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BAYESIAN MODEL OF DECISION-MAKING AS A RESULT OF LEARNING FROM EXPERIENCE[1, 2]

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1. Introduction. A statistical model of decision-making is formally described as follows: given is a set A of possible decisions or actions and a family $\mathcal{P} = \{\nu_\theta : \theta \in \Theta\}$ of probability distributions on a measurable space $(\Xi, \mathfrak{X})$. The decision-maker (statistician) observes a random variable X with values in the space $(\Xi, \mathfrak{X})$ and distributed according to some $\nu_\theta \in \mathcal{P}$, and on the basis of this observation decides for some action $a \in A$. The appropriateness of his decision then depends on the action chosen, and also on an unknown parameter θ (state of Nature), which specifies the distribution ν_θ of the random variable observed, and is measured by a numerical loss function L defined on the product space $\Theta \times A$. A rational decision-maker is then assumed to use a decision function δ, which assigns to every observed value (sample) $x \in \Xi$ an action $a \in A$ in such a manner that the resulting loss is as small as possible. It is clear, however, that no decision function can minimize the loss itself for all values of θ, since a decision function must not depend on this unknown parameter.

In the Bayesian approach, this problem is resolved by assuming that the parameter θ is also a random variable with distribution τ (prior distribution) known to the statistician, and the optimum decision function δ_τ^* (called Bayes decision function again τ) is then defined as that, for which the expected loss $E_\tau\{L(\theta, \delta(X))\}$ attains its minimum—the so-called Bayes risk $\rho(\tau) = \min_\delta E_\tau\{L(\theta, \delta(X))\}$.

To justify this Bayesian model as appropriate for studying decision-making we face a problem concerning both the adequacy of the assumption of randomness of θ and knowledge of the prior distribution as well as the question of interpretation of the minimum expected loss as optimum.

Two essentially different approaches have been taken in this respect. In the first (subjectivistic) approach the loss function is looked upon as a negative utility associated with all pairs (θ, a), $\theta \in \Theta$, $a \in A$, and satisfying Von Neumann-Morgenstern's (or other analogous) axioms. It is then shown that this is tantamount to the existence of a prior distribution, and the optimality of Bayes risk follows from the expected utility hypothesis.

The second (statistical) approach assumes, on the other hand, that the decision problem in question is a typical member of a large population of identical decision situations with parameters θ varying arbitrarily along the population.

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If τ_n is the relative frequency distribution of θ in a population of size n, it follows from the law of large numbers that the difference of an average loss resulting from the use of Bayes decision function against τ_n and the Bayes risk $\rho(\tau_n)$ can be made arbitrarily small (with probability arbitrarily close to one) if n is large enough. Hence a Bayes decision function can be considered as minimizing the actual average loss in a large population.

Objections may still be raised against either approach, concerning mainly the information required for the statistician to be actually able to use the Bayes decision function. In the first approach, the axioms of utility essentially ask the statistician to express his preferences among all probability mixtures of pairs (θ, a), which is quite a strong requirement. In the second approach, the numerical loss must be exactly specified for every pair (θ, a) and known beforehand, as well as the relative frequency distribution τ_n. Moreover, in both cases the family $\mathcal{P}$ must be known too.

In the past decade several papers have appeared, in which some of these requirements were removed, namely those concerning the statistical approach. It was shown that on facing a large population of identical decision situations the decision-maker can "learn" either the relative frequence distributions τ_n (if he knows the distributions ν_θ of the family $\mathcal{P}$—see, e.g., [6]) or even the distributions ν_θ (if he is told the past values of θ—see, e.g., [7]), and in some cases even both ([1]), and still perform asymptotically as well as if he were using the Bayes decision function. However, knowledge of the loss function and some fundamental information about the family $\mathcal{P}$ was always necessary.

In this paper we are going to consider the question of justifying the Bayesian model from a rather different angle. First, we adopt the statistical approach, thus being able to accept the Bayes risk as the desired optimum. We assume, however, that the statistician is aware only of the set A of decisions available to him and of the space $(\Xi, \mathfrak{X})$ of observations. That is, he has no knowledge of the set of parameters Θ, the family $\mathcal{P}$, or even the loss function L. Moreover, we assume that the loss may be random, with distribution depending on θ and a, but also unknown. This will take into account the possible inconsistency or uncertainty in the statistician's judgment of his losses, which was one of the original objections against both subjectivistic and statistical approaches. Further, when dealing with a large population the statistician is supposed to face individual decision situations successively, and after each decision is made he registers the value of random loss incurred.

Next, we try to describe formally what we feel to be a "natural" decision procedure employed by an unskilled rational individual under such conditions (Section 3). Roughly speaking, we assume that the decision-maker guides himself by the following three rules.

(1) For randomly selected test stages of the decision process take any decision $a \in A$ equally likely, regardless of the observed sample x, but remember the sample and the loss thus incurred.

(2) Make these tests infinitely often but with ever-decreasing frequency, putting at the same time increasing weights on the losses registered.

(3) In active stages (i.e., those that are not the test ones) take the decision for which the sum of weighted losses registered through those past test stages, where the observed samples were close to that one observed at present, to be the smallest one.

The last of these rules states what might be called a conservative principle of learning from experience: "Do what turned out to be the best in the past." The first one expresses a progressive principle: "Perpetually try all possibilities to keep up with a changing situation." The second rule then requires that a suitable compromise should be made between these two contradictory principles.

Finally (Section 4), we show that if a decision procedure based on these rules is used it yields asymptotically the same average loss as would the Bayes decision function. More precisely, if $\{\theta_n\}$ is a sequence of parameters and $\{a_n\}$ is a sequence of decisions determined by the rules above, then the average random loss $N^{-1}\sum_{n=1}^{N} L_n(\theta_n, a_n)$ minus the corresponding Bayes risk $\rho(\tau_n)$ approaches zero in probability as N, the size of the population, tends to infinity, and the convergence is uniform over certain regular sequences or parameters. This result, in our opinion, indicates that the Bayesian model of decision-making, together with its components and assumptions, can be considered a result of experience gathered by the decision-maker through a process of learning expressed in a condensed form.

2. Prerequisites.

2.1. *Notation.* Throughout this paper the triplet $(\Omega, \mathcal{S}, P)$ will always denote the basic probability space. Random variables will be designated by capital letters with the argument $\omega \in \Omega$ omitted unless necessary. Sets of elements for which a statement $\mathcal{U}$ is true will be denoted by $\{\mathcal{U}\}$. The indicator function of a set E will be denoted by I_E or by $I(E)$, the complement of E by E^c as long as the universal set is understood from the context. The expectation of a random variable X will be denoted by $E\{X\}$, the conditional expectation given a σ–field $\mathcal{F} \subset \mathcal{S}$ by $E\{X \mid \mathcal{F}\}$. A function f will sometimes be written as $f(\cdot)$ and the same symbol will be used to denote the Euclidean vector with components $f(a)$ in case f is defined on a finite ordered set A. The symbol R^m will stand for m-dimensional Euclidean space; components of vectors will be distinguished by superscripts in parentheses. The inner product of vectors $\xi \in R^m$ and $\eta \in R^m$ will be denoted by $\xi \cdot \eta$.

