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CHARACTERIZATIONS OF THE LINEAR EXPONENTIAL FAMILY IN A PARAMETER BY RECURRENCE RELATIONS FOR FUNCTIONS OF CUMULANTS[1]

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1. Introduction. The linear exponential family was characterized by a recurrence relation in cumulants by Patil [3] and by a recurrence relation in raw moments by Wani [4]. In this paper we present a general approach of characterizing the linear exponential family by a recurrence relation in functions of cumulants under the assumption that cumulants can be in turn expressed as functions of those occurring in the relation. Then, the characterization given by Wani [4] becomes a particular instance of ours since the raw moments can be expressed as functions of cumulants and vice versa.

2. The induced linear exponential family. A family $\mathcal{P}_e = \{P_\omega : \omega \varepsilon \Omega_\mu\}$ of probability distributions is said to be linear exponential in ω over a Euclidean sample space $(\mathfrak{X}, \beta)$ if

$$dP_\omega(x) = \{e^{\omega x}/f(\omega)\}\, d\mu(x) \tag{2.1}$$

where Ω_μ is assumed to be the natural parameter space with a nonvoid interior. It is understood by a natural parameter space that Ω_μ consists of all parameter points ω for which

$$f(\omega) = \int e^{\omega x}\, d\mu(x) \tag{2.2}$$

is positive and finite. Moreover, $f(\omega)$ is analytic in the interior of Ω_μ. If $\mathfrak{X}$ is p-dimensional, then we further assume that Ω_μ is a subset of a p-dimensional Euclidean space so that ωx can be interpreted as a scalar product of two vectors. We may call P_ω a linear exponential distribution in ω, but we bear in mind that P_ω may involve some other parameter in which it may not be linear exponential.

We observe that $f(\omega)$ is not unique; for any positive constant multiple of $f(\omega)$ gives rise to the same distribution P_ω. For example, given any interior point ξ of Ω_μ we may write (2.1) as

$$dP^*_{\theta,\xi}(x) = dP_\omega(x) = \{e^{\theta x}/m(\theta, \xi)\}\, dP_\xi(x) \tag{2.3}$$

where $\theta = \omega - \xi$ and $m(\theta, \xi) = f(\theta + \xi)/f(\xi)$. We can readily see that $m(\theta, \xi)$ is the moment generating function (mgf) of P_ξ with θ as its parameter. Thus we are led to define $P^*_{\theta,\xi}$ as an induced linear exponential distribution in θ of the distribution P_ξ. In fact, we can extend this definition to any distribution, not

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necessarily linear exponential, which admits analytic characteristic function or equivalently for which the mgf exists. So for a distribution P for which the mgf $m(\theta)$ exists in Θ_p with $\theta = 0$ as an interior point, we formally define the family $\mathcal{P}^* = \{P_\theta^*:\theta \varepsilon \Theta_p\}$ as the induced linear exponential family in θ of P over the same Euclidean sample space of P if

$$dP_\theta^*(x) = \{e^{\theta x}/m(\theta)\}\, dP(x). \tag{2.4}$$

We call $\mathcal{P}^*$ briefly as the induced family of P_θ and P^* an induced distribution of P. If P has a parameter ξ, in which P need not be linear exponential, then we may write its induced distribution as in (2.3). Of course, θ should not be in P, but is arbitrary.

Example 2.1. A normal distribution P_ξ with mean 0 and variance ξ is not linear exponential in ξ. However, its induced family $\mathcal{P}^* = \{P_{\theta,\xi}^*:\theta \varepsilon (-\infty, \infty)\}$ where

$$\begin{aligned} dP_{\theta,\xi}^* &= \{e^{\theta x}/\exp\,(1/2\theta^2\xi)\}\, dP_\xi \\ &= (2\pi\xi)^{-\frac{1}{2}} \exp\{-(x - \theta\xi)^2/2\xi\}\, dx, \end{aligned}$$

is again normal with mean $\theta\xi$ and variance ξ. We clearly see that P_ξ is a member of this family when $\theta = 0$ and, therefore, linear exponential in θ. Furthermore we note that the collection of all normal distributions and the collection of their induced distributions coincide.

