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INADMISSIBILITY OF THE BEST INVARIANT ESTIMATOR OF EXTREME QUANTILES OF THE NORMAL LAW UNDER SQUARED ERROR LOSS[1]

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0. Summary. Suppose that independent normally distributed random vectors, $X^{n\times 1}$ and $Y^{k\times 1}$, are observed with $E(X) = 0$, $E(Y) = \mu$, $\mathrm{Cov}(X) = \sigma^2 I$, and $\mathrm{Cov}(Y) = \sigma^2 I$. It is known [2, 5] that the best invariant estimator of μ is admissible if $k \leqq 2$ and inadmissible if $k > 2$. It is also known [1, 7] that the best invariant estimator of σ is inadmissible. In this paper, these results are extended to show that the best invariant estimator of $\theta = A\mu + \eta\sigma$, for a given matrix A and a given vector η, is inadmissible if $|\eta|$ is sufficiently large (when $k = 1, A = 1$, θ is a *quantile*).

1. Introduction. When a statistical decision problem involves a parameter space which is not that of a single real parameter, the problem of establishing either the inadmissibility or admissibility of an estimator becomes much more difficult. Very little is known concerning admissibility in such cases.

When the problem involves an unknown location vector and scale parameter, it may remain invariant under a group of transformations (a subgroup of the full affine group) which takes μ (the location parameter) and σ (the scale) into $c\mu + b$ and $c\sigma$, respectively, where b lies in the range of μ, $c > 0$. This group acts transitively on the parameter space and, consequently, the risk of any invariant estimator is a constant. Among the class of such estimators there is therefore a "best" one. The effect of imposing the principle of invariance, in this case, is to reduce the class of all possible estimators to one.

Under reasonable conditions on the loss function, it can be concluded from Keifer's theorem [3] that this estimator is minimax. Since it possesses this desirable property, it is natural to ask whether it is also admissible. The only known results in this direction concern the estimation of the location parameter or (see Stein [7], Brown [1], Zidek [8]) the estimation of the scale.

In this paper we investigate this question for another sort of estimation problem involving quadratic loss. We are concerned here with the estimation of quantiles of the one-dimensional normal distribution, that is, functions of the form $\mu + \eta\sigma$ where η is specified. It is shown that for quantiles for which η is sufficiently large in magnitude, the best affine invariant procedure is inadmissible.

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The case of quantiles near the median remains an open question. The result is proved by demonstrating that there exists a scale invariant procedure which has a uniformly smaller risk than the estimator in question. How large $|\eta|$ must be before the best affine invariant estimator is inadmissible is not known, nor has attention been devoted here to this question since the only answer we could give would depend on the choice we have made of the dominating scale invariant estimator and would therefore have no intrinsic meaning for the problem. Stein [6] conjectured that the estimator in question was inadmissible as long as $\eta \neq 0$.

The result is actually proved in a more general context where the underlying distribution is multivariate normal with unknown mean μ and covariance $\sigma^2 I$, and $A\mu + \eta\sigma$ is being estimated, A being a matrix and η a vector. However, the result obtained for dimension greater than one is more formal in its interpretation, since our definition of quantile is somewhat artificial in higher dimensions.

2. Preliminaries. If Z is a normally distributed real random variable with mean μ and variance σ^2, the αth quantile of this distribution of Z is that number, z_α, which satisfies $P[Z \leqq z_\alpha] = \alpha$. It follows that $z_\alpha = \mu + \eta_\alpha \sigma$, where η_α denotes the αth percentile of the standard normal distribution.

Let X_i, $i = 1, 2, \cdots, n$, Y_j, $j = 1, 2, \cdots, k$ be independent normally distributed random variables with $\operatorname{Var} Y_j = \operatorname{Var} X_i = \sigma^2$, $E(X_i) = 0$, $E(Y_j) = \mu_j$, for all i, j. The definition of quantile given above can be formally extended to this multivariate case. The extension is achieved by defining $x_1 < x_2$ to mean each co-ordinate of x_1 is less than the corresponding co-ordinate of x_2. In higher dimensional cases z_α is not unique, of course, but once specified, it has an interpretation similar to that of the one dimensional quantile.