2.2. *The decision problem.* Let $\mathcal{P} = \{\nu_\theta : \theta \in \Theta\}$ be a one-parameter family of probability measures on a measurable space $(\Xi, \mathfrak{X})$–the sample space, and let $A = \{a^{(1)}, \cdots, a^{(m)}\}$ be a *finite* set of all possible decisions. Further, let $\mathfrak{A}$ be the $(m-1)$–dimensional probability simplex

$$\mathfrak{A} = \{\alpha \in R^m : \alpha^{(i)} \geqq 0; \quad i = 1, \cdots, m; \quad \sum_{i=1}^{m} \alpha^{(i)} = 1\},$$

let $\mathfrak{I}$ be the σ-field of all subsets of the set of parameters Θ, and let T be the class of finite signed measures on $(\Theta, \mathfrak{I})$ with the property that for each $\tau \in T$ there is a finite subset $\Theta_\tau \subset \Theta$ such that the total variation $\|\tau\|(\Theta_\tau^c) = 0$. In other words, each $\tau \in T$ is discrete with finite number of atoms. Let T_0 be the subclass

of all probability measures in T. If $\tau \varepsilon T_0$ or $\alpha \varepsilon \mathfrak{A}$ is such that $\tau(\{\theta\}) = 1$ for some $\theta \varepsilon \Theta$, or $\alpha^{(i)} = 1$ for some $i = 1, \cdots, m$ we may sometimes write simply θ or $a^{(i)}$ instead of τ and α, respectively.

Next, let $\mathfrak{L}$—the random loss function—be a two-parameter family of real-valued random variables with finite expectations: $\mathfrak{L} = \{L(\theta, a): \theta \varepsilon \Theta, a \varepsilon A\}$, and let us denote $l(\theta, a) = E\{L(\theta, a)\}$. As $\tau \varepsilon T_0$ plays the role of prior distribution of Bayesian theory we define the Bayes risk ρ by

$$\rho(\tau) = \min \int_\Theta \int_\Xi l(\theta, \delta(x))\, d\nu_\theta(x)\, d\tau(\theta), \tag{2.1}$$

where the minimum is taken over all measurable mappings δ from Ξ into A.

As mentioned in the introduction we will study sequences of independent copies of the decision problem described above with the parameter θ varying along the sequence. More precisely, let $\boldsymbol{\Theta}$ be the set of all sequences

$$\boldsymbol{\theta} = \{\theta_n : n = 0, 1, 2, \cdots\}, \tag{2.2}$$

and let

$$\{X_n : n = 0, 1, 2, \cdots\} \tag{2.3}$$

be a sequence of independent random variables with values in $(\Xi, \mathfrak{X})$ such that $PX_n^{-1} = \nu_{\theta_n}$. Further, for any sequence $\{a_n \varepsilon A : n = 0, 1, 2, \cdots\}$ let

$$\{L_n(\theta_n, a_n) : n = 0, 1, 2, \cdots\} \tag{2.4}$$

be a sequence of independent real-valued random variables, where $L_n(\theta_n, a_n)$ is distributed as $L(\theta_n, a_n) \varepsilon \mathfrak{L}$, and let the sequences (2.3) and (2.4) be also mutually independent. Here, the value of X_n represents the sample at the nth stage of the decision process and the value of $L_n(\theta_n, a_n)$ is the loss if the decision a_n had been taken. The decision process starts with $n = 1$. The parameter θ_0 is only a dummy parameter introduced for notational convenience, and it will be assumed that $L(\theta_0, \cdot) = 0$.

One more notational convention will be useful. With every sequence of parameters (2.2) we associate a sequence $\{\tau_n : n = 0, 1, 2, \cdots\}$ of "empirical distributions of θ" defined by $\tau_0(\cdot) = 0$ and

$$\tau_n(E) = n^{-1} \sum_{r=1}^{n} I_E(\theta_r), \qquad E \varepsilon \mathfrak{Z}, \qquad n = 1, 2, \cdots,$$

so that $\tau_n \varepsilon T_0$ for $n = 1, 2, \cdots$. We will use the symbols τ_n exclusively in this sense.

2.3. *Assumptions.* The sample space $(\Xi, \mathfrak{X})$, the family $\mathcal{P}$, the set A, and the random loss function $\mathfrak{L}$ constitute a formal description of the decision problem. We will now impose conditions on these concepts.

First, we require the sample space to be such that:

(C1) *the σ-field $\mathfrak{X}$ has a countable number of generators.* Thus, $(\Xi, \mathfrak{X})$ may be, for example, a separable metric space with Borel σ-field.

Next, let $M > 1$ be an integer and let

$$\{\mathfrak{B}_\kappa : \kappa = 1, 2, \cdots\} \tag{2.5}$$

be a sequence of countable measurable partitions

$$\mathfrak{B}_\kappa = \{B_{\kappa,k} : k = 1, 2, \cdots\} \tag{2.6}$$

of the space $(\Xi, \mathfrak{X})$ such that

(2.7a) *each set* $B_{\kappa,k} \in \mathfrak{B}_\kappa$ *is a union of at most* M *sets from* $\mathfrak{B}_{\kappa+1}$,

and

(2.7b) $\mathfrak{X}$ *is the minimum* σ*-field over the sequence* (2.5).

It is easy to see that such a sequence of partitions exists as long as (C1) is satisfied.

Let $\epsilon > 0$ and let μ be a measure on $(\Xi, \mathfrak{X})$. A set $E \in \mathfrak{X}$ is called an ϵ*-carrier of* μ if $\mu(E^c) < \epsilon$. Let $\tau \in T_0$ and let us denote

$$\nu_\tau(\cdot) = \int_\Theta \nu_\theta(\cdot)\, d\tau(\theta).$$

We now introduce the following definition.

DEFINITION. A sequence $\boldsymbol{\theta} = \{\theta_n\} \in \boldsymbol{\Theta}$ will be called *regular* if for every $\epsilon > 0$ there is an integer $K(\epsilon)$ such that every ν_{τ_n}, $n = 1, 2, \cdots$, possesses an ϵ-carrier that can be covered by at most $K(\epsilon)$ sets from the partition $\mathfrak{B}_1$.