From now on we assume all the distributions we consider possess the mgf's. We also assume that θ denotes the parameter with 0 as an interior point in its natural parameter space and ξ denotes any interior point of its natural parameter space, and that $\theta + \xi$ is always to be taken as an interior point of the parameter space of ξ for all θ in some neighborhood of $\theta = 0$ which lies entirely in the parameter space of θ.

One can immediately see from (2.3) that the induced family of P_ξ, linear exponential in ξ, coincides with the linear exponential family in ξ of P, of which P_ξ is a member. Moreover, when we put $\theta = 0$ in (2.4) P_θ^* reduces to the distribution P, i.e., P is a member of its own induced family. Then we arrive at a remarkable conclusion.

Theorem 2.1. *Every distribution is linear exponential at least in an arbitrary parameter and, if it is already linear exponential in ξ, then its linear exponential family in ξ coincides with that of its induced distributions.*

An immediate and far reaching consequence of Theorem 2.1 is that all the properties, and relations in moments and cumulants of a linear exponential distribution in a parameter ξ can be easily deduced for any distribution and conversely. We shall exploit this important observation to a great measure in what follows.

For the sake of simplicity and clarity we shall consider only univariate distributions, and it is not difficult to extend the results to multivariate distributions analogously. For a distribution P_ξ, not necessarily linear exponential in ξ, we denote the moments about the origin, moments about the mean and cumulants of order r of its induced distribution $P^*_{\theta,\xi}$ by $\mu_r'(\theta, \xi)$, $\mu_r(\theta, \xi)$ and $\kappa_r(\theta, \xi)$ respectively while we denote those of P_ξ correspondingly by $\mu_r'(0, \xi)$, $\mu_r(0, \xi)$ and $\kappa_r(0, \xi)$.

To illustrate the usefulness of Theorem 1, consider, for example, the recurrence relation in cumulants for a linear exponential distribution in θ obtained by Khatri [1] and Patil [3] among others

$$\kappa_{r+1}(\theta, \xi) = \frac{\partial \kappa_r(\theta, \xi)}{\partial \theta}, \qquad r \geqq 1. \tag{2.5}$$

The existence of higher order partial derivatives of the cumulants of $P^*_{\theta,\xi}$ can be easily established from the analyticity of the mgf and, moreover, we can show (Patil [3])

$$\kappa_r(\theta, \xi) = \frac{\partial^r \log m(\theta, \xi)}{\partial \theta^r}, \qquad r \geqq 1. \tag{2.6}$$

Then, if we put $\theta = 0$ in (2.5) and (2.6) we obtain the result,

$$\kappa_{r+1}(0, \xi) = \left[\frac{\partial \kappa_r(\theta, \xi)}{\partial \theta}\right]_{\theta=0}, \qquad r \geqq 1 \tag{2.7}$$

and the well-known result of the cumulant generating function,

$$\kappa_r(0, \xi) = \left[\frac{\partial^r \log m(\theta, \xi)}{\partial \theta^r}\right]_{\theta=0}, \qquad r \geqq 1 \tag{2.8}$$

respectively. If P_ξ is linear exponential in ξ, then we again have

$$\kappa_{r+1}(0, \xi) = \frac{\partial \kappa_r(0, \xi)}{\partial \xi}, \qquad r \geqq 1. \tag{2.9}$$

From (2.7) and (2.8) we obtain an important property of P_ξ being linear exponential in ξ, namely,

$$\left[\frac{\partial \kappa_r(\theta, \xi)}{\partial \theta}\right]_{\theta=0} = \frac{\partial \kappa_r(0, \xi)}{\partial \xi}, \qquad r \geqq 1. \tag{2.10}$$

In fact, we can generalize the relation (2.10) to a function of cumulants as in Theorem 2, which needs the following lemma.

