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THE RELATION OF THE EQUIVALENCE CONDITIONS FOR THE BROWNIAN MOTION TO THE EQUIVALENCE CONDITIONS FOR CERTAIN STATIONARY PROCESSES[1]

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Random processes are called equivalent if the measures they induce on the path space are mutually absolutely continuous. Since Gaussian processes are determined by their mean and covariance functions alone, it should be possible to formulate equivalence conditions involving these functions alone. This has been done in several cases. In particular, Shepp [6] has given the following necessary and sufficient conditions for a Gaussian process with covariance R and mean m to be equivalent to the Brownian motion on the interval $[0, T]$:

(i) $\min(s, t) - R(s, t)$ must be representable as $\int_0^s \int_0^t H(u, v)\, du\, dv$, where H is in L^2 and when considered as the kernel of a Hilbert–Schmidt operator on $L^2([0, T])$, its spectrum does not include the value one;

(ii) $m(t)$ must be representable as $\int_0^t f(u)\, du$, where f is in L^2.

Feldman [1] and Rozanov [4] consider stationary Gaussian processes X and Y with covariance functions $R(u)$ and $S(u)$, respectively, and show that if Y has a spectral density (this condition can be improved, see Feldman [2]), and if the process X has a spectral density f such that

$$0 < \liminf_{\lambda\to\infty} \lambda^2 f(\lambda) \leqq \limsup_{\lambda\to\infty} \lambda^2 f(\lambda) < \infty,$$

then X is equivalent to Y on $[0, T]$ if and only if $R(u) - S(u)$ has a derivative which is absolutely continuous on $(-T, T)$ with $\int_0^T \int_0^T (R - S)''(s - t)^2\, ds\, dt$ finite.

There is an obvious resemblance between these sets of conditions. Both require that the difference of the covariance functions should be signed distribution functions with densities in L^2, but whereas Shepp's result has the difference written as a specific definite integral, Feldman's result does not. On the other hand, Feldman's conditions are more restrictive than Shepp's in that the definite integral $\int_0^s \int_0^t H(u, v)\, du\, dv$ is not necessarily differentiable as a function of s. There is also a difference between the spectral condition on H in Shepp's theorem and the condition that the spectral function of Y be absolutely continuous in the Feldman result. In this paper we show why these similarities and subtle differences occur and equally important extend the results and the theory along the way.

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1. General theory.

Notation. X and Y will always denote mean zero Gaussian processes with covariance functions R and S, respectively.

$Y + m$ will denote the process with covariance S and mean m.

μ_x will denote the measure on R^A corresponding to the process X on $[0, A]$.

$X \sim Y$ means that μ_x and μ_y are mutually absolutely continuous.

$X(t)$ will denote the equivalence class in $L^2(d\mu_x)$ of the random variable $x_t(\omega)$.

H_x will denote the subspace of $L^2(d\mu_x)$ spanned by the functions $X(t)$, t in $[0, A]$.

If T is a linear operator on H_x, TX will denote the process Y such that the random variables $Y(t)$ have the same joint distributions as the random variables $TX(t)$.

$H_x \approx H_y$ means the map taking $X(t)$ into $Y(t)$ extends to a bounded invertible linear operator from H_x onto H_y.

H. Sato [5] gives a version of the Feldman-Hájek Dichotomy Theorem for Gaussian measures from which we begin.

DICHOTOMY THEOREM. *X is equivalent to $Y + m$ if and only if $Y = TX$, where T is a bounded invertible operator from H_x onto H_x with $I - T^*T$ Hilbert–Schmidt, and $m(t) = E(X(t)g)$, where g is in H_x. Otherwise, μ_x and μ_{y+m} are singular.*

This paper rests on the following corollary to this theorem.

PROPOSITION 1. *X is equivalent to Y if and only if $R(s, t) - S(s, t) = E(HX(s)X(t))$, where H is a Hilbert–Schmidt operator with $I - H$ invertible.*

PROOF. Assume X is equivalent to Y. Then $Y = TX$, with T as in the Dichotomy Theorem. Then $R(s, t) - S(s, t) = E(X(s)X(t)) - E(Y(s)Y(t)) = E((I - T^*T)X(s)X(t)) = E(HX(s)X(t))$. Since T is invertible, $I - H = T^*T$ is invertible.