A set of regular sequences $\boldsymbol{\theta}$ for which these integers $K(\epsilon)$ are uniformly bounded for every fixed $\epsilon > 0$ will be called a *regular* subset of $\boldsymbol{\Theta}$.

Clearly, these concepts depend also on the family $\mathcal{P}$. If, for example, $\mathcal{P}$ is the family of all distributions on the real line with zero means and uniformly bounded variances then the set of all sequences $\boldsymbol{\Theta}$ is regular itself.

Finally, we need a condition on the random loss function:

(C2) *there is a finite constant* C_0 *such that for every* $\theta \in \Theta$, $a \in A$, $E|L(\theta, a)|^3 \leqq C_0^3$.

3. Decision procedure. The statistician chooses his decisions $a \in A$ using a decision procedure $\mathfrak{D}$, which tells him at each stage $n = 1, 2, \cdots$ of the decision process the probability $\alpha^{(i)}$ with which to make the decision $a^{(i)}$. Thus, a particular decision procedure $\mathfrak{D}$ is specified by defining a sequence $\{D_n : n = 1, 2, \cdots\}$ of, in general, random vectors with values in $\mathfrak{A}$. Such a sequence then determines a sequence of random decisions $\{\Psi_n : n = 1, 2, \cdots\}$, to be referred to as generated by the decision procedure $\mathfrak{D}$.

According to what was said in the introduction the distribution of D_n may depend on the past decisions $a_1, \cdots, a_{n-1}$, on the past losses, i.e. values of $L_1(\theta_1, a_1), \cdots, L_{n-1}(\theta_{n-1}, a_{n-1})$, on the values of the past and present samples $X_1, \cdots, X_{n-1}, X_n$, and also on the structure of the set A and the space $(\Xi, \mathfrak{X})$. It must not, however, depend on the family $\mathcal{P}$, the loss function $\mathfrak{L}$ or any other characteristic related to them. Besides satisfying these requirements our particular decision procedure should obey the three rules from Section. 1. We will now define such a procedure $\mathfrak{D}^*$ formally.

Let

$$\{U_n : n = 0, 1, 2, \cdots\} \tag{3.1}$$

be a sequence of independent random variables taking values 0 and 1 with

$$P(\{U_n = 1\}) = p_n > 0, \tag{3.2}$$

and let

$$\{V_n : n = 0, 1, 2, \cdots\} \tag{3.3}$$

be a sequence of independent identically distributed random vectors with values in $\mathfrak{A}$ such that for every $i = 1, \cdots, m$, $P(\{V_n = (\delta^{(1)}, \cdots, \delta_i{}^{(m)})\}) = m^{-1}$, where $\delta_i{}^{(j)}$ is the Kronecker delta. The sequences (3.1) and (3.3) are also mutually independent.

Next, let

$$\{Y_n : n = 0, 1, 2, \cdots\} \tag{3.4}$$

be a sequence of random vectors with values in R^m and defined for every $\theta \varepsilon \Theta$ by

$$Y_n{}^{(i)} = mp_n{}^{-1}U_n(1 + L_n(\theta_n, a^{(i)}))V_n{}^{(i)}, \qquad i = 1, \cdots, m. \tag{3.5}$$

Further, let $s^* = (s^{*(1)}, \cdots, s^{*(m)})$ be a mapping from R^m into $\mathfrak{A}$ defined for every $\xi \varepsilon R^m$ by

$$\begin{aligned} s^{*(i)}(\xi) &= \alpha^{(i)} \quad \text{if} \quad \xi^{(i)} = \min_{j=1,\cdots,m}\{\xi^{(j)}\}, \\ &= 0 \quad \text{if} \quad \xi^{(i)} > \min_{j=1,\cdots,m}\{\xi^{(j)}\}, \end{aligned} \tag{3.6}$$

where α is any vector from $\mathfrak{A}$ such that $\sum_{i=1}^m s^{*(i)}(\xi) = 1$.
Let

$$\{g_n(x, x') : n = 0, 1, 2, \cdots\} \tag{3.7}$$

be a sequence of measurable functions on $(\Xi, \mathfrak{X}) \times (\Xi, \mathfrak{X})$ defined for $n = 1, 2, \cdots$ by

$$g_n(x, x') = \sum_{k=1}^{\infty} I_{B\kappa,k}(x)I_{B\kappa,k}(x') \quad \text{whenever} \quad N_{\kappa-1} \leqq n < N_\kappa, \tag{3.8}$$

where

$$\{N_\kappa : \kappa = 0, 1, 2, \cdots\} \tag{3.9}$$

is an increasing sequence of nonnegative integers, $B_{\kappa,k} \varepsilon \mathfrak{B}_\kappa$, and $g_0(\cdot, \cdot) = 0$.

Finally, for every $x \varepsilon \Xi$, $n = 1, 2, \cdots$, let

$$S_n(x) = U_nV_n + (1 - U_n)s^*(\sum_{r=0}^{n-1} Y_r g_n(x, X_r)), \tag{3.10}$$

where $\{X_r\}$ is the sequence (2.3). Clearly, $S_n(x)$ is a random vector for every $x \varepsilon \Xi$ and an $\mathfrak{X}$-measurable vector-valued function for every $\omega \varepsilon \Omega$. The decision procedure $\mathfrak{D}^* = \{D_n{}^* : n = 1, 2, \cdots\}$ is now defined by $D_n{}^* = S_n(X_n)$.

The interpretation of this rather lengthy definition is as follows: random variables U_n determine whether the nth stage of the decision process is a test one ($U_n = 1$) or an active one ($U_n = 0$). Thus at a test stage $D_n = V_n$, that is

a decision $a \varepsilon A$ is selected uniformly at random. The ith component of the vector Y_n can be nonzero only in test stages, where the decision $a^{(i)}$ has been taken and is then proportional to one plus the loss incurred. The proportionality constant (weight) depends on n through p_n. The ith component of the sum $\sum_{r=0}^{n-1} Y_r g_n(X_n, X_r)$ thus represents a weighted sum of losses (each increased by 1) in those past test stages prior to the present nth one, where the samples X_r were equal to X_n modulo the corresponding partition $\mathfrak{B}_\kappa$. These partitions may be considered as slowly increasing refinements of observed data. Finally, the mapping s^* determines the component, that is the decision, for which this sum is minimum. The procedure $\mathfrak{D}^*$ thus satisfies all the requirements discussed in the introduction and at the beginning of this section.

4. Convergence theorem. We will prove now that for the decision procedure $\mathfrak{D}^*$ the average loss approaches asymptotically the corresponding Bayes risk. We will do this first for the special case when the sample space is discrete, and then for the general case.