LEMMA 2.1. *Let g be a function of the functions $\beta_1(\theta, \xi), \cdots, \beta_n(\theta, \xi)$ with partial continuous derivatives $\partial g/\partial \beta_1, \cdots, \partial g/\partial \beta_n$ in a neighborhood of the point $(\beta_1(0, \xi), \cdots, \beta_n(0, \xi))$ of its domain. Assume $[\partial \beta_r(\theta, \xi)/\partial \theta]_{\theta=0}$ and $\partial \beta_r(0, \xi)/\partial \xi$ exist for all $r = 1, \cdots, n$. Then g is a function of θ and ξ and*

$$\left[\frac{\partial \beta_r(\theta, \xi)}{\partial \theta}\right]_{\theta=0} = \frac{\partial \beta_r(0, \xi)}{\partial \xi}, \qquad (r = 1, \cdots, n), \tag{2.11}$$

implies

$$\left[\frac{\partial g(\theta, \xi)}{\partial \theta}\right]_{\theta=0} = \frac{\partial g(0, \xi)}{\partial \xi}. \tag{2.12}$$

Proof. The proof of the lemma is easily seen by applying the chain rule of composite functions of a real variable to both the sides of (2.12) separately and observing them to be equal by virtue of (2.11).

Now we are ready to deduce from Lemma 2.1 with $\beta_r(\theta, \xi) = \kappa_r(\theta, \xi)$, $r \geqq 1$, and $g = \alpha$, the following theorem.

Theorem 2.2. *Let α be a function of the cumulants $\kappa_1(\theta, \xi), \cdots, \kappa_n(\theta, \xi)$ with partial continuous derivatives $\partial\alpha/\partial\kappa_1, \cdots, \partial\alpha/\partial\kappa_n$ in a neighborhood of the point $(\kappa_1(0, \xi), \cdots, \kappa_n(0, \xi))$ of its domain. Then, α is a function of θ and ξ, and for a P_ξ, linear exponential in ξ, we get*

$$\left[\frac{\partial\alpha(\theta, \xi)}{\partial\theta}\right]_{\theta=0} = \frac{\partial\alpha(0, \xi)}{\partial\xi}. \tag{2.13}$$

From Theorem 2.2 we can easily see that (2.13) is true when α stands for a moment of any order about an arbitrary origin, or a factorial moment or cumulant of a discrete distribution, since a moment or a factorial moment or cumulant can be expressed as a polynomial in cumulants which satisfy the conditions of the theorem. In the next section, we show that recurrence relations in the form of (2.13) characterize a linear exponential distribution in ξ.

Instead of choosing the recurrence relation (2.5), we could have alternatively considered for any P_ξ

$$\mu'_{r+1}(0, \xi) = \left[\frac{\partial\mu_r'(\theta, \xi)}{\partial\theta}\right]_{\theta=0} + \mu_1'(0, \xi)\mu_r'(0, \xi) \tag{2.14}$$

given by Noack [2] and, if P_ξ is linear exponential in ξ, we would have arrived at a theorem essentially the same as Theorem 2.2. Incidentally, we have pointed out in (2.7) and (2.14) how a recurrence relation true for a distribution linear exponential in ξ can be so modified to be still true for any distribution by means of its induced family.

Before concluding this section we shall point out that $m(\theta, \xi)$ of (2.3) which is the mgf of P_ξ is usually called the generating function of P_ω since its functional form characterizes a particular family $\mathcal{P}_e$. As we have seen already that the generating function of P_ω is not unique, then all the properties of the mgf hold good for a subclass of generating functions of the family $\mathcal{P}_e$. However, the properties of the mgf which are invariant under a translation of the coordinate axes of the parameter space and a positive constant multiplication of the mgf hold good for all the generating functions. For example, as the mgf is a convex function, so is every generating function. On the other hand, the property that the mgf has a Fourier integral in some neighborhood of zero cannot be simply extended to all generating functions because zero may not be an interior point of the argument of the generating function. However, a generating function can have a Fourier integral in some neighborhood of an interior point of the domain.

3. Characterizations.

We need the following lemma extracted, simplified, and generalized from the proof of Patil's main theorem in [3] before we present the important theorem of this paper.

