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A NOTE ON THE TEST FOR THE LOCATION PARAMETER OF AN EXPONENTIAL DISTRIBUTION

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Suppose that X is distributed according to an exponential distribution with density

$$(1)\qquad f(x) = \theta^{-1} \exp[-\theta^{-1}(x-\gamma)] \quad \text{if} \quad x > \gamma,$$
$$= 0 \quad \text{otherwise}$$

where both θ and γ are unknown parameters.

Let Y be another positive random variable independent of X and distributed according to a continuous distribution with scale parameter θ, and density $g(y/\theta)/\theta$, where g is a known function.

Consider the hypothesis $\gamma = \gamma_0$, and let the procedure be such that the hypothesis is rejected if and only if

$$(2)\qquad (X-\gamma_0)/Y \leqq a \quad \text{or} \quad (X-\gamma_0)/Y \geqq b$$

where $0 \leqq a < b \leqq \infty$. Since $(X-\gamma_0)/Y$ is independent of θ under the hypothesis, a and b can be determined so that

$$\Pr\{a < (X-\gamma_0)/Y < b \mid \gamma_0\} = 1-\alpha \quad \text{for all} \quad \theta.$$

Then the following theorem holds true.

Theorem 1. *For the alternative $\gamma < \gamma_0$, the power of the test* (2) *above is given by*

$$(3)\qquad P(\gamma) = 1-(1-\alpha)\exp[-\theta^{-1}(\gamma_0-\gamma)]$$

i.e. it is independent of the distribution of Y, and also of a or b.

Proof.

$$\begin{aligned}
P(\gamma) &= 1 - P_\gamma\{\gamma_0 + aY < X < \gamma_0 + bY\} \\
&= 1 - P_\gamma\{\gamma_0 - \gamma + aY < X - \gamma < \gamma_0 - \gamma + bY\} \\
&= 1 - \textstyle\int_0^\infty [\int_{\gamma_0-\gamma+ay}^{\gamma_0-\gamma+by} \theta^{-1} \exp(-\theta^{-1}u)\, du]\theta^{-1} g(\theta^{-1}y)\, dy \\
&= 1 - \textstyle\int_0^\infty \theta^{-1} | \exp[-\theta^{-1}(\gamma_0 - \gamma + ay)] - \exp[-\theta^{-1}(\gamma_0-\gamma+by)]| \\
&\qquad g(\theta^{-1}y)\, dy \\
&= 1 - \exp[-\theta^{-1}(\gamma_0-\gamma)] \textstyle\int_0^\infty \theta^{-1}(\exp(-\theta^{-1}ay) - \exp(-\theta^{-1}by)) g(\theta^{-1}y)\, dy \\
&= 1 - \exp[-\theta^{-1}(\gamma_0-\gamma)] \textstyle\int_0^\infty (e^{-au} - e^{-bu}) g(u)\, du
\end{aligned}$$

Since $P(\gamma_0) = \alpha$, the theorem is proved.

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THEOREM 2. *The test procedure is* UMP *among all the tests based on* X *and* Y *against the alternative* $\gamma < \gamma_0$.

PROOF. Fix $\theta = \theta_0$, and consider the test of the simple hypothesis $\theta = \theta_0$, $\gamma = \gamma_0$, against the simple alternative $\theta = \theta_0$, $\gamma = \gamma_1 < \gamma_0$.

Then by the Neyman-Pearson fundamental lemma [3], the most powerful test (which is not unique in this case) must satisfy the following conditions. Considering that $f(x) = 0$, $x \leqq \gamma_0$, when $\gamma = \gamma_0$,

$$\phi(x) = 1 \quad \text{if} \quad x \leqq \gamma_0 \qquad \text{and} \qquad \int_{\gamma_0}^{\infty} \phi(x)\, \theta_0^{-1} \exp\,[-\theta_0^{-1}(x - \gamma_0)]\, dx = \alpha.$$

For such a test the power is given by

$$\begin{aligned} P^*(\gamma_1) &= \int_{\gamma_1}^{\infty} \phi(x)\theta_0^{-1} \exp\,[-\theta_0^{-1}(x - \gamma_1)]\, dx = \int_{\gamma_1}^{\gamma_0} \theta_0^{-1} \exp\,[-\theta_0^{-1}(x - \gamma_1)]\, dx \\ &\qquad + \exp\,[-\theta_0^{-1}(\gamma_0 - \gamma_1)] \int_{\gamma_0}^{\infty} \phi(x)\theta_0^{-1} \exp\,[-\theta_0^{-1}(x - \gamma_0)]\, dx \\ &= (1 - \exp\,[-\theta_0^{-1}(\gamma_0 - \gamma_1)]) + \alpha \exp\,[-\theta_0^{-1}(\gamma_0 - \gamma_1)] \\ &= 1 - (1 - \alpha) \exp\,[-\theta_0^{-1}(\gamma_0 - \gamma_1)] \end{aligned}$$

which is equal to $p(\gamma_1)$ given by (3). Thus the power of the test (2) is equal to the most powerful test of the hypothesis $\theta = \theta_0$, $\gamma = \gamma_0$ against $\theta = \theta_0$, $\gamma = \gamma_1 < \gamma_0$, for any θ_0 and γ_1; hence it is UMP against $\gamma < \gamma_0$.

Now we shall suppose that we have a sample of size n from the exponential population given by (1). Let $X_{(1)} < X_{(2)} < \cdots X_{(n)}$ be the order statistic. Define

$$Y = \textstyle\sum_{i=1}^{n-1} a_i (X_{(i+1)} - X_{(i)}) \tag{4}$$

where a_i are non-negative constants. Then it is well known that Y is independent of $X_{(1)}$, and its distribution has scale parameter θ.

THEOREM 3. *For any* Y *of the form* (4), *the test procedure defined by* (2) *where* $X = X_{(1)}$ *is* UMP *against the alternative* $\gamma < \gamma_0$, *and the power is given by*

$$p(\gamma) = 1 - (1 - \alpha) \exp\,[-n\theta^{-1}(\gamma_0 - \gamma)]. \tag{5}$$

PROOF. Since $X_{(1)}$ is distributed according to an exponential distribution with scale parameter θ/n, the power is given from Theorem 1. And if we put $a_i = n - i + 1$, the pair $(X_{(1)}, Y)$ gives a sufficient statistic; hence the UMP test based on this pair is UMP among all the tests. But since the power is independent of the choice of Y, the test is UMP against the alternative $\gamma < \gamma_0$ irrespective of the choice of Y.

This theorem can be regarded as an extension of Dubey's results [1], [2]. It should be remarked that the power against $\gamma > \gamma_0$ does depend on the distribution of Y, but it is intuitively clear and can easily be proved that the power is decreasing with respect to b. Hence, if we consider the two sided alternative $\gamma \neq \gamma_0$, a and b can be determined so that $a = 0$, and (4) holds true.

REFERENCES

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