4.1. *The discrete case.* Let the sample space be discrete, that is, let Ξ be a countable set and $\mathfrak{X}$ be the σ-field of all subsets of Ξ. The condition (C1) of Section 2 is then automatically satisfied and the partitions (2.6) can be chosen identical and consisting of all one-element subsets of Ξ:

The functions (3.7) then become

$$g_n(x, x') = 1 \quad \text{if} \quad x = x', \qquad = 0 \quad \text{if} \quad x \neq x', \tag{4.1}$$

for all $n = 1, 2, \cdots$, and the Bayes risk (2.1) is given by

$$\rho(\tau) = \sum_{x \varepsilon \Xi} \min_{a \varepsilon A} \int_\Theta l(\theta, a) v_\theta(\{x\})\, d\tau(\theta). \tag{4.2}$$

We refer to this choice of partitions $\mathfrak{B}_x$ as the discrete case.

THEOREM 1. *Let, in the discrete case,* $\{\Psi_n : n = 1, 2, \cdots\}$ *be the sequence of decisions generated by the decision procedure* $\mathfrak{D}^*$ *of Section* 3., *and let the condition* (C2) *be satisfied.*

If the probabilities (3.2) *are such that as* $n \to \infty$

$$p_n \downarrow 0 \quad \textit{and} \quad np_n \uparrow \infty \tag{4.3}$$

then for any regular sequence $\boldsymbol{\theta} = \{\theta_n\}$

$$N^{-1} \sum_{n=1}^{N} L_n(\theta_n, \Psi_n) - \rho(\tau_N) \to 0 \quad \textit{in probability as} \quad N \to \infty$$

and the convergence is uniform over any regular subset of $\boldsymbol{\Theta}$.

PROOF. Let $\boldsymbol{\theta} = \{\theta_n\} \varepsilon \boldsymbol{\Theta}_{\text{reg}}$, where $\boldsymbol{\Theta}_{\text{reg}}$ is a regular subset of $\boldsymbol{\Theta}$, and let $\mathfrak{F}_n$ be the σ-field induced by the family $\{X_0, \cdots, X_n; L_0(\theta_0, \cdot), \cdots, L_n(\theta_n, \cdot); U_c, \cdots, U_{n+1}; V_0, \cdots, V_{n+1}\}$ so that

$$E\{L_n(\theta_n, \Psi_n) \mid \mathfrak{F}_{n-1}\} = \sum_{x \varepsilon \Xi} l(\theta_n, S_n(x)) v_{\theta_n}(\{x\}) \text{ a.s.} \tag{4.4}$$

Since, by the Extended Bienaymé equality ([3], page 386) and (C2),

$$E|N^{-1}\sum_{n=1}^{N}[L_n(\theta_n, \Psi_n) - E\{L_n(\theta_n, \Psi_n) \mid \mathfrak{F}_{n-1}\}]| \leqq C_0 N^{-\frac{1}{2}},$$

the statement will follow by the Markov inequality ([3], page 158), (4.2) and (4.4) if we prove that

$$E|\sum_{x\in\Xi} W_N(x)| \to 0 \quad \text{uniformly in} \quad \boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{reg}}, \tag{4.5}$$

where we denoted

$$W_N(x) = N^{-1}\sum_{n=1}^{N} w_x(\theta_n, S_n(x)) - \varphi_x(\tau_N), \tag{4.6}$$

$$w_x(\tau, \alpha) = \int_\Theta \sum_{i=1}^{m} l(\theta, a^{(i)})\alpha^{(i)}\nu_\theta(\{x\})\, d\tau(\theta), \tag{4.7}$$

$$\varphi_x(\tau) = \min_{a\in A}\{w_x(\tau, a)\}. \tag{4.8}$$

Let $\epsilon > 0$ and let $B(\epsilon, \tau)$ denote an ϵ-carrier of ν_τ. Since $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{reg}}$ is regular, $B(\epsilon, \tau_N)$ can be found for every τ_N such that it contains at most $K(\epsilon)$ elements of Ξ. Since by (C2) $|W_N(x)| \leqq 2C_0\nu_{\tau_N}(\{x\})$, we have then

$$\sum_x E|W_N(x)| \leqq K(\epsilon)\sup_x E|W_N(x)| + 2C_0\epsilon,$$

whence we conclude that (4.5) will follow if we prove

$$E|W_N(x)| \to 0 \quad \text{uniformly in both} \quad \boldsymbol{\theta} \in \boldsymbol{\Theta} \quad \text{and} \quad x \in \Xi. \tag{4.9}$$

Let $Y_{n,x} = Y_n g_n(x, X_n)$, where Y_n and g_n are defined by (3.5) and (4.1) respectively; let

$$\hat{S}_{n,x} = s^*(\sum_{r=0}^{n} Y_{r,x}), \tag{4.10}$$

and let us denote the right-hand side of (4.6) with $S_n(x)$ replaced by $\hat{S}_{n,x}$ and $\hat{S}_{n-1,x}$ by $W_N'(x)$ and $W_N''(x)$, respectively. Since by (3.7), (3.10) and (4.10), $U_n = 0 \Rightarrow S_n(x) = \hat{S}_{n-1,x} = \hat{S}_{n,x}$, we have by (C2) and (4.3)

$$\max\{E|W_N(x) - W_N'(x)|, E|W_N(x) - W_N''(x)|\} \leqq 2C_0 N^{-1}\sum_{n=1}^{N} p_n \downarrow 0. \tag{4.11}$$

We are now going to prove that the random variables $W_N'(x)$ are bounded from above and the random variables $W_N''(x)$ are bounded from below by random variables that converge to zero in the mean uniformly in $\boldsymbol{\theta}$ and x. This, in view of (4.11), will prove (4.9).

Let us begin with $W_N''(x)$. This can be written as

$$W_N''(x) = N^{-1}\sum_{n=1}^{N} n[w_x(\tau_n, \hat{S}_{n-1,x}) - w_x(\tau_n, \hat{S}_{n,x})] + [w_x(\tau_N, \hat{S}_{N,x}) - \varphi_x(\tau_N)]. \tag{4.12}$$

Next, noticing that

$$E\{Y_{n,x}\} = (1 + l(\theta_n, \cdot))\nu_{\theta_n}(\{x\}), \tag{4.13}$$

and denoting

$$\tilde{Y}_{n,x} = Y_{n,x} - E\{Y_{n,x}\}, \tag{4.14}$$

$$Z_{n,x} = n^{-1/2}\sum_{r=0}^{n} \tilde{Y}_{r,x}, \; n = 1, 2, \cdots; \qquad Z_{0,x} = 0, \tag{4.15}$$

we have by (3.6) and (4.10)

$$\hat{S}_{n,x} = s^*(w_x(\tau_n, \cdot) + n^{-1/2}Z_{n,x}). \tag{4.16}$$

Here we need the following simple lemma. *Let* $\tau \varepsilon T$, $\xi \varepsilon R^m$, $\alpha_1 = s^*(w_x(\tau, \cdot) + \xi)$, $\alpha_2 \varepsilon \mathfrak{A}$. *Then*

$$w_x(\tau, \alpha_1) - w_x(\tau, \alpha_2) \leqq \xi\cdot(\alpha_2 - \alpha_1). \tag{4.17}$$

