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ON A CLASS OF RANK ORDER TESTS FOR THE PARALLELISM OF SEVERAL REGRESSION LINES[1]

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0. Summary. For the regression model $Y_{\nu i} = \alpha + \beta c_{\nu i} + \epsilon_{\nu i}$, $i = 1, \cdots, N_\nu$, where the $\epsilon_{\nu i}$ are independent and identically distributed random variables (iidrv), optimum rank order tests for the hypothesis that $\beta = 0$ are due to Hoeffding (1950), Terry (1952) and Hájek (1962), among others. In the present paper, the theory is extended to the problem of testing the homogeneity of the regression coefficients from $k(\geqq 2)$ independent samples. Allied efficiency results are also presented.

1. Introduction and preliminary notions. For each positive integer $\nu(1 \leqq \nu < \infty)$, consider a sequence of $N_\nu(= \sum_{i=1}^k n_{\nu i})$ independent random variables $Y_{\nu ij}$, $j = 1, \cdots, n_{\nu i}$, $i = 1, \cdots, k$, where we assume that

$$P[Y_{\nu ij} \leqq x] = F_{\nu ij}(x) = F(x - \alpha_i - \beta_i c_{\nu ij}), \qquad F \varepsilon \mathfrak{F}, \tag{1.1}$$

where $\mathbf{c}_{\nu i} = (c_{\nu i1}, \cdots, c_{\nu in_{\nu i}})$, $i = 1, \cdots, k$, are vectors of known constants, $\alpha_1, \cdots, \alpha_k$ are nuisance parameters, $\beta_1, \cdots, \beta_k$ are the regression parameters, and $\mathfrak{F}$ is the class of all absolutely continuous (univariate) cumulative distribution functions (cdf) for which the square root of the density function possesses a quadratically integrable derivative. That is, $\mathfrak{F} = \{F : I(F) < \infty\}$, where

$$I(F) = \int_{-\infty}^{\infty} [f'(x)/f(x)]^2 \, dF(x); \qquad f(x) = (d/dx)F(x) \quad \text{and} \quad f'(x) = (d/dx)f(x). \tag{1.2}$$

Our problem is to test the null hypothesis

$$H_0 : \beta_1 = \cdots = \beta_k = \beta \text{ (unknown)}, \tag{1.3}$$

against the set of alternatives that $\beta_1, \cdots, \beta_k$ are not all equal.

Let us define

$$\bar{c}_{\nu i} = n_{\nu i}^{-1} \sum_{j=1}^{n_{\nu i}} c_{\nu ij}, \qquad C_{\nu i}^2 = \sum_{j=1}^{n_{\nu i}} (c_{\nu ij} - \bar{c}_{\nu i})^2, \qquad i = 1, \cdots, k; \tag{1.4}$$

$$C_\nu^2 = \sum_{i=1}^k C_{\nu i}^2 \quad \text{and} \quad \gamma_{\nu i} = C_{\nu i}^2 / C_\nu^2, \qquad \text{for } i = 1, \cdots, k. \tag{1.5}$$

It is assumed that as $\nu \to \infty$, $n_{\nu i}$, $C_{\nu i}^2$, $i = 1, \cdots, k$, all tend to ∞, satisfying

$$\gamma_{\nu i} \to \gamma_i : 0 < \gamma_0 \leqq \gamma_1, \cdots, \gamma_k \leqq 1 - \gamma_0 < 1, \quad \text{where} \quad \gamma_0 \leqq 1/k, \tag{1.6}$$

and the Noether condition:

$$\lim_{\nu\to\infty} [\max_{1 \leqq j \leqq n_{\nu i}} |c_{\nu ij} - \bar{c}_{\nu i}| / C_{\nu i}] = 0, \qquad \text{for all } i = 1, \cdots, k. \tag{1.7}$$

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Further, without any loss of generality, we assume that for each $\nu(1 \leqq \nu < \infty)$,

$$(1.8) \quad c_{\nu i1} \leqq c_{\nu i2} \leqq \cdots \leqq c_{\nu i n_{\nu i}} \text{ (where all the equality signs are not strict)}, \quad i = 1, \cdots, k.$$

Let now $\phi(u)$ be an absolutely continuous and non-decreasing function of $u: 0 < u < 1$, and assume that $\phi(u)$ is square integrable over $(0, 1)$. Also, let $U_{\nu 1} < \cdots < U_{\nu n_{\nu i}}$ be the ordered random variables in a random sample of size $n_{\nu i}$ from a rectangular distribution over $(0, 1)$, and define

$$(1.9) \qquad E_{\nu,j}^{(i)} = E[\phi(U_{\nu j})], \qquad j = 1, \cdots, n_{\nu i}, \qquad i = 1, \cdots, k.$$

Alternatively, we may also define the scores

$$(1.10) \qquad E_{\nu,j}^{(i)} = \phi(j/[n_{\nu i} + 1]), \qquad j = 1, \cdots, n_{\nu i}, \qquad i = 1, \cdots, k.$$

It is well known that the scores in (1.9) and (1.10) lead to asymptotically equivalent statistics [cf. Hájek (1962, 1968)]. Two particular scores are worth mentioning, and are used in later sections. If $\phi(u) = u: 0 < u < 1$, then the corresponding scores in (1.9) or (1.10) are given by $j/[n_{\nu i} + 1]$, $j = 1, \cdots, n_{\nu i}$, and are known as the Wilcoxon scores, and if $\phi(u)$ is the inverse of the standard normal cdf, then (1.9) relates to the expected values of the order statistics of a sample of size $n_{\nu i}$ from the standard normal distribution, and are known as the normal scores.

Let us now define

$$(1.11) \qquad \bar{E}_{\nu}^{(i)} = (1/n_{\nu i}) \textstyle\sum_{j=1}^{n_{\nu i}} E_{\nu,j}^{(i)}, \qquad i = 1, \cdots, k;$$

$$(1.12) \qquad \phi^* = \textstyle\int_0^1 \phi(u)\, du \quad \text{and} \quad A^2 = \int_0^1 \phi^2(u)\, du - (\phi^*)^2.$$

Consider then the statistics

$$(1.13) \qquad T_{\nu,i} = [\textstyle\sum_{j=1}^{n_{\nu i}} (c_{\nu ij} - \bar{c}_{\nu i}) E_{\nu R_{\nu ij}}^{(i)}]/[A \cdot C_{\nu i}], \qquad i = 1, \cdots, k,$$

where $R_{\nu ij}$ is the rank of $Y_{\nu ij}$ among $Y_{\nu i1}, \cdots, Y_{\nu i n_{\nu i}}$, and $C_{\nu i}$ is defined by (1.4). Such statistics are studied in detail by Hoeffding (1950), Terry (1952) and Hájek (1962) for testing the hypothesis of no regression in the single sample case. For convenience, we write

$$(1.14) \quad T_{\nu,i} = T_{\nu,i}(\mathbf{Y}_{\nu i}), \quad \text{where} \quad \mathbf{Y}_{\nu i} = (Y_{\nu i1}, \cdots, Y_{\nu i n_{\nu i}}), \quad i = 1, \cdots, k.$$

Also, we denote by $T_{\nu,i}(\mathbf{Y}_{\nu i} + b \cdot \mathbf{c}_{\nu i})$ the statistic in (1.13) when the observations $Y_{\nu ij}$ are replaced by $Y_{\nu ij} + b \cdot c_{\nu ij}$, where b is a real number. It follows from Theorem 6.1 (to follow) that under (1.8) through (1.14)

$$(1.15) \quad T_{\nu,i}(\mathbf{Y}_{\nu i} - b \cdot \mathbf{c}_{\nu i}) \text{ is } \downarrow \text{ in } b(-\infty < b < \infty), \quad \text{for all } i = 1, \cdots, k.$$

Also, for later use we define here

$$(1.16) \qquad \psi(u) = -[f'(F^{-1}(u))/f(F^{-1}(u))], \qquad 0 < u < 1,$$

so that

$$(1.17) \qquad \textstyle\int_0^1 \psi(u)\, du = 0 \quad \text{and} \quad \int_0^1 \psi^2(u)\, du = I(F) < \infty.$$

Finally, let

$$\rho(\psi, \phi) = [\int_0^1 \psi(u)\phi(u)\,du]/[A^2 \cdot I(F)]^{\frac{1}{2}} \tag{1.18}$$

and assume that $\rho(\psi, \phi)$ is strictly positive. It may be noted that the assumption of monotonicity of $\phi(u)$ (made earlier) holds if we take

$$\phi(u) = -[g'(G^{-1}(u))/g(G^{-1}(u))], \qquad 0 < u < 1, \qquad G \varepsilon \mathfrak{F}, \tag{1.19}$$

when $g(x)$ is a strongly unimodal density function [cf. Hájek and Šidák (1967, page 20, Lemma c)]. Further, if both $f(x)$ and $g(x)$ are strongly unimodal, it can be shown that $\rho(\psi, \phi)$, defined by (1.19), is positive whenever both A^2 and $I(F)$ are so.