LEMMA 3.1. *Let $g(\theta, \xi)$ be analytic in θ in a neighborhood of $\theta = 0$ and in ξ in some neighborhood of ξ. Then $g(\theta, \xi) = g(0, \theta + \xi)$ if and only if*

$$\left[\frac{\partial^r g(\theta, \xi)}{\partial \theta^r}\right]_{\theta=0} = \left[\frac{\partial^r g(0, \xi)}{\partial \xi^r}\right], \qquad \textit{for all } r \geqq 0.$$

PROOF. The proof follows from Taylor's expansions

$$g(\theta, \xi) = \sum_{r=0}^{\infty} \left[\frac{\partial^r g(\theta, \xi)}{\partial \theta^r}\right]_{\theta=0} \theta^r/r!$$

and

$$g(0, \theta + \xi) = \sum_{r=0}^{\infty} \frac{\partial^r g(0, \xi)}{\partial \xi^r} \theta^r/r!\,.$$

THEOREM 3.1. *Let P_ξ and $P^*_{\theta,\xi}$ denote a distribution and its induced distribution respectively. Let $\kappa_1(\theta, \xi)$, $\kappa_2(\theta, \xi)$ $\cdots$ denote the cumulants of $P^*_{\theta,\xi}$ as before. Let α_n, $n = 1, 2, \cdots$ be a function of $\kappa_1(\theta, \xi), \cdots, \kappa_n(\theta, \xi)$ with continuous partial derivatives in a neighborhood of the point $(\kappa_1(0, \xi), \cdots, \kappa_r(0, \xi))$ such that κ_n, $n = 1, 2, \cdots$ can be in turn expressed as a function of $\alpha_1(\theta, \xi), \cdots, \alpha_n(\theta, \xi)$ with continuous partial derivatives $\partial\kappa_n/\partial\alpha_1, \cdots, \partial\kappa_n/\partial\alpha_n$ in a neighborhood of the point $(\alpha_1(0, \xi), \cdots, \alpha_n(0, \xi))$. Then, P_ξ is linearly exponential in ξ if and only if any one of the following statements is true:*

(a) $$P^*_{\theta,\xi} = P_{\theta+\xi} \quad \textit{for all} \quad \xi,$$

(b) $$\kappa_1(\theta, \xi) = \kappa_1(0, \theta + \xi),$$

(c) $$\frac{\partial \kappa_r(0, \xi)}{\partial \xi} = \left[\frac{\partial \kappa_r(\theta, \xi)}{\partial \theta}\right]_{\theta=0}, \qquad r \geqq 1,$$

(d) $$\frac{\partial \alpha_r(0, \xi)}{\partial \xi} = \left[\frac{\partial \alpha_r(\theta, \xi)}{\partial \theta}\right]_{\theta=0}, \qquad r \geqq 1.$$

PROOF. (a) The necessity is proved in (2.3) and the sufficiency obviously follows from the fact that $P_{\theta+\xi}$ linear exponential in θ for all ξ (being interior points in a neighborhood of $\theta = 0$) implies P_ξ is linear exponential in ξ.

(b) For any distribution P_ξ we note from (2.6)

$$\kappa_1(\theta, \xi) = \frac{\partial \log m(\theta, \xi)}{\partial \theta} \tag{3.1}$$

where $m(\theta, \xi)$ is the mgf of P_ξ. The necessity is easily verified when we recall $m(\theta, \xi) = f(\theta + \xi)/f(\xi)$ from (2.3). The sufficiency is established by letting $\int \kappa_1(0, \theta + \xi)\, d\theta = g(\theta + \xi)$ and then applying the inversion of the Fourier transform to $m(it, \xi)$ where $\log m(\theta, \xi) = g(\theta + \xi) - g(\xi)$.

(c) The necessity is already proved in (2.10). The sufficiency is established as follows: from (2.7) and part (c) we get

$$\kappa_{r+1}(0, \xi) = \frac{\partial \kappa_r(0, \xi)}{\partial \xi}, \qquad r \geqq 1$$

which can be written as

$$\kappa_{r+1}(0, \xi) = \frac{\partial^r \kappa_1(0, \xi)}{\partial \xi^r}, \qquad r \geqq 1 \tag{3.2}$$

while from (2.6) and part (c)

$$\kappa_{r+1}(0, \xi) = \left[\frac{\partial^r \kappa_1(\theta, \xi)}{\partial \theta^r}\right]_{\theta=0}. \tag{3.3}$$

We complete the proof of part (c) by observing that (3.2) and (3.3) imply part (b) by Lemma 3.1.