Conversely, assume $R(s, t) - S(s, t) = E(HX(s)X(t))$ with H Hilbert–Schmidt and $I - H$ invertible. Then since R and S are symmetric, H is self-adjoint. Also $E(Y(s)Y(t)) = E((I - H)X(s)X(t))$ implies that $I - H$ automatically has no negative eigenvalues. It follows that $Y = (I - H)^{\frac{1}{2}}X$ with $I - (I - H)^{\frac{1}{2}}(I - H)^{\frac{1}{2}} = H$ being Hilbert–Schmidt. Furthermore, since $I - H$ is invertible, $(I - H)^{\frac{1}{2}}$ is invertible. $\square$

There are several ways to verify the invertibility condition on $I - H$ in Proposition 1. Assume it has been shown that $R(s, t) - S(s, t) = E(HX(s)X(t))$ with H Hilbert–Schmidt. Then $I - H$ is invertible if and only if $I - H$ or $(I - H)^{\frac{1}{2}}$ has kernel zero. But this says that the kernel of the linear operator taking $X(t)$ into $Y(t)$ is zero. If, furthermore, it is assumed that $H_x \approx H_y$, then $(I - H)^{\frac{1}{2}}$ and hence $I - H$ is automatically invertible.

COROLLARY (Shepp's Theorem). *Let X be Brownian motion. Then $X \sim Y + m$ if and only if $\min(s, t) - S(s, t) = \int_0^s \int_0^t H(u, v)\, du\, dv$ with H in L^2 and one not in the spectrum of the operator on $L^2([0, T])$ with kernel H, and $m(t) = \int_0^t f(u)\, du$, f in L^2.*

PROOF. The map taking $X(t)$ into the characteristic function of $[0, t]$ in

$L^2([0, T])$ is an isomorphism of Hilbert spaces and the equivalence conditions in Proposition 1 only depend on the Hilbert space structure of H_x. □

Shepp proved the necessity of the conditions in his theorem from general principles, but needed a separate proof of their sufficiency. The non-trivial half of Proposition 1 ensures the existence of an operator T as in the Dichotomy Theorem from the existence of the operator H in the expression for $R - S$.

Additional notation. M_x will denote the set of functions $m(t) = E(X(t)g)$ with g in H_x.

a_x will denote the set of functions $A(s, t) = E(HX(s)X(t))$, H Hilbert–Schmidt on H_x.

The equivalence conditions now take the simple form.

PROPOSITION 1A. *$X \sim Y + m$ if and only if $H_x \approx H_y$, $R - S \varepsilon a_x$ and $m \varepsilon M_x$.*

In this form we can say that equivalence conditions for homeomorphic processes take the same form. More precisely, we have

PROPOSITION 2. *$H_x \approx H_y$ implies $M_x = M_y$ and $a_x = a_y$.*

PROOF. Let $T : Y(s) \to X(s)$ be the linear homeomorphism taking H_y to H_x. Then $E(HX(s)X(t)) = E(T^*HTY(s)Y(t)) = E(H_2Y(s)Y(t))$, where $H_2 = T^*HT$ is Hilbert–Schmidt when H is.

Similarly $E(X(s)\varnothing) = E(Y(s)T^*\varnothing)$, where $T^*\varnothing$ is in H_y. □

This proposition will be used in the following way. Assume we can describe the set a_x and can show $H_x \approx H_y$. Then we can say $Z \sim Y$ if and only if $H_z \approx H_y$ and $S - T \varepsilon a_x$.

The next proposition gives a method for deciding when processes are homeomorphic.

PROPOSITION 3. *If there is an isometry of H_x taking $X(t)$ into $f_t(\cdot)$ in $L^2(d\mu)$ and an isometry of H_y taking $Y(t)$ into $f_t(\cdot)$ in $L^2(d\nu)$ with $c_1 < d\mu/d\nu < c_2$, 0 less then c_1, then $H_x \approx H_y$.*

PROOF. Obvious.