In the sequel, it will be necessary for us to consider a consistent estimator of $\int_0^1 \psi(u)\phi(u)\,du$, and for this, we require the following assumption on $\phi(u)$:

(1.20) $(d/dx)\phi(F(x))$ is continuous and bounded for all $x(-\infty < x < \infty)$.

In the classical two-sample location problem, (1.20) is often assumed to be true for the general Chernoff-Savage (1958) type of statistics [cf. Puri (1964), and others]. Finally, in the sequel, we use the symbols $O(1)$ [or $O_p(1)$] and $o(1)$ [or $o_p(1)$] in the same sense as in Mann and Wald (1943).

Under the above notations and assumptions, in Section 2, asymptotic properties of the likelihood ratio test—including its asymptotic optimality as a parametric test—and the variance-ratio test are derived. Section 3 presents asymptotically optimum non-parametric tests. In Section 4, the model (1.1) is generalized to distributions not necessarily identical, and suitable large sample tests for (1.3) are constructed. The last section is devoted to the study of an optimum property of a pooled sample nonparametric estimator of β when (1.3) holds.

2. Asymptotically optimum parametric tests. Under (1.1), let $p(\mathbf{Y}_\nu; \boldsymbol{\alpha}, \boldsymbol{\beta})$ be the joint density function of $\mathbf{Y}_\nu = (\mathbf{Y}_{\nu 1}, \cdots, \mathbf{Y}_{\nu k})$, where $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_k)$ and $\boldsymbol{\beta} = (\beta_1, \cdots, \beta_k)$. Let Ω_{2k} be the $2k$-dimensional real space of the parameters $(\boldsymbol{\alpha}, \boldsymbol{\beta})$, and let $\Omega^*_{(k+1)}$ be the subspace of Ω_{2k} for which $\beta_1 = \cdots = \beta_k$. Denote the likelihood ratio statistic by

$$\lambda_\nu = [\sup_{\Omega^*_{(k+1)}} p(\mathbf{Y}_\nu; \boldsymbol{\alpha}, \boldsymbol{\beta})/\sup_{\Omega_{2k}} p(\mathbf{Y}_\nu; \boldsymbol{\alpha}, \boldsymbol{\beta})]. \tag{2.1}$$

It follows from the results of Wald (1943, Theorem 9, page 480) that under H_0 in (1.3), $-2 \log \lambda_\nu$ has asymptotically a chi-square distribution with $k - 1$ degrees of freedom (df). Hence, for large sample sizes, the likelihood ratio test may be constructed as follows:

(2.2) reject or accept the null hypothesis (1.3) according as $-2 \log \lambda_\nu$ is at least as large as $\chi^2_{\epsilon,k-1}$ or not, where $\chi^2_{\epsilon,k-1}$ is the upper $100\,\epsilon\%$ point of the chi-square distribution with $k - 1$ df.

Let now $\hat{\beta}_{\nu i}$ be the maximum likelihood estimator of β_i, for $i = 1, \cdots, k$, and let

$$Q_\nu = I(F) \cdot [\sum_{i=1}^k C^2_{\nu i}(\hat{\beta}_{\nu i} - \hat{\beta}_{\nu 0})^2], \quad \text{where} \quad \hat{\beta}_{\nu 0} = \sum_{i=1}^k \gamma_{\nu i}\hat{\beta}_{\nu i}. \tag{2.3}$$

Then, using again the results of Wald (1943, page 478), it follows that

$$Q_\nu + 2 \log \lambda_\nu \rightarrow_p 0 \quad \text{as} \quad \nu \rightarrow \infty. \tag{2.4}$$

Since $\hat{\beta}_{\nu i}$ consistently estimates β_i and $C_{\nu i}^2 \rightarrow \infty$ as $\nu \rightarrow \infty$, for all $i = 1, \cdots, k$, Q_ν can be made arbitrarily large (as $\nu \rightarrow \infty$) when $\beta_1, \cdots, \beta_k$ are not all equal. Hence the test in (2.2) is consistent. Thus, for the study of its asymptotic properties, we shall consider the following sequence of alternative hypotheses for which the power of the test is bounded away from 1:

$$H_\nu: \beta_i = \beta + C_\nu^{-1}\theta_i, \qquad i = 1, \cdots, k; \qquad \sum_{i=1}^k \gamma_i \theta_i = 0, \tag{2.5}$$

where $\theta_1, \cdots, \theta_k$ are all real and β is a nuisance parameter. Then, from the results of Wald (1943, page 480), it follows that under $\{H_\nu\}$, $-2 \log \lambda_\nu$ (or Q_ν) has asymptotically a non-central chi-square distribution with $k - 1$ df and the non-centrality parameter

$$\Delta_Q = I(F) \cdot [\sum_{i=1}^k \gamma_i \theta_i^2], \tag{2.6}$$

where $\gamma_1, \cdots, \gamma_k$ are defined by (1.6).

Now, for each positive a and each point $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_k) \varepsilon R^k$, we define the surface $S_a(\boldsymbol{\theta})$ by the equation

$$S_a(\boldsymbol{\theta}) = \{\boldsymbol{\theta}: \sum_{i=1}^k \gamma_i \theta_i^2 = a\}. \tag{2.7}$$

Consider now a transformation $\boldsymbol{\theta}^* = \mathbf{D}\boldsymbol{\theta}$, where the first row of $\mathbf{D}$ is $(\gamma_1, \cdots, \gamma_k)$ and

$$\mathbf{D}\mathbf{D}' = \mathbf{\Gamma} = \text{diag}\ (\gamma_1, \cdots, \gamma_k). \tag{2.8}$$

This transformation transforms the surface $S_a(\boldsymbol{\theta})$ into the sphere $S_a'(\boldsymbol{\theta}^*)$ given by

$$\sum_{i=2}^k (\theta_i^*)^2 = a, \quad \text{where by definition} \quad \theta_1^* = 0. \tag{2.9}$$

For any point $\boldsymbol{\theta}_0$ and any positive δ, consider the set $\omega(\boldsymbol{\theta}_0, \delta)$ consisting of all points $\boldsymbol{\theta}$ which lie on the same $S_a(\boldsymbol{\theta})$ as $\boldsymbol{\theta}_0$ and for which $|\boldsymbol{\theta} - \boldsymbol{\theta}_0| < \delta$. Let then

$$\eta(\boldsymbol{\theta}) = \lim_{\delta \to 0} \{\mathcal{A}[\omega'(\boldsymbol{\theta}, \delta)]/\mathcal{A}[\omega(\boldsymbol{\theta}, \delta)]\}, \tag{2.10}$$

where $\omega'(\boldsymbol{\theta}, \delta)$ is the image of $\omega(\boldsymbol{\theta}, \delta)$ by the transformation $\boldsymbol{\theta}^* = \mathbf{D}\boldsymbol{\theta}$, and $\mathcal{A}(\omega)$ denotes the area of the set ω. Then, from Theorem 8 of Wald (1943, page 478), we obtain the following theorem.