By "quantile", we will mean a function of the form $\mu + \eta\sigma$ (η specified) where $\mu = (\mu_1, \cdots, \mu_k)'$ and $\eta = (\eta_1, \eta_2, \cdots, \eta_k)'$. By "extreme quantile" we simply mean a quantile, $\mu + \eta\sigma$, for which $\|\eta\|$ is large, where $\|\cdot\|$ denotes the usual Euclidean norm.

In this paper we shall be concerned with the estimation of quantiles of the distribution of $(X_1, \cdots, X_n, Y_1, \cdots, Y_k)$, or rather, more generally, with the estimation of the function $A\mu + \eta\sigma$, where $\eta = (\eta_1, \cdots, \eta_m)'$, A is an $m \times k$ matrix, and both A and η are specified. The loss incurred in estimating $A\mu + \eta\sigma$, say, $\hat{\theta}$ is assumed given by $\|\hat{\theta} - A\mu - \eta\sigma\|^2/\sigma^2$.

Suppose $\eta = (0, \cdots, 0)' \varepsilon R^k$. If $k \geqq 3$, $m = k$, $A = I$ the well-known result of James and Stein [2] implies the inadmissibility of the best affine invariant estimator, Y, of μ. If $k = 1$ or 2, $m = k$, $A = I$, it is known that Y is an admissible estimator of μ. In fact, this can be concluded from the work of Stein [5], and James and Stein [2], which pertains to the case where σ is known. Since the form of the best invariant estimator, Y, is independent of the value of the nuisance parameter σ, if it were known, we can use the results, just quoted, together with the necessary and sufficient condition given in [4] to obtain the asserted conclusion.

In the case where A is the matrix each of whose elements are 0, the best affine invariant estimator of the resulting quantity, $\eta\sigma$ ($\|\eta\| \neq 0$) is inadmissible. This result follows from an argument given by Stein [7] in obtaining the corresponding result in the estimation of σ^2. Since this result is fundamental to the result of this paper, the argument, adapted to the present case, is given below, where without loss of generality we have assumed $\eta = 1$. Brown [1] generalized Stein's result to apply to more general loss functions where the underlying distribution either has compact support or is the univariate normal (actually, a slightly more general family including the normal is considered). These remarks summarize all that appears to be known concerning the admissibility of the best affine invariant estimator of $A\mu + \eta\sigma$. In the next section we deduce that for fixed A and all n for which $\|\eta\|$ is sufficiently large, the estimator in question is inadmissible.

We turn now to the proof of the counterpart of Stein's result, for the estimation of σ. A sufficient statistic in this problem is $(S, Y_1, \cdots, Y_k)$ where $S = \sum_{i=1}^n X_i^2$ and we can assume any estimator is a function of this statistic. The problem remains invariant under the transformations:

$$(2.1) \qquad \begin{aligned} (s, y) &\rightarrow (a^2 s, a\alpha y + b) \\ \mu &\rightarrow a\alpha\mu + b \\ \sigma^2 &\rightarrow a^2\sigma^2 \\ \hat{\sigma} &\rightarrow a\hat{\sigma}, \end{aligned}$$

where $0 < a < \infty$, $b \in R^k$, $y = (y_1, \cdots, y_k)'$, α is a $k \times k$ orthogonal matrix, and $\hat{\sigma}$ is an estimator of σ. It follows that any invariant estimator of σ is of the form $CS^{\frac{1}{2}}$ and the optimal choice for the constant, C, is $C = C_{n+1}$ where $C_{n+1} = 2^{\frac{1}{2}}\Gamma(\frac{1}{2}(n + 1))/(n\Gamma(\frac{1}{2}n))$. However, the resulting estimator is inadmissible. To see this, consider the class of estimators invariant under the subgroup of transformations which is obtained from that just described by setting $b = 0$ wherever it appears in (2.1). Such an estimator, ϕ, must be a function of (S, T) alone, where $T = \sum Y_i^2$ and, in addition, satisfy $\phi(a^2S, a^2T) = a\phi(S, T)$, $a > 0$ so that $\phi(S, T) = (S + T)^{\frac{1}{2}}\phi((1 + T/S)^{-1}, 1 - (1 + T/S)^{-1})$. Thus $\phi(S, T) = \Psi(S/(S + T)) \times (S + T)^{\frac{1}{2}}$, where $\Psi(t) = \phi(t, 1 - t)$, $t > 0$. Define ϕ^* by $\phi^*(S, T) = \Psi^*(S/(S + T))(S + T)^{\frac{1}{2}}$, where $\Psi^*(t) = \min\{\Psi(t), C_{n+k+1}\}$. It will now be shown that

$$(2.2) \qquad E(\phi^*(S, T) - \sigma)^2/\sigma^2 \leqq E(\phi(S, T) - \sigma)^2/\sigma^2,$$

for all σ, unless $\phi^* \neq \phi$ when strict inequality holds.

