4] Erdélyi, A. (1953). *Higher Transcendental Functions,* **2**. McGraw Hill Company, New York.

[5] Kibble, W. F. (1941). A two-variate gamma type distribution. *Sankhyā* **5** 137–150.

[6] Krishnaiah, P. R. and Rao, M. M. (1961). Remarks on a multivariate gamma distribution. *Amer. Math. Monthly* **68** 342–346.

[7] Krishnamoorthy, A. S. and Parthasarathy, M. (1951). A multivariate gamma-type distribution. *Ann. Math. Statist.* **22** 549–557 and **31** 229.

[8] Lancaster, H. O. (1958). The structure of bivariate distributions. *Ann. Math. Statist.* **29** 719–736.

[9] Moran, P. A. P. (1967). Testing for correlation between two non-negative variates. *Biometrika* **54** 385–394. Testing for serial correlation with exponentially distributed variates. *Biometrika* **54** 395–401.

[10] Sarmanov, O. V. (1966). Generalized normal correlation and two-dimensional classes of Frechět. Doklady Akad. Nauk. S.S.S.R. **168** 32–35.

[11] Vere-Jones, D. (1967). The infinite divisibility of a bivariate gamma distribution. *Sankhyā Ser. A* **29** 421–422.