THEOREM 2.1. *For testing the null hypothesis* (1.3), *the likelihood ratio test in* (2.2) (i) *has asymptotically best average power with respect to the surfaces* $S_a(\boldsymbol{\theta})$ *and weight functions* $\eta(\boldsymbol{\theta})$, (ii) *has asymptotically best constant power on the surfaces* $S_a(\boldsymbol{\theta})$, *and* (iii) *is an asymptotically most stringent test.*

By virtue of Theorem 2.1, we shall regard the likelihood ratio test in (2.2) as an asymptotically optimum parametric test. In practice, the most commonly used test is based on the variance-ratio criterion involving the least squares estimators, viz.,

$$Z_\nu = [\sum_{i=1}^k C_{\nu i}^2(\bar{\beta}_{\nu i} - \bar{\beta}_{\nu 0})^2/(k - 1)s_e^2], \tag{2.11}$$

where $\bar{\beta}_{\nu i} = \sum_{j=1}^{n_{\nu i}} Y_{\nu ij}(c_{\nu ij} - \bar{c}_{\nu i})/C_{\nu i}^2$, $i = 1, \cdots, k$, $\bar{\beta}_{\nu 0} = \sum_{i=1}^{k} \gamma_{\nu i}\bar{\beta}_{\nu i}$, and s_e^2 is the (pooled) within sample mean square due to error. It is easy to verify that for any $F(x)$ having a finite variance $\sigma^2(F)$, $(k - 1)Z_\nu$ has asymptotically (under (1.3)) a chi-square distribution with $k - 1$ df. Also, under (2.5), it has asymptotically a non-central chi-square distribution with $k - 1$ df. and the non-centrality parameter

$$\Delta_Z = [\textstyle\sum_{i=1}^{k} \gamma_i \theta_i^2]/\sigma^2(F). \tag{2.12}$$

Now, under (2.5), $-2 \log \lambda_\nu$ and $(k - 1)Z_\nu$ have asymptotically non-central chi-square distributions with $k - 1$ df and the non-centrality parameters Δ_Q and Δ_Z, respectively. Hence, according to the conventional measure of the asymptotic relative efficiency (A.R.E.) (cf. Hájek and Šidák (1967, p. 270)], the A.R.E. of the Z_ν-test with respect to the λ_ν-test is

$$e_{Z,\lambda} = \Delta_Z/\Delta_Q = [\sigma^2(F)I(F)]^{-1} \leqq 1, \tag{2.13}$$

by the classical Rao-Cramér inequality. In Sections 3 and 4, when the proposed rank order tests will be compared with the Z_ν-test or the λ_ν-test, the same definition of the A.R.E. will be employed.

3. Asymptotically most powerful rank order tests. Looking at (2.3) and (2.11), we observe that the λ_ν and Z_ν-tests are respectively based on the discrepancies of the maximum likelihood and the least squares estimators of $\beta_1, \cdots, \beta_k$. In the non-parametric case, estimates of the regression coefficients are considered by Mood and Brown (1951), Theil (1950), Adichie (1967) and Sen (1968), among others. A suitable quadratic form in these estimates can be used to construct test statistics as in (2.3) or (2.11). Unfortunately, this requires the estimation of all the individual $\beta_1, \cdots, \beta_k$ as well as $\int_0^1 \psi(u)\phi(u)\,du$ (all of which usually require trial and error solutions, cf. [1]). In the alternative procedure considered below, we only require a pooled sample estimate of the regression coefficient. For this, define

$$T_\nu^* = \textstyle\sum_{i=1}^{k} C_{\nu i} T_{\nu,i}/C_\nu = [\sum_{i=1}^{k} \sum_{j=1}^{n_{\nu i}} (c_{\nu ij} - \bar{c}_{\nu i}) E_{\nu, R_{\nu ij}}^{(i)}]/[AC_\nu]. \tag{3.1}$$

For later convenience, we express T_ν^* as $T_\nu^*(\mathbf{Y}_\nu)$, and write $\mathbf{c}_\nu = (\mathbf{c}_{\nu 1}, \cdots, \mathbf{c}_{\nu k})$. It follows from (1.15) that

$$T_\nu^*(\mathbf{Y}_\nu - b\cdot\mathbf{c}_\nu) \quad \text{is} \downarrow \text{in} \quad b(-\infty < b < \infty). \tag{3.2}$$

Also, it follows from (1.9), (1.10) and (1.13) that

$$E(T_{\nu,i}(\mathbf{Y}_{\nu i}) \mid \beta_i = 0) = 0, \qquad \text{for all } i = 1, \cdots, k. \tag{3.3}$$

Consequently, under (1.3),

$$E(T_\nu^*(\mathbf{Y}_\nu - \beta\cdot\mathbf{c}_\nu) \mid \beta_1 = \cdots = \beta_k = \beta) = 0. \tag{3.4}$$

[Note that the cdf of $T_\nu^*(\mathbf{Y}_\nu - \beta\mathbf{c}_\nu)$ may or may not be symmetric about 0. A sufficient condition for this symmetry is that $\phi(u) + \phi(1 - u) = 2\phi(\frac{1}{2})$ for all $0 < u < 1$. But, for our purpose, we do not want to impose symmetry on $\phi(u)$ or $\psi(u)$.] We shall now estimate β [under (1.3)] by equating $T_\nu^*(\mathbf{Y}_\nu - b\cdot\mathbf{c}_\nu)$ to

0. For this, let

$$(3.5)\qquad \beta^*_{\nu,(1)} = \sup\{b: T_\nu^*(\mathbf{Y}_\nu - b\cdot\mathbf{c}_\nu) > 0\},$$
$$\beta^*_{\nu,(2)} = \inf\{b: T_\nu^*(\mathbf{Y}_\nu - b\cdot\mathbf{c}_\nu) < 0\}.$$

Then, the proposed pooled sample estimator of β is

$$(3.6)\qquad \beta_\nu^* = (\beta^*_{\nu,(1)} + \beta^*_{\nu,(2)})/2.$$

[For the single sample case, such an estimate has been considered in detail by Adichie (1967) under the additional assumption that F and G, defined by (1.1) and (1.19), are both symmetric (which is not needed for our purpose)]. Let us now define

$$(3.7)\qquad \hat{T}_{\nu,i} = T_{\nu,i}(\mathbf{Y}_{\nu i} - \beta_\nu^*\cdot\mathbf{c}_{\nu i}), \qquad i = 1, \cdots, k \quad \text{and} \quad L_\nu = \textstyle\sum_{i=1}^k \hat{T}^2_{\nu,i}.$$

Our proposed test is based on the statistic L_ν and the following theorem, whose proof follows as a special case of Theorem 3.2 (to follow).

THEOREM 3.1. *Under* (1.3) *and the conditions of Section* 1, L_ν *has asymptotically a chi-square distribution with* $k - 1$ df.

By virtue of Theorem 3.1, we have the following asymptotically size ϵ test:

(3.8) reject or accept H_0 in (1.3) according as L_ν is at least as large as $\chi^2_{\epsilon,k-1}$ or not, where $\chi^2_{\epsilon,k-1}$ is defined by (2.2).

As in Section 2, we shall consider here the sequence of alternatives in (2.5) and study the asymptotic power of the test in (3.8). For this, we have the following.

THEOREM 3.2. *Under the conditions of Section 1 and the sequence of alternatives in* (2.5), L_ν *has asymptotically a non-central chi-square distribution with* $k - 1$ df *and the non-centrality parameter*

$$(3.9)\qquad \Delta_L = [\rho(\psi, \phi)]^2 I(F)[\textstyle\sum_{i=1}^k \gamma_i\theta_i^2],$$

where $\rho(\psi, \phi)$ *is defined by* (1.18).

The proof of the theorem rests on the following lemmas.