Observe that S/σ^2 has the central χ^2 distribution with n degrees of freedom, while T/σ^2 has a noncentral χ^2 distribution with k degrees of freedom and noncentrality parameter $\|\mu\|^2/\sigma^2$. Alternatively, T/σ^2 has the same distribution as the random variable $W = \chi^2_{k+2L}$, where L has a Poisson distribution with parameter $\|\mu\|^2/(2\sigma^2)$, and given L, W has a central χ^2 distribution with $k + 2L$ degrees of freedom. As we shall see, we may without loss of generality let $\sigma = 1$.

Then

$$\begin{aligned} E(\phi(S, T) - 1)^2 &= E(\Psi[S/(T + S)](S + T)^{\frac{1}{2}} - 1)^2 \\ &= E(\Psi^2[S/(T + S)]E(S + T \mid L) \\ &\quad -2\Psi[S/(T + S)]E((S + T)^{\frac{1}{2}} \mid L) + 1) \\ &= E(\Psi^2[S/(S + T)](n + k + 2L) \\ &\quad - (2(n + k + 2L)\Psi[S/(S + T)]C_{n+k+2L+1} + 1). \end{aligned}$$

From this calculation, it follows that

$$\begin{aligned} E(\phi(S, T) - 1)^2 = E((n + k + 2L)\{\Psi[S/(T + S)] - C_{n+k+2L+1}\}^2 \\ - (n + k + 2L)C^2_{n+k+2L+1} + 1). \end{aligned}$$

A similar result holds for ϕ^* and inequality (2.2) is obtained since

$$\{\Psi[S/(S + T)] - C_{n+k+2L+1}\}^2 \geqq \{\Psi^*[S/(S + T)] - C_{n+k+2L+1}\}^2,$$

for all values of L. Since

$$C_{n+1}S^{\frac{1}{2}} = C_{n+1}[S/(S + T)]^{\frac{1}{2}}(S + T)^{\frac{1}{2}} = \Psi[S/(S + T)](S + T)^{\frac{1}{2}},$$

the required result follows from inequality (2.2).

Observe that the estimator which dominates $C_{n+1}S^{\frac{1}{2}}$ essentially modifies that estimator when T is small. Brown [1] also improves on the corresponding estimate for the variance in his model by making a modification under similar circumstances, but his estimator and proof are different.

3. Inadmissibility of the best invariant estimator. Now consider the problem of estimating $A\mu + \eta\sigma$. It remains invariant under the following transformations

$$\begin{aligned} (s, y) &\rightarrow (a^2 s, ay + b) \\ \mu &\rightarrow a\mu + b \\ \sigma^2 &\rightarrow a^2\sigma^2 \\ A\mu + \eta\sigma &\rightarrow a(A\mu + \eta\sigma) + Ab \\ \hat{\theta} &\rightarrow a\hat{\theta} + Ab, \end{aligned}$$

where $0 < a < \infty$, $b \varepsilon R^k$, $y = (y_1, \cdots, y_k)$, and $\hat{\theta}$ is an invariant estimator of $A\mu + \eta\sigma$.

The best invariant estimator is, as is easily shown,

$$AY + \eta C_{r+1}S^{\frac{1}{2}}. \tag{3.1}$$

Given A, it is inadmissible for all sufficiently large values of $\|\eta\|$, and, in fact, a uniformly better estimate, in that case, is

$$AY + \eta\phi_1(S, Y), \tag{3.2}$$

where

$$\phi_1(S, Y) = \min \{C_{n+1}S^{\frac{1}{2}}, C_{n+k+1}[S + \|Y\|^2]^{\frac{1}{2}}\}.$$

This will be proved with the help of inequality (2.2) and the following rather technical lemma. For convenience let $\phi_2(S, Y) = C_{n+1}S^{\frac{1}{2}}$.