To prove this we write

$$\begin{aligned} w_x(\tau, \alpha_1) - w_x(\tau, \alpha_2) &= \sum_{i=1}^{m} \left(\int_\Theta l(\theta, a^{(i)})\nu_\theta(\{x\})\, d\tau(\theta)\right)(\alpha^{(i)} - \alpha^{(i)}) \\ &= w_x(\tau, \cdot)\cdot(\alpha_1 - \alpha_2) \\ &= (w_x(\tau, \cdot) + \xi)\cdot(\alpha_1 - \alpha_2) - \xi\cdot(\alpha_1 - \alpha_2) \\ &\leqq -\xi\cdot(\alpha_1 - \alpha_2), \end{aligned}$$

where the last inequality follows from the definition of α_1.

Now applying this lemma to the summands in (4.12), and using the fact that the last bracket in (4.12) is nonnegative, we obtain

$$\begin{aligned} W_N''(x) &\geqq N^{-1}\sum_{n=1}^{N} n^{1/2}Z_{n,x}\cdot(\hat{S}_{n,x} - \hat{S}_{n-1,x}) \\ &= N^{-1}\sum_{n=1}^{N} [(n-1)^{1/2}Z_{n-1,x} - n^{1/2}Z_{n,x}]\cdot\hat{S}_{n-1,x} \\ &\quad + N^{-1/2}Z_{N,x}\cdot\hat{S}_{N,x} \\ &\geqq -N^{-1}\sum_{n=1}^{N}\tilde{Y}_{n,x}\cdot\hat{S}_{n-1,x} - \max_{i=1,\cdots,m} |N^{-1}\sum_{n=1}^{N}\tilde{Y}_{n\,x}^{(i}|, \end{aligned} \tag{4.18}$$

where the last inequality follows from (4.15) and $\hat{S}_{N,x} \varepsilon \mathfrak{A}$. Next, from (4.13), (4.14), (C2) and C_r-inequality ([3], page 155) we have for all $n = 0, 1, 2, \cdots$; $x \varepsilon \Xi$, and $i = 1, \cdots, m$,

$$E|\tilde{Y}_{n,x}^{(i)}|^\lambda \leqq (2C_1)^\lambda m^{\lambda-1}p_n^{\,1-\lambda}\nu_{\theta_n}(\{x\}), \qquad \lambda = 1, 2, 3 \tag{4.19}$$

where C_1 depends only on the constant C_0 from (C2). Hence

$$N^{-2}\sum_{n=1}^{N} E|\tilde{Y}_{n,x}^{(i)}|^2 \leqq (2C_1)^2 m(Np_N)^{-1}, \tag{4.20}$$

and since the random variables (4.14) are independent and centered at expectations, the last term in (4.18) goes to zero by (4.20) and (4.3). Further, denoting by $\mathcal{Y}_{n,x}$ the σ-field induced by $\{Y_{0,x}, \cdots, Y_{n,x}\}$ and realizing that, by (4.10),

$$E\{\tilde{Y}_{n,x}\cdot\hat{S}_{n-1,x} \mid \mathcal{Y}_{n-1,x}\} = E\{\tilde{Y}_{n,x}\}\cdot\hat{S}_{n-1,x} = 0 \quad \text{a.s.},$$

we conclude by (4.20) that the next to the last term in (4.18) goes to zero as well. Hence

$$\liminf_{N\to\infty} E|W_N''(x)| = 0 \quad \text{uniformly in} \quad \boldsymbol{\theta} \quad \text{and} \quad x,$$

and it remains to prove that also

$$\lim\sup_{N\to\infty} E|W_N'(x)| = 0 \quad \text{uniformly in} \quad \theta \quad \text{and} \quad x. \tag{4.21}$$

Writing again $W_N'(x) = N^{-1}\sum_{n=1}^{N-1} n[w_x(\tau_n, \hat{S}_{n,x}) - w_x(\tau_n, \hat{S}_{n+1,x})] + [w_x(\tau_N, \hat{S}_{N,x}) - \varphi_x(\tau_N)]$, and applying (4.17) to the summands with $\alpha_2 = \hat{S}_{n+1,x}$ and also to the last bracket with $\alpha_2 = s^*(w_x(\tau_N, \cdot))$, we obtain, similarly as in (4.18), the inequality

$$W_N'(x) \leqq -N^{-1}\sum_{n=1}^{N} \tilde{Y}_{n,x}\cdot \hat{S}_{n,x} + \max_{i=1,\cdots,m} |N^{-1}\sum_{n=1}^{N} \tilde{Y}_{n,x}^{(i)}|.$$

Let $\tilde{Y}_{n,x}^{(ij)} = \tilde{Y}_{n,x}^{(i)} - \tilde{Y}_{n,x}^{(j)}$, $i = 1, \cdots, m, j = 1, \cdots, m$. Since $\hat{S}_{n,x} \varepsilon \mathfrak{A}$ we have the identity

$$N^{-1}\sum_{n=1}^{N} \tilde{Y}_{n,x}\cdot \hat{S}_{n,x} = m^{-1}\sum_{i=1}^{m}\sum_{j=1}^{m} N^{-1}\sum_{n=1}^{N} \tilde{Y}_{n,x}^{(ij)}\hat{S}_{n,x}^{(i)} + m^{-1}\sum_{i=1}^{m} N^{-1}\sum_{n=1}^{N} \tilde{Y}_{n,x}^{(i)},$$

and similarly as before we conclude that (4.21) will follow if for any $i \neq j$,

$$E|N^{-1}\sum_{n=1}^{N} E\{\tilde{Y}_{n,x}^{(ij)}\hat{S}_{n,x}^{(i)} \mid \mathfrak{Y}_{n-1,x}\}| \to 0 \tag{4.22}$$

uniformly in θ and x. Let

$$H_{n,x}^{(i)} = \{\sum_{r=0}^{n} Y_{r,x}^{(i)} < \min_{\lambda=1,\cdots,m;\lambda\neq i} \sum_{r=0}^{n} Y_{r,x}^{(\lambda)}\}$$

and

$$G_{n,x}^{(i)} = \{\sum_{r=0}^{n} Y_{r,x}^{(i)} \leqq \min_{\lambda=1,\cdots,m} \sum_{r=0}^{n} Y_{r,x}^{(\lambda)}\}.$$

By (4.10) and (3.61) we have for every $i = 1, \cdots, m$, $I_{H_{n,x}^{(i)}} \leqq \hat{S}_{n,x}^{(i)} \leqq I_{G_{n,x}^{(i)}}$, so that (4.22) is implied by

$$E|N^{-1}\sum_{n=1}^{N} E\{\tilde{Y}_{n,x}^{(ij)} I_{H_{n,x}^{(i)}} \mid \mathfrak{Y}_{n-1,x}\}| \to 0 \tag{4.23}$$

and

$$E|N^{-1}\sum_{n=1}^{N} E\{\tilde{Y}_{n,x}^{(ij)} I_{G_{n,x}^{(i)}} \mid \mathfrak{Y}_{n-1,x}\}| \to 0, \tag{4.24}$$

both uniformly in $\theta \varepsilon \Theta$ and $x \varepsilon \Xi$.