LEMMA 3.1. *For the sequence of alternatives in* (2.5),

$$(3.10)\qquad |C_\nu(\beta_\nu^* - \beta)| = O_p(1), \quad as \quad \nu \to \infty.$$

PROOF. It follows from (3.5) and (3.6) that for any real x, $[\beta_\nu^* < x] \Rightarrow [\beta^*_{\nu,(1)} < x]$. Also, by definition in (3.5), $[\beta^*_{\nu,(1)} < x] \Rightarrow [T_\nu^*(\mathbf{Y}_\nu - x\mathbf{c}_\nu) \leqq 0]$. Thus, for any positive a,

$$(3.11)\qquad P_{H_\nu}[C_\nu(\beta_\nu^* - \beta) < -a] \leqq P_{H_\nu}[T_\nu^*(\mathbf{Y}_\nu - [\beta - a/C_\nu]\mathbf{c}_\nu) \leqq 0].$$

Now, under H_ν in (2.5)

$$(3.12)\qquad P[Y_{\nu ij} - (\beta - a/C_\nu)c_{\nu ij} \leqq x] = F(x - \alpha_i - [(a + \theta_i)/C_\nu]c_{\nu ij}),$$

for $j = 1, \cdots, n_{\nu i}$, $i = 1, \cdots, k$. Thus, by virtue of (1.6) and (1.7), (3.12) conforms to the basic model considered by Hájek (1962). Hence, on using the same technique as in his Theorem 6.1, it follows that under $\{H_\nu\}$ in (2.5), $T_{\nu i}(\mathbf{Y}_{\nu i} - [\beta - a/C_\nu]\mathbf{c}_{\nu i})$ converges in law (as $\nu \to \infty$) to a normal distribution with mean

$\gamma_i^{\frac{1}{2}}(\theta_i + a)[\int_0^1 \psi(u)\phi(u)\,du]/A$ and unit variance, for all $i = 1, \cdots, k$. Since the different $T_{\nu,i}$'s are stochastically independent, it follows from the above result that under (2.5), $T_\nu^*(\mathbf{Y}_\nu - [\beta - a/C_\nu]\mathbf{c}_\nu)$ converges in law (as $\nu \to \infty$) to a normal distribution with mean $a[\int_0^1 \psi(u)\phi(u)\,du]/A$ and unit variance. Thus, the right hand side of (3.11) converges (as $\nu \to \infty$) to $\Phi(-a[\int_0^1 \psi(u)\phi(u)\,du]/A) = \Phi(-a\rho(\psi, \phi)[I(F)]^{\frac{1}{2}})$, where $\Phi(x)$ is the standard normal cdf at x $(-\infty < x < \infty)$. Since in Section 1, both $\rho(\psi, \phi)$ and $I(F)$ are assumed to be positive, by choosing a adequately large, say greater than a_δ, $\Phi(-a_\delta\rho(\psi, \phi)[I(F)]^{\frac{1}{2}})$ can be made smaller than $\delta/2$, where δ is an arbitrarily small positive number. It can be shown similarly that for large ν, $P_{H_\nu}[C_\nu(\beta_\nu^* - \beta) \geqq a_\delta]$ can be made smaller than $\delta/2$. Consequently,

$$\lim_{\nu\to\infty} P_{H_\nu}[|c_\nu(\beta_\nu^* - \beta)| \leqq a_\delta] \geqq 1 - \delta. \quad \square \tag{3.13}$$

LEMMA 3.2. *Under the conditions of Section* 1 *and for any real* (a, b)

$$\begin{aligned}&[T_{\nu,i}(\mathbf{Y}_{\nu i} - [\beta_i - a/C_{\nu i}]\mathbf{c}_{\nu i}) - T_{\nu,i}(\mathbf{Y}_{\nu i} - [\beta_i - b/C_{\nu i}]\mathbf{c}_{\nu i})] \\ &\qquad = \rho(\psi, \phi)(a - b)[I(F)]^{\frac{1}{2}} + o_p(1) \quad \textit{as} \quad \nu \to \infty.\end{aligned} \tag{3.14}$$

PROOF. For notational simplicity we assume (without any loss of generality) that in (1.1) and (1.4), $\alpha_i = \beta_i = \bar{c}_{\nu i} = 0$. Then, it follows from (1.13) and Theorem 6.1 of Hájek (1962) that for any real and finite t

$$\mathcal{L}[T_{\nu,i}(\mathbf{Y}_{\nu i} + C_{\nu i}^{-1}\cdot t\cdot \mathbf{c}_{\nu i})] \to \mathfrak{N}(t\rho(\psi, \phi)[I(F)]^{\frac{1}{2}}, 1), \quad \text{as} \quad \nu \to \infty. \tag{3.15}$$

Consequently, it suffices to show that the joint distribution of $T_{\nu,i}(\mathbf{Y}_{\nu i} + C_{\nu i}^{-1}\cdot a\cdot \mathbf{c}_{\nu i})$ and $T_{\nu,i}(\mathbf{Y}_{\nu i} + C_{\nu i}^{-1}\cdot b\cdot \mathbf{c}_{\nu i})$ asymptotically (as $\nu \to \infty$) degenerates on a straight line in the two dimensional plane. Let us now define

$$V_\nu^{(t)} = [AC_{\nu i}]^{-1}\textstyle\sum_{j=1}^{n_{\nu i}} c_{\nu ij}\phi(F[Y_{\nu ij} + tc_{\nu ij}/C_{\nu i}]). \tag{3.16}$$

Then, from the results of Hájek (1962) (viz., his Section 5 and the basic contiguity arguments), it follows that

$$[T_{\nu,i}(\mathbf{Y}_{\nu i} + C_{\nu i}^{-1}\cdot t\cdot \mathbf{c}_{\nu i}) - V_\nu^{(t)}] \to_p 0 \quad \text{as} \quad \nu \to \infty. \tag{3.17}$$

Thus, it is enough to show that the joint cdf of $(V_\nu^{(a)}, V_\nu^{(b)})$ asymptotically (as $\nu \to \infty$) contracts on a straight line, and a sufficient condition for this is that $\operatorname{Var}[V_\nu^{(a)} - V_\nu^{(b)}]$ tends to 0 as $\nu \to \infty$. Now, by (3.16), $V_\nu^{(t)}$ is a linear function of independent random variables. Hence,

$$\begin{aligned}
&\operatorname{Var}[V_\nu^{(a)} - V_\nu^{(b)}] \\
&\quad = [AC_{\nu i}]^{-2}\textstyle\sum_{j=1}^{n_{\nu i}} c_{\nu ij}^2\cdot\operatorname{Var}[\phi(F[Y_{\nu ij} + ac_{\nu ij}/C_{\nu i}]) \\
&\qquad - \phi(F[Y_{\nu ij} + bc_{\nu ij}/C_{\nu i})] \\
&\quad \leqq A^{-2}\{\max_{1\leqq j\leqq n_{\nu i}}[E\{\phi(F[Y_{\nu ij} + ac_{\nu ij}/C_{\nu i}]) \\
&\qquad - \phi(F[Y_{\nu ij} + bc_{\nu ij}/C_{\nu i}])\}^2]\} \\
&\quad = A^{-2}\{\max_{1\leqq j\leqq n_{\nu i}}[(c_{\nu ij}^2/C_{\nu i}^2)(a - b)^2 \\
&\qquad \cdot E\{\phi_x'(F[Y_{\nu ij} + c\cdot c_{\nu ij}/C_{\nu i}])\}^2]\},
\end{aligned} \tag{3.18}$$

where $\phi_x' = (d/dx)\phi(F[x])$ and c lies between a and b. Hence, by (1.7) and (1.20), the right hand side of (3.18) converges to 0 as $\nu \to \infty$. This completes the proof.

REMARK. Let $I(a_0) = \{a: |a| \leqq a_0, a_0 > 0\}$. Then, (3.14) actually holds simultaneously for all $(a, b) \varepsilon I(a_0)$. To show this, let $a_r = b_r = -a_0 + 2a_0 r/k_\nu$, $r = 0, 1, \cdots, k_\nu$, where $k_\nu \to \infty$ but $k_\nu[\max_{1 \leqq j \leqq n_{\nu i}} c_{\nu ij}/C_{\nu i}] \to 0$ as $\nu \to \infty$. Then, as in Section 5 of [4] and our Lemma 3.2, it can be shown that (3.14) holds simultaneously for all (a_r, b_s), $r, s = 0, 1, \cdots, k_\nu$. Since, for any $a \varepsilon (a_r, a_{r+1})$, $a - a_r = o(1)$, and for any $b \varepsilon (b_s, b_{s+1})$, $b - b_s = o(1)$, the rest of the proof can be completed by using the monotonicity in (1.15) and some standard computations.

Now, in (3.5) and (3.6), if we work with $T_{\nu,i}$ instead of T_ν^*, the corresponding estimator will be denoted by $\beta_{\nu,i}^*$, for $i = 1, \cdots, k$. These individual sample estimators are already studied in detail by Adichie (1967) under the additional assumptions that (i) the distributions F and G defined by (1.1) and (1.19) are both symmetric and (ii) $C_{\nu i}^2/n_{\nu i}$ converges to a positive (finite) limit $c_i^2 > 0$, as $\nu \to \infty$. We note that the first assumption is not at all needed here, while the second may be safely replaced by (1.5) and (1.6).