2. Applying the principles.

PROPOSITION 4. *Let Y be a stationary Gaussian process with covariance $\frac{1}{2}e^{-|u|}$. If X is a process with corresponding random variables $X(t)$ with the same joint distributions as the random variables $Y(t) - Y(0)$, then X is equivalent to Brownian motion b on $[0, T]$ for all finite T.*

PROOF. The covariance of X is $\frac{1}{2}(e^{-|s-t|} - e^{-|s|} - e^{-|t|} + 1)$, which we denote by R. We see that $\min(s, t) - R(s, t) = \int_0^s \int_0^t \frac{1}{2}e^{-|u-v|}\, du\, dv$, which is in a_b, where b stands for Brownian motion. The problem is to show that one is not in the spectrum of the operator H on $L^2[0, T]$ with kernel $\frac{1}{2}e^{-|u-v|}$. Since H is Hilbert–Schmidt it suffices to show that the null space of $I - H$ is zero.

Since the set of differentiable φ vanishing off $[0, T]$ are dense in $L^2[0, T]$, it is enough to show that $((I - H)\varphi, \varphi)$ is uniformly bounded from 0 for all such φ with $\|\varphi\| = 1$. Let $\hat{f}$ denote the inverse Fourier transform $(2\pi)^{-\frac{1}{2}} \int e^{-i\lambda x} f(x)\, dx$. Then the inverse transform of $\frac{1}{2}e^{-|u|}$ is $(2\pi)^{-\frac{1}{2}}(1 + \lambda^2)^{-1}$ and

$$((I - H)\varphi, \varphi) = 1 - ((1 + \lambda^2)^{-1}\hat{\varphi}, \hat{\varphi}) = \int \lambda^2(1 + \lambda^2)^{-1}|\hat{\varphi}|^2 \, d\lambda.$$

$$|\hat{\varphi}| \leqq (2\pi)^{-\frac{1}{2}} \int_0^T |\varphi(u)| \, du \leqq (T/2\pi)^{\frac{1}{2}}.$$

Hence $\int_{|\lambda| \leqq a} |\hat{\varphi}|^2 \, d\lambda \leqq aT/\pi$ and $\int_{|\lambda| \geqq a} |\hat{\varphi}|^2 \, d\lambda > 1 - aT/\pi$. Thus

$$\int \lambda^2(1 + \lambda^2)^{-1} |\hat{\varphi}|^2 \, d\lambda > a^2(1 + a^2)^{-1}(1 - aT/\pi).$$

For any T by choosing a so that aT is less than one, we obtain a lower bound. □

More notation. A process X derived from a process Y as in Proposition 4 will be denoted by $Y - Y(0)$.

$H_{x-x(0)}$ will denote the Hilbert space spanned by the random variables $X(t) - X(0)$, $t \, \varepsilon \, [0, T]$.

$H \oplus \langle \xi \rangle$ is the Hilbert space spanned by H and ξ.

Proposition 5. *If* $X(0) \not\varepsilon H_{x-x(0)}$ *and* $Y(0) \not\varepsilon H_{y-y(0)}$ *then*

(i) $X - X(0) \sim Y - Y(0) \rightarrow X \sim Y$ *and*

(ii) $X - X(0) \sim X - X(0) + m \rightarrow X \sim X + m + c$ *for any constant* c.

Proof. Let H_1 and H_2 be the Hilbert spaces generated by $X(t) - X(0)$ and $Y(t) - Y(0)$, respectively, $t \, \varepsilon \, [0, T]$. Let $X(0) = \xi_1 + \eta_1$, $\xi_1 \perp H_1$ with $\eta_1 \, \varepsilon \, H_1$ and $Y(0) = \xi_2 + \eta_2$, $\xi_2 \perp H_2$ with $\eta_2 \, \varepsilon \, H_2$. Let $T_0 : H_1 \rightarrow H_2$ be the extension of the map taking $X(t) - X(0)$ into $Y(t) - Y(0)$. By hypothesis we have that T_0 is a linear homeomorphism with $I - T_0^*T_0$ Hilbert–Schmidt. Let T extend T_0 with $T(\xi_1) = \xi_2 + \eta_2 - \eta_1$ so that $TX(0) = Y(0)$. That is, T becomes the extension of the identity taking $H_1 \oplus \langle \xi_1 \rangle$ to $H_2 \oplus \langle \xi_2 \rangle$.