To prove (4.23) let for $\lambda = 1, \cdots, m$,

$$\Gamma_{n,x}^{(i\lambda)} = \{|\tilde{Y}_{n,x}^{(i\lambda)}| \leqq 2C_0 n^{1/6} p^{-5/6}\} \quad \text{and} \quad \Gamma_{n,x}^{(i)} = \bigcap_{\lambda=1}^{m} \Gamma_{n,x}^{(i\lambda)}.$$

Clearly $H_{n,x}^{(i)} - \Gamma_{n,x}^{(i)} \subset (\Gamma_{n,x}^{(i)})^c$, and for any $j = 1, \cdots, m, j \neq i$,

$$(\Gamma_{n,x}^{(i)})^c \subset (\{|\tilde{Y}_{n,x}^{(ij)}| > C_0 n^{1/6} p^{-5/6}\} \cup \{|\tilde{Y}_{n,x}^{(ij)}| \leqq 2C_0\}) \cap \{U_n = 1\}.$$

The latter inclusion follows from the fact that if $|\tilde{Y}_{n,x}^{(i,\lambda)}| > 2C_0 n^{1/6} p^{-5/6} \geqq 2C_0$ for some $\lambda = 1, \cdots, m$ then by the definition of $\tilde{Y}_{n,x}^{(i,\lambda)}$ and by (C2) either $Y_{n,x}^{(i)} \neq 0$ or $Y_{n,x}^{(\lambda)} \neq 0$, so that $U_n = 1$ and either $V_n^{(i)} = 1$ or $V_n^{(\lambda)} = 1$. If $V_n^{(i)}$ or if $\lambda = j$ then

$$\tilde{Y}_{n,x}^{(ij)} = \tilde{Y}_{n,x}^{(i\lambda)}, \quad \text{and} \quad |\tilde{Y}_{n,x}^{(ij)}| = E|\tilde{Y}_{n,x}^{(ij)}| \quad \text{otherwise.}$$

Hence from the fact that $|\tilde{Y}_{n,x}^{(ij)}| I_{\Gamma_{n,x}^{(i)c}}$ is independent of $\mathfrak{Y}_{n-1,x}$ we have almost

surely

$$|E\{\tilde{Y}^{(ij)}_{n,x} I_{H^{(i)}_{n,x}} \mid \mathcal{Y}_{n-1,x}\} - E\{\tilde{Y}^{(ij)}_{n,x} I_{H^{(i)}_{n,x} \cap \Gamma^{(i)}_{n,x}} \mid \mathcal{Y}_{n-1,x}\}|$$
$$\leqq \int_{\{|\tilde{Y}^{(ij)}_{n,x}| > c_0 n^{1/6} p_n^{-5/6}\}} |\tilde{Y}^{(ij)}_{n,x}| \, dP + 2C_0 p_n .$$

The last term in this inequality tends to zero by (4.3) and the integral is bounded by $C_0^{-2} n^{-1/3} p_n^{5/3} E|\tilde{Y}^{(ij)}_{n,x}|^3$, which again tends to zero by (4.20) and (4.3). Thus it remains to prove that for every $i \neq j$,

$$E\,|N^{-1} \textstyle\sum_{n=1}^{N} E\{\tilde{Y}^{(ij)}_{n,x} I_{H^{(i)}_{n,x} \cap \Gamma^{(i)}_{n,x}} \mid \mathcal{Y}_{n-1,x}\}| \to 0 \tag{4.25}$$

uniformly in θ and x. For this let

$$H^{(i\lambda)}_{n,x} = \{\textstyle\sum_{r=0}^{n-1} \tilde{Y}^{(i\lambda)}_{r,x} < -\tilde{Y}^{(i\lambda)}_{n,x} - \sum_{r=0}^{n} \Delta_r^{(i\lambda)} \nu_{\theta_r}(\{x\})\},$$
$$\bar{H}^{(i\lambda)}_{n,x} = \{\textstyle\sum_{r=0}^{n-1} \tilde{Y}^{(i\lambda)}_{r,x} < 2C_0 n^{1/6} p_n^{-5/6} - \sum_{r=0}^{n} \Delta_r^{(i\lambda)} \nu_{\theta_r}(\{x\})\},$$

and

$$\underline{H}^{(i\lambda)}_{n,x} = \{\textstyle\sum_{r=0}^{n-1} \tilde{Y}^{(i\lambda)}_{r,x} < -2C_0 n^{1/6} p_n^{-5/6} - \sum_{r=0}^{n} \Delta_r^{(i\lambda)} \nu_{\theta_r}(\{x\})\},$$

where $\Delta_r^{(i\lambda)} = E\{Y_r^{(i)} - Y_r^{(\lambda)}\}$; and let $\bar{H}^{(i)}_{n,x} = \bigcap_{\lambda=i,\,\lambda\neq i}^{m} \bar{H}^{(i\lambda)}_{n,x}$, $\underline{H}^{(i)}_{n,x} = \bigcap_{\lambda=1,\,i\neq\lambda}^{m} \underline{H}^{(i\lambda)}_{n,x}$. It is easily seen that also $H^{(i)}_{n,x} = \bigcap_{\lambda=1,\lambda\neq i}^{m} H^{(i\lambda)}_{n,x}$ and

$$\underline{H}^{(i)}_{n,x} \subset H^{(i)}_{n,x} \cap \Gamma^{(i)}_{n,x} \subset \bar{H}^{(i)}_{n,x}$$

so that we have the inequality

$$-|\tilde{Y}^{(ij)}_{n,x}| \, I(\bar{H}^{(i)}_{n,x} - \underline{H}^{(i)}_{n,x}) + \tilde{Y}^{(ij)}_{n,x} I(\underline{H}^{(i)}_{n,x}) \leqq \tilde{Y}^{(ij)}_{n,x} I(H^{(i)}_{n,x} \cap \Gamma^{(i)}_{n,x})$$
$$\leqq |\tilde{Y}^{(ij)}_{n,x}| \, I(\bar{H}^{(i)}_{n,x} - \underline{H}^{(i)}_{n,x}) + \tilde{Y}^{(ij)}_{n,x} I(\underline{H}^{(i)}_{n,x}). \tag{4.26}$$