LEMMA 3.3. *Under* (2.5) *and the conditions of Section* 1,

$$(3.19) \qquad |C_\nu(\beta_\nu^* - \textstyle\sum_{j=1}^k \gamma_{\nu i}\beta_{\nu,i}^*)| = o_p(1) \quad as \quad \nu \to \infty.$$

PROOF. Precisely on the same line as in Lemma 3.1, it follows that

$$(3.20) \qquad |C_{\nu i}(\beta_{\nu,i}^* - \beta_i)| = O_p(1) \quad \text{as} \quad \nu \to \infty, \qquad \text{for all } i = 1, \cdots, k.$$

Now, using (3.5), (3.6) and some simple arguments, it follows that

$$(3.21) \qquad |T_\nu^*(\mathbf{Y}_\nu - \beta_\nu^* \cdot \mathbf{c}_\nu)| = o_p(1),$$

and it can be shown similarly that

$$(3.22) \qquad |T_{\nu,i}(\mathbf{Y}_{\nu i} - \beta_{\nu,i}^* \mathbf{c}_{\nu i})| = o_p(1), \qquad \text{for all } i = 1, \cdots, k.$$

Also, by (3.1) and (3.22)

$$\begin{aligned} & T_\nu^*(\mathbf{Y}_\nu - \beta_\nu^* \mathbf{c}_\nu) \\ (3.23) \qquad &= \textstyle\sum_{i=1}^k (C_{\nu i}/C_\nu) T_{\nu,i}(\mathbf{Y}_{\nu i} - \beta_\nu^* \mathbf{c}_{\nu i}) \\ &= \textstyle\sum_{i=1}^k \gamma_{\nu i}^{\frac{1}{2}}[T_{\nu,i}(\mathbf{Y}_{\nu i} - \beta_\nu^* \mathbf{c}_{\nu i}) - T_{\nu,i}(\mathbf{Y}_{\nu i} - \beta_{\nu,i}^* \mathbf{c}_{\nu i})] + o_p(1). \end{aligned}$$

Now, by (3.21), the left hand side of (3.23) converges in probability to 0 as $\nu \to \infty$, while by (3.20), Lemma 3.1 and the remark following Lemma 3.2, the right hand side is equivalent (in probability) to

$$\begin{aligned} (3.24) \quad & [\textstyle\sum_{i=1}^k C_{\nu,i}^2\{(\beta_{\nu,i}^* - \beta_\nu^*) + (\beta_i - \beta)\}][\rho(\psi, \phi)\{I(F)\}^{\frac{1}{2}}]/C_\nu \\ & \qquad \sim_p C_\nu[\textstyle\sum_{i=1}^k \gamma_{\nu i}(\beta_{\nu,i}^* - \beta_\nu^*)][\rho(\psi, \phi)\{I(F)\}^{\frac{1}{2}}], \end{aligned}$$

as under (2.5) and (1.6), $C_\nu[\sum_{i=1}^k \gamma_{\nu i}(\beta_i - \beta)] \to 0$ as $\nu \to \infty$. □

We have the following lemma whose proof follows along the line of approach of Section 5 of Adichie (1967), though we note that here F and G need not be symmetric and his condition that $C_{\nu i}^2/n_{\nu i} \to c_i^2(>0)$ as $\nu \to \infty$, may be replaced by $\lim_{\nu\to\infty} C_{\nu i}^2 = \infty$.

LEMMA 3.4. *Under the conditions of Section* 1, *the independent random variables* $C_{\nu i}(\beta_{\nu,i}^* - \beta_i)$, $i = 1, \cdots, k$ *are asymptotically normally distributed with means zero and (common) variance* $([\rho(\psi, \phi)]^2 \cdot I(F))^{-1}$.

A direct consequence of the preceding two lemmas is the following lemma.

LEMMA 3.5. *Under* (2.5) *and the conditions of Section 1,*

$$L_\nu^* = [\rho^2(\psi, \phi)\, I(F)][\textstyle\sum_{i=1}^k C_{\nu i}^2(\beta_{\nu,i} - \beta_\nu^*)^2] \tag{3.25}$$

has asymptotically a non-central chi-square distribution with $k - 1$ df *and the non-centrality parameter* Δ_L, *defined by* (3.9).

Returning now to the proof of Theorem 3.2, we observe that by virtue of (3.6), (3.7), Lemmas 3.2 and 3.3, (3.21) and (3.22), we have under $\{H_\nu\}$ in (2.5)

$$L_\nu \sim_p L_\nu^*. \tag{3.26}$$

Hence, the proof of the theorem follows readily from Lemma 3.5 and (3.26).

From the results of Section 2 and Theorem 3.2, it follows that (i) the A.R.E. of the L_ν-test with respect to the likelihood ratio test is equal to

$$e_{L,Q} = \Delta_L/\Delta_Q = [\rho(\psi, \phi)]^2, \tag{3.27}$$

and the A.R.E. of the L_ν-test with respect to the Z_ν-test is equal to

$$e_{L,Z} = \Delta_L/\Delta_Z = \sigma^2(F)[\textstyle\int_0^1 \psi(n)\phi(u)\, du]^2/A^2. \tag{3.28}$$

Now, (3.27) agrees with the efficiency studied in detail by Hájek (1962). Hence, referring to his Section 6, we omit the details here. We only note that if the assumed score $\phi(u)$ agrees with the score $\psi(u)$, (3.27) will be equal to 1. Concerning (3.28), we again note that this coincides with the A.R.E. of the Chernoff-Savage (1958) two-sample location test with respect to Student's t-test, if we rewrite $[\int_0^1 \psi(u)\phi(u)\, du]$ in the Chernoff-Savage form $[\int_{-\infty}^{\infty} \{(d/dx) \cdot \phi(F(x))\}\, dF(x)]$. The details of the A.R.E. therefore follow from the well known results of Chernoff and Savage (1958). In particular, for the test based on the normal scores, it follows from their results that (3.28) is bounded below by 1, where the lower bound is attained only when $F(x)$ in (1.1) is itself normal. This explains the supremacy of the normal scores test over the classical variance ratio test.

4. Test for a more general model. Instead of (1.1), we consider the following model:

$$P[Y_{\nu ij} \leqq x] = F_i(x - \alpha_i - \beta_i c_{\nu ij}), \qquad j = 1, \cdots, n_{\nu i}, i = 1, \cdots, k, \tag{4.1}$$

where $F_i \varepsilon \mathfrak{F}$, for all $i = 1, \cdots, k$ and $c_{\nu ij}$'s satisfy the conditions of Section 1. Here we desire to test the null hypothesis (1.3), without assuming that $F_1, \cdots, F_k$ have the same form.

We define the statistics $T_{\nu,i}$ as in (1.13) and also the individual sample estimates $\beta_{\nu,i}^*$ as in Section 3. Now, under the hypothesis that $\beta_i = 0$, it is possible to locate two values, say, $T_{\nu,i}^{(1)}$ and $T_{\nu,i}^{(2)}$, such that

$$P[T_{\nu,i}^{(1)} \leqq T_{\nu,i} \leqq T_{\nu,i}^{(2)} \mid \beta_i = 0] = 1 - \alpha, \qquad \text{for all } F_i \varepsilon \mathfrak{F} \tag{4.2}$$

where $\alpha (0 < \alpha < 1)$ is preassigned. Define then

$$\beta_{\nu,i,L}^{*} = \inf \{b : T_{\nu,i}(\mathbf{Y}_{\nu i} - b\mathbf{c}_{\nu i}) \leqq T_{\nu,i}^{(2)}\}, \tag{4.3}$$

$$\beta_{\nu,i,U}^{*} = \sup \{b : T_{\nu,i}(\mathbf{Y}_{\nu i} - b\mathbf{c}_{\nu i}) \geqq T_{\nu,i}^{(1)}\}. \tag{4.4}$$

Then, from (1.15), (4.1), (4.2), (4.3) and (4.4), we obtain that

$$P[\beta_{\nu,i,L}^{*} \leqq \beta_i \leqq \beta_{\nu,i,U}^{*} \mid \beta_i] = 1 - \alpha. \tag{4.5}$$

We shall use these confidence intervals to estimate certain parameters which we define below. In (1.16), if we replace F by F_i, the corresponding score will be denoted by $\psi_i(u)$, and in (1.18), replacing $\psi(u)$ by $\psi_i(u)$, we define $\rho(\psi_i, \phi)$, $i = 1, \cdots, k$. We need to estimate the parameters

$$B_i = B(\psi_i, \phi) = [\textstyle\int_0^1 \psi_i(u)\phi(u)\, du], \qquad i = 1, \cdots, k. \tag{4.6}$$

For this, we first note that by virtue of the asymptotic normality of $T_{\nu,i}$ under $\beta_i = 0$ [cf. Hájek (1962)], we have on using symmetric tails in (4.2),

$$T_{\nu,i}^{(j)} \to (-1)^j \tau_{\alpha/2}, \qquad \text{for } j = 1, 2 \text{ and } i = 1, \cdots, k, \text{ (as } \nu \to \infty) \tag{4.7}$$

where τ_α is the upper $100\alpha\,\%$ point of the standard normal distribution.