LEMMA 3.1. *The quantity*, $E_\mu[(Y_i - \mu_i)(\phi_1(S, Y) - 1)]/\{E_\mu(\phi_2 - 1)^2 - E_\mu(\phi_1 - 1)^2\}$, *is bounded as a function of* μ *for each* $i = 1, 2, \cdots, k$ (*and* $\sigma = 1$).

PROOF. Suppose $i = 1$. Let $K = C^2_{n+k+1}/(C^2_{n+1} - C^2_{n+k+1})$. Observe that C_{n+k+1} is decreasing in $k \geqq 1$ and hence $K > 0$. Define functions g and g_1 by $g(\mu) = E(Y_1 - \mu_1)(\phi_1(S, Y) - 1)$ and $g_1(\mu) = E_\mu(\phi_2 - 1)^2 - E_\mu(\phi_1 - 1)^2$. Then

$$(3.3) \quad g(\mu) = (2\pi)^{-\frac{1}{2}k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\frac{1}{2}\|y-\mu\|^2}(y_1 - \mu_1)f^*(y_1, \cdots, y_k)\, dy_1 \cdots dy_k,$$

where

$$f^*(y) = [2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)]^{-1} \int_{K\|y\|^2}^{\infty} e^{-s/2}s^{n/2-1}[C_{n+k+1}(s + \|y\|^2)^{\frac{1}{2}} - C_{n+1}s^{\frac{1}{2}}]\, ds.$$

For simplicity, we write

$$h(s) = e^{-s/2}s^{n/2-1}/[2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n)], \qquad s > 0.$$

Define a function f by

$$f(y) = \frac{\partial}{\partial y_1} f^*(y) = y_1 \int_{K\|y\|^2}^{\infty} h(s)C_{n+k+1}/(s + \|y\|^2)^{\frac{1}{2}}\, ds.$$

By integrating the innermost integral in equation (3.3) by parts, we obtain

$$g(\mu) = e^{-\frac{1}{2}\|\mu\|^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}k}e^{\mu' y}e^{-\frac{1}{2}\|y\|^2}f(y)\, dy_1, \cdots, dy_k.$$

At the same time,

$$g_1(\mu) = e^{-\frac{1}{2}\|\mu\|^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (2\pi)^{-\frac{1}{2}k}e^{\mu' y}e^{-\frac{1}{2}\|y\|^2}f_1(y)\, dy_1, \cdots, dy_k,$$

where

$$f_1(y) = \int_{K\|y\|^2}^{\infty} h(s)\{(C_{n+1}s^{\frac{1}{2}} - 1)^2 - (C_{n+k+1}(s + \|y\|^2)^{\frac{1}{2}} - 1)^2\}\, ds.$$

Thus,

$$g(\mu)/g_1(\mu) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\mu' y}e^{-\frac{1}{2}\|y\|^2}f(y)\, dy_1 \cdots dy_k \div \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{\mu' y}e^{-\frac{1}{2}\|y\|^2}f_1(y)\, dy_1 \cdots dy_k.$$

Observe that g is an odd function of μ_1 and an even function of each of its other arguments and g_1 is an even function of each of its arguments. Thus, in considering $|g/g_1|$, as we must in proving the boundedness of g/g_1, we can assume, without loss of generality, that $\mu_i \geqq 0$, $i = 1, 2, \cdots, k$. Another useful observation is that f_1 is positive when $\|y\|^2 > 1/\{C^2_{n+k+1}(1 + K)\}$.

A straightforward computation yields the asymptotic equalities,

$$(3.4) \quad \Gamma(\tfrac{1}{2}n)f(y) \sim C^2_{n+k+1}(2^{\frac{1}{2}}C_{n+1})^{-1}y_1 \exp(-\tfrac{1}{2}K\|y\|^2(\tfrac{1}{2}K\|y\|^2)^{\frac{1}{2}(n-3)}$$

and

(3.5) $$\Gamma(\tfrac{1}{2}n)f_1(y) \sim 2(C^2_{n+1} - C^2_{n+k+1}) \exp(-\tfrac{1}{2}K\|y\|^2)(\tfrac{1}{2}K\|y\|^2)^{\frac{1}{2}(n-2)},$$

as $\|y\|^2 \to \infty$. They are obtained in the following manner. It is easy to show that, as $p \to \infty$,