By definition, both $\bar{H}^{(i)}_{n,x}$ and $\underline{H}^{(i)}_{n,x}$ are $\mathcal{Y}_{n-1,x}$-measurable sets so that

$$E\{\tilde{Y}^{(ij)}_{n,x} I(\underline{H}^{(i)}_{n,x}) \mid \mathcal{Y}_{n-1,x}\} = E\{\tilde{Y}^{(ij)}_{n,x}\} I(\underline{H}^{(i)}_{n,x}) = 0 \quad \text{a.s.},$$

and (4.25) reduces to showing that for $i \neq j$,

$$N^{-1} \textstyle\sum_{n=1}^{N} E\{|\tilde{Y}^{(ij)}_{n,x}| \, I(\bar{H}^{(i)}_{n,x} - \underline{H}^{(i)}_{n,x})\} \to 0 \tag{4.27}$$

uniformly in θ and x. Let us denote the expectations in (4.27) by $q^{(ij)}_{n,x}$. By (4.19), and since $\bar{H}^{(i)}_{n,x} - \underline{H}^{(i)}_{n,x} \subset \bigcup_{\lambda=1,\lambda\neq i}^{m} (\bar{H}^{(i)}_{n,x} - \underline{H}^{(i)}_{n,x})$, we have

$$q^{(ij)}_{n,x} \leqq 4C_1 \textstyle\sum_{\lambda=1,\lambda\neq i}^{m} P(\bar{H}^{(i\lambda)}_{n,x} - \underline{H}^{(i\lambda)}_{n,x}).$$

Next, the set-theoretical differences in (4.28) can be written as

$$\bar{H}^{(i\lambda)}_{n,x} - \underline{H}^{(i\lambda)}_{n,x} = \{-2C_0 n^{1/6}(n-1)^{-1} p_n^{-5/6} - \textstyle\sum_{r=1}^{n} \Delta_r^{(i\lambda)} \nu_{\theta_r}(\{x\})$$
$$\leqq Z^{(i\lambda)}_{n-1,x} < 2C_0 n^{1/6}(n-1)^{-1} - \textstyle\sum_{r=1}^{n} \Delta_r^{(i\lambda)} \nu_{\theta_r}(\{x\})\},$$

where $Z^{(i\lambda)}_{n,x} = n^{-\frac{1}{2}} \sum_{r=1}^{n} \tilde{Y}^{(i\lambda)}_{r,x}$, and since the random variables are independent and centered at expectations the Berry-Esseen normal approximation theorem ([3], page 288), together with the well-known property $|\phi(\xi) - \phi(\xi')| \leqq |\xi - \xi'|$

of the normal distribution function, $\phi(\xi)$ yields

$$(4.29)\quad P(\bar{H}_{n,x}^{(i\lambda)} - \underline{H}_{n,x}^{(i\lambda)}) \leqq 4C_0 n^{1/6} p_n^{-5/6} (\sum_{r=0}^{n-i} E\,|\tilde{Y}_{r,x}^{(i\lambda)}|^2)^{-\frac{1}{2}} + 2\beta \sum_{r=0}^{n-1} E\,|\tilde{Y}_{r,x}^{(i\lambda)}|^3 (\sum_{r=0}^{n-1} E\,|\tilde{Y}_{r,x}^{(i\lambda)}|^2)^{-3/2},$$

where β is an absolute constant.

To find an upper bound to (4.29) we need first a lower bound to $E\,|\tilde{Y}_{n,x}^{(ij)}|^2$. Direct computation gives

$$E\,|\tilde{Y}_{n,x}^{(ij)}|^2 = [mp_n^{-1}(E\,|1 + L_n(\theta_n, a^{(i)})|^2 + E\,|1 + L_n(\theta_n, a^{(j)})|^2) - |\Delta_n^{(ij)}|^2]\nu_{\theta_n}(\{x\}) \geqq (2mp_n^{-1} - 1)\nu_{\theta_n}(\{x\}),$$

since

$$(mp_n^{-1} - 1)(l^2(\theta_n, a^{(i)}) + l^2(\theta_n, a^{(j)})) + 2mp_n^{-1}(l(\theta_n, a^{(i)}) + l(\theta_n, a^{(j)})) + 2l(\theta_n, a^{(i)})l(\theta_n, a^{(j)}) \geqq -1$$

and since $mp_n^{-1} - 1 \geqq 0$ we obtain for $i \neq j$:

$$(4.30)\qquad E\,|\tilde{Y}_{n,x}^{(ij)}|^2 \geqq mp_n^{-1}\nu_{\theta_n}(\{x\}).$$

Substituting now from (4.19) and (4.30) into (4.29) we have by (4.28) and since $\tilde{Y}_{0,x}^{(i\lambda)} = 0$:

$$(4.31)\quad q_{n,x}^{(ij)} \leqq 16C_0 C_1 n^{1/6} p_n^{-5/6} m^{\frac{1}{2}} (\sum_{r=1}^{n-1} p_r^{-1}\nu_{\theta_r}(\{x\}))^{-\frac{1}{2}} + 32\beta C_1^4 m^{3/2} (\sum_{r=1}^{n-1} p_r^{-2}\nu_{\theta_r}(\{x\}))(\sum_{r=1}^{n-1} p_r^{-1}\nu_{\theta_r}(\{x\}))^{-3/2}.$$

By (4.3), $p_r^{-2} \leqq p_n^{-1}p_r^{-1}$ for $r = 1, \cdots, n-1$, so that the last term in (4.31) becomes $32\beta C_1^4 m^{3/2} p_n^{-1}(\sum_{r=1}^{n-1} p_r^{-1}\nu_{\theta_r}(\{x\}))^{-\frac{1}{2}}$. Next, let $F_{1,x}, \cdots, F_{n-1,x}$ be random variables taking values 0 and 1 with $P(\{F_{r,x} = 1\}) = \nu_{\theta_r}(\{x\})$. By (4.3)

$$\sum_{r=1}^{n-1} p_r^{-1}\nu_{\theta_r}(\{x\}) \geqq np_n^{-1} \sum_{r=1}^{n-1} rn^{-2}\nu_{\theta_r}(\{x\}) = np_n^{-1}E\{n^{-2} \sum_{r=1}^{n-1} rF_{r,x}\};$$

and denoting $\Phi_{n,x} = \sum_{r=1}^{n-1} F_{r,x}$ we have

$$n^{-2} \sum_{r=1}^{n-1} rF_{r,x} \geqq n^{-2} \sum_{r=1}^{\Phi_{n,x}} r \geqq \tfrac{1}{2}(n^{-1}\Phi_{n,x})^2 = \tfrac{1}{2}(n^{-1} \sum_{r=1}^{n-1} F_{r,x})^2.$$