(d) This part is proved when we observe that it is equivalent to the part (c) by Lemma 2.1 in which $\beta_r(\theta, \xi)$ plays roles of $\kappa_r(\theta, \xi)$ as in Theorem 2.1 and $\alpha_r(\theta, \xi)$ to obtain the converse of Theorem 2.1. This completes the proof of Theorem 3.1.

From (2.7) and part (c) of Theorem 3.1 we deduce

COROLLARY 3.1. (Patil [3]) *P_ξ is a linear exponential in ξ if and only if*

$$\kappa_{r+1}(0, \xi) = \frac{\partial \kappa_r(0, \xi)}{\partial \xi}, \qquad r \geqq 1. \tag{3.4}$$

From (2.14) and part (d) of Theorem 3.1 we deduce

COROLLARY 3.2. (Wani [4]) *P_ξ is linear exponential in ξ if and only if*

$$\mu'_{r+1}(0, \xi) = \frac{\partial \mu_r'(0, \xi)}{\partial \xi} + \mu_1'(0, \xi)\mu_r'(0, \xi), \qquad r \geqq 1. \tag{3.5}$$

At first sight (3.5) may look different from the recurrence relation Wani [4] employed to characterize a linear exponential family in ξ, but his form is equivalent to (3.5) with $\mu_1'(0, \xi) = df(\xi)/d\xi$.

By Theorem 3.1 and corollaries one can prove that all the recurrence relations given by Khatri [1] and Noack [2] except the one which we consider in the next section characterize a linear exponential family in a parameter. One can relax the conditions on the functions of α in Theorem 3.1 as long as parts (c) and (d) of the theorem are equivalent.

4. Discussion. The recurrence relation in the moments about the mean given by Noack [2],

$$\mu_{r+1}(0, \xi) = \frac{\partial \mu_r(0, \xi)}{\partial \xi} + r\mu_2(0, \xi)\mu_{r-1}(0, \xi), \tag{4.1}$$

of a linear exponential P_ξ in ξ gives rise to

$$\frac{\partial \mu_r(0, \xi)}{\partial \xi} = \left[\frac{\partial \mu_r(\theta, \xi)}{\partial \theta}\right]_{\theta=0}, \qquad r \geqq 1. \tag{4.2}$$

Since $\mu_1(\theta, \xi) \equiv 0$, $\kappa_1(\theta, \xi)$ cannot be expressed as a function of the moments about the mean. Therefore, by Theorem 3.1, (4.2) alone cannot characterize a linear exponential family in ξ, but we need one more relation

$$\mu_2(0, \xi) = \frac{\partial \mu_1'(0, \xi)}{\partial \xi}. \tag{4.3}$$

If (4.3) is not given, one may wonder what family of distributions (4.2) alone can characterize. In order to discover this family let us assume the mean of a member to be a. Denoting a particular solution of the equation

$$\frac{\partial \mu'(0, \xi)}{\partial \xi} = \mu_2(0, \xi) \tag{4.4}$$

by $\mu'(0, \xi)$, we get a general solution of (4.4) as $\mu'(0, \xi) + c$ where c is an arbitrary constant. Then, $m(\theta, \xi)$, the mgf of the distribution of the family in question, can be composed as

$$m(\theta, \xi) = m'(\theta + \xi)e^{b(\xi)\theta} \tag{4.5}$$

where $m'(\theta + \xi)$ is the mgf of a distribution P_ξ, linear exponential in ξ, with the mean $\mu'(0, \xi)$ and the moments about the mean given by the recurrence relation (4.2), and $b(\xi) = a - \mu'(0, \xi)$. Note, $b(\xi)$ is an arbitrary point since a is arbitrary. Then we conclude that (4.2) alone characterizes a family of all distributions each of which is obtained by a convolution of a member of $\mathcal{P}_e$ linear exponential in ξ and an arbitrary point distribution. In other words, (4.2) characterizes a family of all distributions which can be made linear exponential in ξ by a translation of the coordinate axes of the sample space.

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