T remains a homeomorphism since $\|\xi_1\|$ and $\|\xi_2\|$ are both greater than zero. We need only show $I - T^*T$ is Hilbert–Schmidt.

Choose an orthonormal basis $\{y_k\}$ for H_1. Then $\xi_1/\|\xi_1\|$ and $\{y_k\}$ form an orthonormal basis for H_x. Denoting $\xi_1/\|\xi_1\|$ by y_0 we must show that

$$\sum_{k,j} ((I - T^*T)y_k, y_j)^2 < \infty.$$

This becomes

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} ((I - T^*T)y_k, y_j)^2 + 2\sum_{k=1}^{\infty} ((I - T^*T)y_0, y_k)^2 + ((I - T^*T)y_0, y_0)^2.$$

The first summation is finite since $I - T_0^*T_0$ is Hilbert–Schmidt. The second summation is finite since $\sum ((I - T^*T)y_0, y_k)^2$ is less than or equal to $\|(I - T^*T)y_0\|^2$.

To prove (ii), assume $X - X(0) + m \sim X - X(0)$. Then

$$m(t) = E[(X(t) - X(0))\phi]$$

for $\phi \, \varepsilon \, H_{x-x(0)}$ with $X(0) = \xi_1 \oplus \eta_1$ as above.

Then

$$E[X(t)(\phi + k\xi_1)] = E[(X(t) - X(0) + \xi_1 + \eta_1)(\phi + k\xi_1)]$$
$$= m(t) + k\|\xi_1\|^2 + E[\eta_1\phi].$$

Since $\|\xi_1\|^2 \neq 0$ we can choose k so that $k\|\xi_1\|^2 + E(\eta_1\phi) = c$ for any constant c. Thus $X \sim X + m + c$ for any constant c. □

These propositions lead to a partial extension of the Feldman–Rozanov result. Addenda then complete the extension.

THEOREM (Extended Feldman–Rozanov). *Let X be stationary with spectral density $f(\lambda)$ satisfying the relations $c_1 \leqq (1 + \lambda^2)f(\lambda) \leqq c_2$ for c_1 and c_2 positive numbers. Then $Y + m$ is equivalent to X on a given interval $[0, A]$ if and only if:*

(i) *$R - S$ is a signed distribution function in $[0, A] \times [0, A]$ with density in L^2.*

(ii) *$m(t) = m(0) + \int_0^t g(u)\,du$, g in L^2.*

(iii) *The map T taking H_x to H_y with $X(t)$ going into $Y(t)$ has kernel zero.*

PROOF. Consider the process X_1 on $[0, A]$ with covariance $R_1(u) = \frac{1}{2}e^{-|u|}$ and spectral density $(2\pi)^{-\frac{1}{2}}(1 + \lambda^2)$. We show $X_1(0) \not\in H_{X_1 - X_1}(0)$ by breaking the process into the sum of a non-zero variable ξ and an independent process X_2. $R_1(u)$ is convex and positive in $[0, A]$ and hence can be written as $R_1(A) + (R_1(u) - R_1(A))$, where $R_1(A)$ is positive and $R_1(u) - R_1(A)$ is convex and hence a covariance. Let $E(\xi^2) = R(A)$ and let X_2 be a process with covariance $R_1(u) - R_1(A)$.

By Proposition 4, $X_1 - X_1(0)$ is equivalent to Brownian motion b on any finite interval so in particular $H_{X_1 - X_1(0)} \approx H_b$. But by the assumption on the spectral function of an arbitrary X in the hypothesis of the Theorem and Proposition 3 it follows that $X(0)$ is not in $H_{X - X(0)}$ and $H_{X - X(0)} \approx H_b$.

Now applying Proposition 2, $Y - Y(0)$ is equivalent to $X - X(0)$ if and only if $(R - S)(s, t) + (R - S)(0, 0) - (R - S)(0, s) - (R - S)(0, t)$ is in a_b and the kernel of T_0 taking $X(t) - X(0)$ into $Y(t) - Y(0)$ is zero.