(3.6) $$\int_p^\infty e^{-s}s^r\,ds \sim e^{-p}(p^r + rp^{r-1})$$

(3.7) $$\int_p^\infty (s + cp)^{\frac{1}{2}}e^{-s}s^r\,ds \sim e^{-p}[p^{r+\frac{1}{2}}(1 + c)^{\frac{1}{2}} + p^{r-\frac{1}{2}}(r\{1 + c\}^{\frac{1}{2}} + \tfrac{1}{2}/\{1 + c\}^{\frac{1}{2}})]$$

and

(3.8) $$\int_p^\infty e^{-s}s^r(s + cp)^{-\frac{1}{2}}\,ds \sim e^{-p}p^{r-\frac{1}{2}}(1 + c)^{-\frac{1}{2}},$$

where c is any positive constant and $r > -1$. Using equation (3.8),

$$\Gamma(\tfrac{1}{2}n)f(y) = -2^{-\frac{1}{2}}y_1C_{n+k+1}\int_{\frac{1}{2}K\|y\|^2}^\infty e^{-s}s^{\frac{1}{2}n-1}/[s + \tfrac{1}{2}K\|y\|^2/K]^{\frac{1}{2}}\,ds$$
$$\sim -2^{\frac{1}{2}}y_1C_{n+k+1}/(1 + K^{-1})^{\frac{1}{2}} \times (\tfrac{1}{2}K\|y\|^2)^{\frac{1}{2}(n-3)}e^{-(\frac{1}{2}K\|y\|^2)},$$

as $\|y\|^2 \to \infty$, and this yields (3.4). To obtain (3.5) observe that

$$\Gamma(\tfrac{1}{2}n)f_1(y)$$
$$= 2(C^2_{n+1} - C^2_{n+k+1})\{\int_{\frac{1}{2}K\|y\|^2}^\infty e^{-s}s^{\frac{1}{2}n}\,ds - (\tfrac{1}{2}K\|y\|^2)\int_{\frac{1}{2}K\|y\|^2}^\infty e^{-s}s^{\frac{1}{2}n-1}\,ds\}$$
$$- 2(2)^{\frac{1}{2}}C_{n+1}\int_{\frac{1}{2}K\|y\|^2}^\infty e^{-s}s^{(\frac{1}{2}(n-1))}\,ds + 2^{3/2}C_{n+k+1}\int_{\frac{1}{2}K\|y\|^2}^\infty e^{-s}s^{\frac{1}{2}n-1}$$
$$[s + \tfrac{1}{2}K\|y\|^2/K]^{\frac{1}{2}}\,ds.$$

On applying (3.6) and (3.7), we obtain (3.5).

Let M be chosen subject to $M > 1/[C^2_{n+k+1}(1 + K)]$. Suppose $\mu_i \geqq 0$, $i = 1, 2, \cdots, k$. Then $|g(\mu)|$ is bounded by $g^*(\mu)$ which is obtained from $g(\mu)$ by replacing f with $|f|$ in the integrand of g. We shall now show $g^*(\mu)/|g_1(\mu)|$ is bounded.

From

$$\int_{-\infty}^M \cdots \int_{-\infty}^M e^{\mu'y}e^{-\frac{1}{2}\|y\|^2}|f(y)|\,dy_1 \cdots dy_k$$
$$\div \{\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty - \int_{-\infty}^M \cdots \int_{-\infty}^M e^{\mu'y}e^{-\frac{1}{2}\|y\|^2}|f(y)|\,dy_1 \cdots dy_k$$
$$\leqq \exp(-M\textstyle\sum_{1=1}^k \mu_i)\int_{-\infty}^M \cdots \int_{-\infty}^M e^{-\frac{1}{2}\|y\|^2}|f(y)|\,dy_1 \cdots dy_k$$
$$\div \int_{2M}^\infty \cdots \int_{2M}^\infty e^{-\frac{1}{2}\|y\|^2}|f(y)|\,dy_1 \cdots dy_k,$$

we conclude

(3.9) $$(2\pi)^{\frac{1}{2}k}g^*(\mu) \sim \{\int_{-\infty}^\infty \cdots \int_{-\infty}^\infty - \int_{-\infty}^M \cdots \int_{-\infty}^M\}e^{\mu'y}e^{-\frac{1}{2}\|y\|^2}|f(y)|\,dy_1 \cdots dy_k,$$

as $|\mu_i| \to \infty$ for at least one i. Since these results hold for all values of M satisfying $M > 1/\{C^2_{n+k+1}(1 + K)\}$, it follows from asymptotic equalities (3.4), (3.5), (3.6), and an expression for g_1 similar to (3.9) for g^* (with $|f(y)|$ replaced by