THEOREM 4.1. *For the model* (4.1) *and under the conditions of Section* 1,

$$\hat{B}_{\nu i} = A[T_{\nu,i}^{(2)} - T_{\nu,i}^{(1)}]/[C_{\nu i}(\beta_{\nu,i,U}^{*} - B_{\nu,i,L}^{*})] \to_p B_i, \quad \textit{for all } i = 1, \cdots, k. \tag{4.8}$$

PROOF. By the same technique as in Lemma 3.1, it can be shown that $|C_{\nu i}(\beta_{\nu,i,U}^{*} - \beta_i)|$ and $|C_{\nu i}(\beta_{\nu,i,L}^{*} - \beta_i)|$ are both bounded in probability (as $\nu \to \infty$), for all $i = 1, \cdots, k$. Hence, the proof of the theorem follows directly from (4.2) through (4.8) in conjunction with the remark following Lemma 3.2. Therefore the details are omitted. Further, as a direct extension of Lemma 3.4 we have the following.

THEOREM 4.2. *Under* (4.1) *and the conditions of Section* 1, *the independent random variables* $B_i C_{\nu i}(\beta_{\nu,i}^{*} - \beta_i)/A$, $i = 1, \cdots, k$ *are asymptotically normally distributed with zero means and unit variances.*

Let us now define

$$\gamma_{\nu i}^{*} = C_{\nu i}^2 \hat{B}_{\nu,i}^2 / C_\nu^2, \qquad i = 1, \cdots, k; \qquad \beta_\nu^{**} = \textstyle\sum_{i=1}^k \gamma_{\nu i}^{*} \beta_{\nu,i}^{*} / \sum_{i=1}^k \gamma_{\nu i}^{*}. \tag{4.9}$$

Then, the proposed test is based on the statistic

$$S_\nu = [\textstyle\sum_{i=1}^k C_{\nu i}^2 \hat{B}_{\nu i}^2 (\beta_{\nu,i}^{*} - \beta_\nu^{**})^2]/A^2. \tag{4.10}$$

From Theorem 4.1, Theorem 4.2, (4.9), (4.10) and some routine computations, we obtain the following.

THEOREM 4.3. *Under* H_0 *in* (1.3) *and the conditions of Section* 1, S_ν *has asymptotically a chi-square distribution with* $k - 1$ df.

By virtue of this result, an asymptotically distribution-free test for H_0 in (1.3), under the model (4.1), can be constructed as in (3.8). Now, by (1.5), (1.6) and (4.8), it follows that as $\nu \to \infty$, $\gamma_{\nu i}^*$ converges in probability to γ_i^*, where

$$\gamma_i^* = \gamma_i B_i^2, \qquad i = 1, \cdots, k. \tag{4.11}$$

We define $\theta_1, \cdots, \theta_k$ as in (2.5), and let $\bar{\theta} = \sum_{i=1}^k \gamma_i^* \theta_i / \sum_{i=1}^k \gamma_i^*$. Then, from Theorem 4.2, (4.8) through (4.11), it follows by some standard arguments that under $\{H_\nu\}$, in (2.5) and the conditions of Section 1, S_ν has asymptotically a non-central chi-square distribution with $k - 1$ df and the non-centrality parameter

$$\Delta_S = [\textstyle\sum_{i=1}^k \gamma_i^* (\theta_i - \bar{\theta})^2]/A^2. \tag{4.12}$$

If the cdf's $F_1, \cdots, F_k$ in (4.1) are all known, we may construct the likelihood ratio test as in Section 2. In this case, let the likelihood ratio criterion be denoted by λ_ν^*. Then, the test may be constructed as in (2.2), and also, under (2.5), it can be shown that $-2 \log \lambda_\nu^*$ has asymptotically a non-central chi-square distribution with $k - 1$ df and the non-centrality parameter

$$\Delta_Q^* = [\textstyle\sum_{i=1}^k \gamma_i (\theta_i - \bar{\theta}^*)^2 I(F_i)] \tag{4.13}$$

where $\bar{\theta}^* = \sum_{i=1}^k I(F_i)\gamma_i\theta_i / \sum_{i=1}^k I(F_i)\gamma_i$. Therefore, upon noting that

$$\textstyle\sum_{i=1}^k \gamma_i^* (\theta_i - \bar{\theta})^2 \leqq \sum_{i=1}^k \gamma_i^* (\theta_i - \bar{\theta}^*)^2; \tag{4.14}$$

$$\gamma_i^* = \gamma_i B_i^2 = \gamma_i I(F_i) A^2 \rho^2(\psi_i, \phi) \quad \text{for all } i = 1, \cdots, k, \tag{4.15}$$

we obtain from (4.12) through (4.15) that

$$\Delta_S/\Delta_Q^* \leqq \frac{\sum_{i=1}^k \gamma_i I(F_i) \rho^2(\psi_i, \phi)(\theta_i - \bar{\theta}^*)^2}{\sum_{i=1}^k \gamma_i I(F_i)(\theta_i - \bar{\theta}^*)^2} \leqq 1, \tag{4.16}$$

(as $\rho^2(\psi_i, \phi) \leqq 1$ for all i), and that the equality sign holds only when $\rho(\psi_i, \phi) = 1$ for all $i = 1, \cdots, k$. A particular case when (4.16) is independent of $\theta_1, \cdots, \theta_k$ is of some importance and is considered below. Consider the heteroscedastic model:

$$F_i(x) = F(x/\sigma_i) \qquad \text{for all } i = 1, \cdots, k, \tag{4.17}$$

where $\sigma_i, \cdots, \sigma_k$ are all positive scale factors. Then,

$$I(F_i) = I(F)/\sigma_i^2 \text{ and } \rho(\psi_i, \phi) = \rho(\psi, \phi) \qquad \text{for all } i = 1, \cdots, k, \tag{4.18}$$

where $I(F)$ and $\rho(\psi, \phi)$ are defined by (1.2) and (1.18) respectively. In this case, it readily follows that (4.16) equals to $\rho^2(\psi, \phi)$ independently of $\theta_1, \cdots, \theta_k$ as well as $\sigma_1, \cdots, \sigma_k$. Thus, for the heteroscedastic model (4.17) (in conjunction with (4.1)), the A.R.E. of the S_ν-test with respect to the likelihood ratio test is equal to the A.R.E. for the basic cdf F, independently of the scale factors.