Finally, applying Proposition 5, X is equivalent to Y if and only if $X - X(0)$ is equivalent to $Y - Y(0)$ and the kernel of T taking $X(t)$ into $Y(t)$ is zero. Since $\ker T = \{0\}$ implies $\ker T_0$ is automatically zero, the result takes its stated form.

The result about the mean follows since $m - m(0)$ must equal $\int_0^t g(u)\,du$. □

This theorem is an extension of the original in that Y is not assumed stationary. The description of M_x should be compared to the general result of Grenander [3] for translates of stationary processes, which says that $m(t)$ must be expressible as $\int e^{i\lambda t} g(\lambda)/(1 + \lambda^2)\,d\lambda$ for some g in $L^2[(1/1 + \lambda^2)\,d\lambda]$. The following lemma shows the Theorem is truly an extension of the Feldman–Rozanov Theorem, for if Y is assumed stationary it reduces to that form.

LEMMA. *If A is continuous with*

$$\int_a^b \int_a^b A(s - t) f'(s)\, g'(t)\, ds\, dt = \int_a^b \int_a^b K(s, t) f(s) g(t)\, ds\, ds$$

for all f and g infinitely differentiable with compact support in (a, b), where K is in L^2, then A has an absolutely continuous derivative in $(a - b, b - a)$ such that $\int_a^b \int_a^b (A''(s - t))^2\, ds\, dt$ is finite.

PROOF. Define $B(s, t) = \int_t^s A(u - t)\,du = \int_0^{s-t} A(\xi)\,d\xi$. Then $(\partial B/\partial s)(s, t) = A(s - t)$ and $(\partial B/\partial t) = -A(s - t)$. Hence on integrating by parts twice we obtain

$$-\int_a^b \int_a^b A(s-t)f''(s)g(t)\,ds\,dt = \int_a^b \int_a^b K(s,t)f(s)g(t)\,ds\,dt.$$

Thus $-\int_a^b A(s-t)f''(s)\,ds = \int_a^b K(s,t)f(s)\,ds$, almost all t. Hence by a theorem of distribution theory (see Rozanov [4] page 228) we have that for those t, $A(s-t)$ has an absolutely continuous derivative for s in (a, b) and $\partial^2/\partial s^2 A(s-t) = K(s,t)$. This says that for almost all t in (a, b), $A'(u)$ exists and is absolutely continuous in $(a-t, b-t)$. By choosing t arbitrarily near b and a we see A' is absolutely continuous in $(a-b, b-a)$ and $-A''(s-t) = K(s,t)$ for almost all t. □

To apply this we note that if

$$A(s-t) + A(0) - A(s) - A(t) = \int_0^s \int_0^s H\,du\,dv$$

then

$$\begin{aligned}\int_0^T \int_0^T A(s-t)f'(s)g'(t)\,ds\,dt &= \int_0^T \int_0^T (A(s-t) + A(0) - A(s) - A(t))f'(s)g'(t)\,ds\,dt \\ &= \int_0^T \int_0^T H(s,t)f(s)g(t)\,ds\,dt\end{aligned}$$

for f and g satisfying the conditions of the Lemma. Letting $A = R - S$ we see that (i) takes the form that $\int_0^T \int_0^T ((R-S)''(s-t))^2\,ds\,dt$ must be finite.

(iii) is obviously satisfied if the spectral function of Y is absolutely continuous. Thus the spectral restriction on H in Shepp's Theorem has become the condition that Y have a spectral density.

To weaken the hypothesis on the spectral density of X to say

$$0 < \liminf_{\lambda\to\infty} \lambda^2 f(\lambda) \leqq \limsup_{X\to\infty} f(\lambda) < \infty$$

rather than $c_1 \leqq (1+\lambda^2)f(\lambda) \leqq c_2$ we note that if $f_2 = f$ off of a compact set then process X_2 with spectral density f_2 can be shown equivalent to X. (This follows easily from [2].) By the general method of Proposition 2 we can then transfer the equivalence conditions (i), (ii), and (iii) to the process X_2.

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