$f_1(y))$,

$$
\begin{aligned}
|g^*(\mu)/g_1(\mu)| \sim \{ & \textstyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} - \int_{-\infty}^{M} \cdots \int_{-\infty}^{M}\} \exp(\mu' y - \frac{1}{2}\|y\|^2) \\
& \times C_{n+k+1}^2 (2^{\frac{1}{2}} C_{n+1})^{-1} |y_1| e^{-\frac{1}{2}K\|y\|^2} (\tfrac{1}{2}K\|y\|^2)^{\frac{1}{2}(n-3)}\, dy_1 \cdots dy_k \\
& \div \{\textstyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} - \int_{-\infty}^{M} \cdots \int_{-\infty}^{M}\} \exp(\mu' y - \frac{1}{2}\|y\|^2) \\
& \times 2\{C_{n+1}^2 - C_{n+k+1}^2\} e^{-\frac{1}{2}K\|y\|^2} (\tfrac{1}{2}K\|y\|^2)^{\frac{1}{2}n-1}\, dy_1 \cdots dy_k
\end{aligned} \tag{3.10}
$$

as $|\mu_1| \to \infty$ for some i. The quantity on the right of asymptotic equality (3.10) is bounded, for all μ, by

$$C_{n+k+1}/[2C_{n+1}(C_{n+1}^2 - C_{n+k+1}^2)^{\frac{1}{2}}].$$

Since, on any finite rectangle, $g(\mu)/g_1(\mu)$ is bounded, the required conclusion follows.

It will now be shown that the estimator given in (3.2) is uniformly better, with A fixed, for all sufficiently large values of $\|\eta\|$, than the estimator given in (3.1). We continue to assume $\sigma = 1$. Consider the difference of the risks of the two estimators, namely $E_\mu\|AY + \eta\phi_2 - A\mu - \eta\|^2 - E_\mu\|AY + \eta\phi_1 - A\mu - \eta\|^2$. This is just

$$
\begin{aligned}
&\|\eta\|\{E_\mu(\phi_2 - 1)^2 - E_\mu(\phi_1 - 1)^2\}[\|\eta\| \\
&\qquad -2E_\mu(Y - \mu)'(A'\eta/\|\eta\|)(\phi_1 - 1)/\{E_\mu(\phi_2 - 1)^2 - E_\mu(\phi_1 - 1)^2\}].
\end{aligned}
$$

By inequality (2.2), the second factor in this expression is positive for all μ. Furthermore,

$$
\begin{aligned}
& E_\mu(Y - \mu)'(A'\eta/\|\eta\|)(\phi_1 - 1)/\{E_\mu(\phi_2 - 1)^2 - E_\mu(\phi_1 - 1)^2\} \\
& = \textstyle\sum_{i=1}^k [\sum_{j=1}^m a_{ij}(\eta_j)/\|\eta\|)] \\
& \qquad \cdot (E_\mu(Y_i - \mu_i)(\phi_1 - 1)/\{E_\mu(\phi_2 - 1)^2 - E_\mu(\phi_1 - 1)^2\}]
\end{aligned} \tag{3.11}
$$

where $A = (a_{ij})$ and $\eta = (\eta_1, \cdots, \eta_m)'$. Since,

$$|\textstyle\sum_{i=1}^m a_{ij}(\eta_j)/\|\eta\|| \leqq (\sum_{j=1}^m a_{ij}^2)^{\frac{1}{2}},$$

by Lemma 3.1, the quantity appearing in equation (3.11) is uniformly bounded in μ and η. Provided $\|\eta\|$ exceeds the upper bound for this expression, the difference in the risks of the two estimators will be positive for all values of μ. It follows that the best invariant estimator is inadmissible for all sufficiently large values of $\|\eta\|$ when A is fixed.

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