Let us now consider the least squares estimators $\bar{\beta}_{\nu i}$, $i = 1, \cdots, k$, defined just after (2.11). Also, let $s_{e,i}^2$ be the mean square due to error from the ith sample, $i = 1, \cdots, k$, and

$$\hat{\gamma}_{\nu i}^0 = \gamma_{\nu i}/s_{e,i}^2, \qquad i = 1, \cdots, k. \tag{4.19}$$

Thus, by (1.5), (1.6) and the well known convergence property of the mean squares $s_{e,i}^2$, it follows that as $\nu \to \infty$,

$$\hat{\gamma}_{\nu i}^0 \to_p \gamma_i/\sigma_i^2 = \gamma_i^0, \qquad \text{for all } i = 1, \cdots, k. \tag{4.20}$$

Finally, let $\bar{\beta}_\nu^* = \sum_{i=1}^k \gamma_{\nu i}^0 \bar{\beta}_{\nu i} / \sum_{i=1}^k \gamma_{\nu i}^0$ and $\bar{\theta}^{**} = \sum_{i=1}^k \gamma_i^0 \theta_i / \sum_{i=1}^k \gamma_i^0$. Then, it follows by standard arguments that the following statistic based on the least squares estimates,

$$Z_\nu^* = C_\nu^2[\sum_{i=1}^k \gamma_{\nu i}^0 (\bar{\beta}_{\nu i} - \bar{\beta}_\nu^*)^2], \tag{4.21}$$

has asymptotically [under (1.3) and (4.1)] a chi-square distribution with $k - 1$ df, and hence a large sample test may be constructed as in (2.2). It also is seen by some routine analysis that under (2.5), Z_ν^* follows asymptotically a non-central chi-square distribution with $k - 1$ df and the non-centrality parameter

$$\Delta_Z^* = [\sum_{i=1}^k \gamma_i^0 (\theta_i - \bar{\theta}^{**})^2]. \tag{4.22}$$

Again, upon noting that

$$\sum_{i=1}^k \gamma_i^0 (\theta_i - \bar{\theta}^{**})^2 \leqq \sum_{i=1}^k \gamma_i^0 (\theta_i - \bar{\theta})^2 \tag{4.23}$$

and that

$$\gamma_i^* = \gamma_i B_i^2 = \gamma_i^0 \sigma_i^2 B_i^2, \qquad \text{for } i = 1, \cdots, k, \tag{4.24}$$

we obtain from (4.12), (4.22) through (4.24) that the A.R.E. of the S_ν-test with respect to the Z_ν^*-test is equal to

$$\Delta_S/\Delta_Z^* \geqq \frac{\sum_{i=1}^k \gamma_i^0 e_i (\theta_i - \bar{\theta})^2}{\sum_{i=1}^k \gamma_i^0 (\theta_i - \bar{\theta})^2} \geqq \min_i e_i, \tag{4.25}$$

where

$$e_i = \sigma_i^2 B_i^2 / A^2 = \text{A.R.E. } (\beta_{\nu,i}^* / \bar{\beta}_{\nu i}), \quad \text{for} \quad i = 1, \cdots, k. \tag{4.26}$$

[When we speak of the A.R.E. of two sequences of estimates, we interpret it as the reciprocal of the ratio of their asymptotic variances. See also, Theorem 5.2, (5.6) and (5.8)]. Now, under the heteroscedastic model (4.17), $e_1 = \cdots = e_k = e$ is given by (3.28), and hence, (4.25) equals (3.28) for all $\sigma_1, \cdots, \sigma_k$. Also, for the normal scores test, $e_i \geqq 1$, for all $i = 1, \cdots, k$, and hence, (4.25) is uniformly bounded below by 1, for all $F_1, \cdots, F_k$.

5. Optimality of β_ν^* based on T_ν^*. Suppose now that (1.3) holds and we desire to estimate the common value of β. In the parametric case, the combined sample least squares estimator $\bar{\beta}_\nu$, defined just after (2.11), is a linear compound of the individual sample least squares estimators. In the nonparametric case, either we may use the pooled sample estimator in (3.6), or we may use a linear compound of the individual sample estimators, i.e.,

$$\beta_\nu^{0*} = \sum_{i=1}^k \gamma_{\nu i} \beta_{\nu,i}^*, \tag{5.1}$$

where $\beta_{\nu i}^*$ is the ith sample estimate of β_i, considered by Adichie (1967). The properties of invariance and symmetry of β_ν^* and β_ν^{0*} follow exactly on the same line as in Adichie (1967), and hence, the discussion is confined to their availability as practical estimates and their asymptotic efficiencies. We note that explicit expressions for β_ν^* and β_ν^{0*} in terms of $\mathbf{Y}_\nu$ are not available, unless we impose some restrictions on $\mathbf{c}_\nu$ [e.g., the two-sample location problem where the $c_{\nu ij}$ can assume the values 1 and 0 only]. Adichie (1967) uses a trial and error method for finding the $\beta_{\nu,i}^*$. Since T_ν^* is a simple linear function of the $T_{\nu,i}$, the same trial and error procedure is applicable for the β_ν^*. The question is therefore about the orders of the labor involved in the two trial and error procedures. We note that for the computation of β_ν^{0*}, we require $\beta_{\nu,i}^*$, $i = 1, \cdots, k$, and hence, k trial and error procedures for the k samples. On the other hand, the computation of β_ν^* requires a single trial and error procedure on T_ν^*. Hence, from this stand point, β_ν^* appears to be no less practical than β_ν^{0*}. Now, it follows from Lemma 3.3 that under the model (1.1), the two estimates are asymptotically equivalent. Hence, we shall compare them only under the model (4.1). Then, we have the following theorem.

THEOREM 5.1. *Under* (4.1) *and the conditions of Section* 1, *both the estimators* β_ν^* *and* β_ν^{0*} *are asymptotically normally distributed, and*

$$\text{A.R.E. } (\beta_\nu^*/\beta_\nu^{0*}) \geqq 1, \tag{5.2}$$

where the equality sign holds iff $B_1 = \cdots = B_k = B$, *and the* B_i *are defined by* (4.6).

PROOF. It follows from Theorem 4.2 and (5.1) that under (4.1) and (1.3)

$$\mathcal{L}[C_\nu(\beta_\nu^{0*} - \beta)/A] \rightarrow \mathfrak{N}(0, \textstyle\sum_{i=1}^k \gamma_i/B_i^2). \tag{5.3}$$

Now, using Theorem 6.1 of Hájek (1962) for the individual $T_{\nu,1}, \cdots, T_{\nu,k}$, we obtain by convolution that under (4.1) and the following sequence of alternative hypotheses:

$$H_\nu^*: \beta_i = \beta_{\nu i} = a/C_\nu, \quad i = 1, \cdots, k, \text{ (where } a \text{ is real and finite)}, \ (1 \leqq \nu < \infty), \tag{5.4}$$

T_ν^*, defined by (3.1), has asymptotically a normal distribution with mean $a(\sum_{i=1}^k \gamma_i B_i)$ and unit variance. Hence, proceeding as in Section 5 of Adichie (1967), we obtain after some routine analysis that under (1.3) and (4.1)

$$\mathcal{L}[C_\nu(\beta_\nu^* - \beta)/A] \rightarrow \mathfrak{N}(0, [\textstyle\sum_{i=1}^k \gamma_i B_i]^{-2}). \tag{5.5}$$

Hence, it follows from (5.3) and (5.5) that

$$\text{A.R.E. } (\beta_\nu^*/\beta_\nu^{0*}) = [\textstyle\sum_{i=1}^k \gamma_i/B_i^2][\sum_{i=1}^k \gamma_i B_i]^2 \geqq 1, \tag{5.6}$$

by virtue of some elementary moment inequalities among non-negative quantities. The equality sign in (5.6) holds only when $B_1 = \cdots = B_k$. Hence the theorem.

This suggests that as a pooled estimate of β, β_ν^* is asymptotically at least as good as the linear compound estimator β_ν^{0*}. Assume now that σ_i^2, the variance of the cdf F_i, is finite, for all $i = 1, \cdots, k$. Then, it is easy to show that for the combined sample least squares estimator $\bar{\beta}_\nu$[defined just after (2.11)]

$$\mathcal{L}[C_\nu(\bar{\beta}_\nu - \beta)] \rightarrow \mathfrak{N}(0, \textstyle\sum_{i=1}^k \gamma_i \sigma_i^2), \tag{5.7}$$

where the γ_i are defined by (1.5). Hence, from (5.5) and (5.7), we obtain on using the same moment inequalities that

$$(5.8)\quad \text{A.R.E. } (\beta_\nu^*/\bar{\beta}_\nu) = [\textstyle\sum_{i=1}^k \gamma_i\sigma_i^2][\sum_{i=1}^k \gamma_i B_i]^2/A^2$$
$$= [\textstyle\sum_{i=1}^k \gamma_i\sigma_i^2][\sum_{i=1}^k \gamma_i e_i^{\frac{1}{2}}/\sigma_i]^2 \geqq \min_i e_i,$$

where the e_i are defined by (4.26); also, the A.R.E. $\leqq \bar{e} = \sum_{i=1}^k \gamma_i e_i$, when $\sigma_i = \cdots = \sigma_k$. Thus, for the heteroscedastic model (4.17), we recall that $e_1 = \cdots = e_k = e$ [defined by (3.28)], and hence, the A.R.E. in (5.8) is bounded below by e, where the equality sign holds only when $\sigma_1 = \cdots = \sigma_k$. Also, for the normal scores, e_i, defined by (4.26), is bounded below by 1, and hence, (5.8) is bounded below by 1, uniformly in $F_1, \cdots, F_k$. Finally, we remark that under (1.3) and (4.1), T_ν^* in (3.1) can provide an exact confidence interval for β, whatever be $F_1, \cdots, F_k$. On the other hand, the other method fails to do so. The procedure for obtaining the confidence interval is exactly the same as in (4.2) through (4.5), where we have only to change $T_{\nu,i}$ to T_ν^*. Since, under (4.1) and the hypothesis that $\beta_1 = \cdots = \beta_k = \beta = 0$, T_ν^* is a linear compound of k distribution-free statistics, its distribution is also independent of $F_1, \cdots, F_k$. Hence, we can always find two values, say, $T_\nu^{*(i)}$, $i = 1, 2$, such that (4.2) holds with $T_{\nu,i}$ replaced by T_ν^*, The rest of the procedure is the same. [For large ν, we note again that (4.7) extends to the situation where T_ν^* is used.]

6. Appendix: monotonicity of regression rank statistics. Let $E_1 \leqq \cdots \leqq E_n$ (not all equal) be n rank scores, and $c_1 \leqq \cdots \leqq c_n$ (not all equal) be known constants. Also, let $Y_1, \cdots, Y_n$ be independent random variables with continuous cdf's $F_1, \cdots, F_n$, respectively. Further, let $R_i(b)$ be the rank of $Z_i(b) = Y_i - bc_i$ among $Z_1(b), \cdots, Z_n(b)$, for $i = 1, \cdots, n$. Finally, let

$$(6.1)\quad T_n(b) = \textstyle\sum_{i=1}^n (c_i - \bar{c})E_{R_i(b)}; \qquad \bar{c} = n^{-1}\sum_{i=1}^n c_i, \qquad -\infty < b < \infty.$$

THEOREM 6.1. *Under the conditions stated above, there exists* $n^*[1 \leqq n^* \leqq \binom{n}{2}]$ *points* $b_1 < \cdots < b_{n^*}$, *such that* (i) $T_n(b) = T_n(b_s + 0)$ *for all* $b_s < b < b_{s+1}$, $s = 0, 1, \cdots, n^*$ (*where* $b_0 = -\infty$, $b_{n^*+1} = \infty$), (ii) $T_n(b_s - 0) \geqq T_n(b_s) \geqq T_n(b_s + 0)$, $s = 1, \cdots, n^*$, *and* (iii) $T_n(b)$ *is necessarily positive* (*negative*) *for* $b < b_1 (> b_{n^*})$. *Thus,* $T_n(b)$ *is* $\downarrow$ *in* b $(-\infty < b < \infty)$.

PROOF. Let $c_i^* = c_i - c_1 (\geqq 0)$, $i = 1, \cdots, n$, and rewrite $T_n(b)$ as

$$(6.2)\qquad T_n(b) = \textstyle\sum_{i=1}^n c_i^* E_{R_i(b)} + (c_1 - \bar{c})\sum_{i=1}^n E_i,$$

where the last term does not depend on $b(-\infty < b < \infty)$. Let then

$$(6.3)\qquad Z_i^*(b) = Y_i - bc_i^*, \qquad i = 1, \cdots, n(-\infty < b < \infty).$$

Now, in (6.3), we have n straight lines in b. The (i, i')th lines are either parallel (if $c_i = c_{i'}$) or they intersect at a single point $b_{ii'}$ (if $c_i \neq c_{i'}$), $1 \leqq i < i' \leqq n$. As not all the c_i are equal, the number of distinct pairs $[(c_i, c_{i'}): c_{i'} > c_i, 1 \leqq i < i' \leqq n]$ is equal to n^*, where $1 \leqq n^* \leqq \binom{n}{2}$. We denote the ordered values of these n^* points of intersection of the n lines in (6.3) by $b_1, \cdots, b_{n^*}$. As $F_1, \cdots, F_n$ are all assumed to be continuous, ties among $Y_1, \cdots, Y_n$, and hence, among

$b_1, \cdots, b_{n^*}$, can be neglected with probability 1. Thus, $b_1 < \cdots < b_{n^*}$, with probability one. Now, for any $s(=0, \cdots, n^*)$, consider the open interval $b_s < b < b_{s+1}$. Since in this interval, no two lines in (6.3) intersect, the ranks of $Z_1^*(b), \cdots, Z_n^*(b)$ are the same as those of $Z_1^*(b_s+0), \cdots, Z_n^*(b_s+0)$, respectively. Hence,

$$(6.4)\qquad T_n(b) = T_n(b_s+0), \quad \text{for all} \quad b_s < b < b_{s+1}, \qquad s = 0, 1, \cdots, n^*.$$

At $b = b_s$, let the two intersecting lines be the (i, i')th ones, and let $R_i(b_s - 0) = q_1$, $R_{i'}(b_s - 0) = q_2$. As $c_{i'}^* > c_i^*$ (otherwise the two lines do not intersect), we must have (i) $q_1 = q_2 - 1 = q - 1$ (say), (ii) $R_i(b_s+0) = q = 1 + R_{i'}(b_s+0)$, and (iii) $R_j(b_s - 0) = R_j(b_s + 0)$ for the remaining $n - 2$ values of j. Thus,

$$(6.5)\quad T_n(b_s+0) = T_n(b_s-0) - (c_{i'}^* - c_i^*)(E_q - E_{q-1}) \leqq T_n(b_s - 0),$$

as $c_{i'}^* > c_i^*$ and $E_q \geqq E_{q-1}$. Again, at $b = b_s$, $Z_i^*(b_s) = Z_{i'}^*(b_s)$. Hence, according to the usual convention of mid-ranks for tied observations, the scores for each of these two variables are $E_q^* = E_{q-1}^* = [E_q + E_{q-1}]/2$. Thus,

$$(6.6)\quad \begin{aligned} T_n(b_s) &= T_n(b_s - 0) - (c_{i'}^* - c_i^*)(E_q - E_{q-1})/2 \\ &= [T_n(b_s - 0) + T_n(b_s + 0)]/2, \end{aligned}$$

and hence, from (6.5) and (6.6), we have

$$(6.7)\qquad T_n(b_s - 0) \geqq T_n(b_s) \geqq T_n(b_s + 0), \quad \text{for all } s = 1, \cdots, n^*.$$

Finally, we rewite $T_n(b)$ as

$$(6.8)\quad T_n(b) = n^{-1} \textstyle\sum_{1 \leqq i < i' \leqq n} (c_{i'} - c_i)(E_{R_{i'}(b)} - E_{R_i(b)}), \qquad -\infty < b < \infty.$$

Now, if $c_{i'} > c_i$, then for $b < b_{ii'}$, $Z_i^*(b) < Z_{i'}^*(b)$ [which implies $R_i(b) < R_{i'}(b)$ and hence $E_{R_i(b)} \leqq E_{R_{i'}}(b)$]. Also, $b_1 = \min_{i,i'} b_{ii'}$, and hence, for $b < b_1$, all the $E_{R_{i'}(b)} - E_{R_i(b)}$ are non-negative. Let now $Y_M = \max_i [Y_i : c_i = c_n]$ and $Y_m = \min_i [Y_i : c_i = c_1]$. Then, for $b < b_1$, $Z_M(b)$ and $Z_m(b)$ have respectively the rank n and 1. Also, $E_n > E_1$ and $c_n > c_1$ (by the hypothesis that the scores or the constants are not all equal), and hence, from (6.8), we obtain that

$$(6.9)\qquad T_n(b) \geqq n^{-1}(c_n - c_1)(E_n - E_1) > 0, \qquad \text{for all } b < b_1.$$

It can be shown in the same manner that $T_n(b) < 0$, for all $b > b_{n^*}$. Hence the theorem.

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