The Schwarz inequality yields

$$(E\{n^{-2} \sum_{r=1}^{n-1} rF_{r,x}\})^{\frac{1}{2}} \geqq n^{-1} \sum_{r=1}^{n-1} \nu_{\theta_r}(\{x\}).$$

Hence

$$(\sum_{r=1}^{n-1} p_r^{-1}\nu_{\theta_r}(\{x\}))^{-\frac{1}{2}} \leqq (2n^{-1}p_n)^{\frac{1}{2}}(n^{-1} \sum_{r=1}^{n-1} \nu_{\theta_r}(\{x\}))^{-1},$$

so that (4.31) becomes

$$q_{n,x}^{(ij)} \leqq [16C_0 C_1(2m)^{\frac{1}{2}}(np_n)^{-\frac{1}{2}} + 16\beta C_1^4(2m)^{3/2}(np_n)^{-\frac{1}{2}}](n^{-1} \sum_{r=1}^{n-1} \nu_{\theta_r}(\{x\}))^{-1}.$$

However, by (4.19) we have also $q_{n,x}^{(ij)} \leqq 4C_0\nu_{\theta_n}(\{x\})$, so that there is a finite constant C_2 such that for all $n = 2, 3, \cdots$,

$$q_{n,x}^{(j)} \leqq C_2(np_n)^{-\frac{1}{2}}(n^{-1}\sum_{r=1}^{n-1}\nu_{\theta_r}(\{x\}))^{-1},$$

and

$$\bar{q}_{n,x}^{(ij)} \leqq C_2 n^{-1}\sum_{r=1}^{n-1}\nu_{\theta_r}(\{x\}),$$

where $\bar{q}_{n,x}^{(ij)} = n^{-1}\sum_{r=1}^{n-1} q_{r,x}^{(ij)}$, $n = 2, 3, \cdots$. Hence, setting $\bar{q}_{1,x}^{(ij)} = 0$ we have for $n = 1, 2, \cdots$ $q_{n,x}^{(ij)}\bar{q}_{n,x}^{(ij)} \leqq C_2^2(np_n)^{-\frac{1}{2}}$ so that

$$(\bar{q}_{N,x}^{(ij)})^2 = N^{-2}\sum_{n=1}^{N-1}(q_{n,x}^{(ij)})^2 + N^{-2}\sum_{n=1}^{N-1} n\bar{q}_{n,x}^{(ij)}q_{n,x}^{(ij)}$$
$$\leqq C_2(N-1)N^{-2} + C_2^2N^{-1}\sum_{n=1}^{N-1}(np_n)^{-\frac{1}{2}},$$

which goes to zero by (4.3). Hence (4.27), and consequently (4.23), is proved. Finally, it is easy to see that exactly the same reasoning applies if the sets $H_{n,x}^{(i)}$ and other sets related to them are replaced by the sets $G_{n,x}^{(i)}$ and similarly related analogs. Therefore, (4.24) holds as well, and the theorem is proved.

4.2. *The general case.* In the general case, the sample space $(\Xi, \mathfrak{X})$ may be any abstract measurable space satisfying condition (C1).

THEOREM 2. *Let* $\{\Psi_n : n = 1, 2, \cdots\}$ *be the sequence of decisions generated by the decision procedure* $\mathfrak{D}^*$ *of Section* 3 *with the probabilities* (3.2) *defined by*:

$$p_n = N^{-\gamma}, \qquad 0 < \gamma < 1, \tag{4.32}$$

and the sequence (3.9) *defined by*

$$N_\kappa = ((\kappa + 1)!)^2, \tag{4.33}$$

and let the conditions (C1), (C2) *be satisfied. Then the conclusion of Theorem* 1 *is true.*

PROOF. By the same reasoning as in the proof of Theorem 1 it is enough to prove that as $N \to \infty$

$$E|N^{-1}\sum_{n=1}^{N} E\{L_n(\theta_n, \Psi_n) \mid \mathfrak{F}_{n-1}\} - \rho(\tau_N)| \to 0$$

uniformly in $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{reg}}$. As before we have almost surely

$$E\{L_n(\theta_n, \psi_n) \mid \mathfrak{F}_{n-1}\} = \int_{\Xi} l(\theta_n, S_n(x))\, d\nu_{\theta_n}(x). \tag{4.34}$$

With a slight abuse of notation let us denote

$$S_{n,\kappa}(x) = U_nV_n + (1 - U_n)s^*(\sum_{r=0}^{n-1} Y_r\, g_\kappa(x, X_r)), \tag{4.35}$$

$$\rho_\kappa(\tau) = \sum_{k=1}^{\infty} \min_{a\in A}\int_\Theta l(\theta, a)\nu_\theta(B_{\kappa,k})\, d\tau(\theta), \tag{4.36}$$

and

$$\bar{W}_{n,\kappa} = n^{-1}\sum_{r=1}^{n}\int_\Xi l(\theta_n, S_{n,\kappa}(x))\, d\nu_{\theta_n}(x) - \rho_\kappa(\tau_n), \tag{4.37}$$

and let the integers N and κ be such that $N_\kappa < N \leqq N_{\kappa+1}$. Then by (4.34),

(4.35) and (4.37) we have

$$
\begin{aligned}
&|N^{-1}\textstyle\sum_{n=1}^{N} E\{L_n(\theta_n, \psi_n) \mid \mathfrak{F}_{n-1}\} - \rho(\tau_N)| \\
(4.38)\quad &\leqq \textstyle\sum_{r=1}^{\kappa-1} N_r N^{-1} |N_r^{-1} \sum_{n=1}^{N} \int_{\Xi} [l(\theta_n, S_{n,r}(x)) \\
&\quad - l(\theta_n, S_{n,r+1}(x))] f_{\theta_n}(x)\, d\mu(x)| + |\bar{W}_{N_\kappa,\kappa}| + |\bar{W}_{N_\kappa,\kappa+1}| + |\bar{W}_{N,\kappa+1}| \\
&\quad + |\rho(\tau_N) - \rho_{\kappa+1}(\tau_N)| + |\rho_\kappa(\tau_{N_\kappa}) - \rho_{\kappa+1}(\tau_{N_\kappa})|.
\end{aligned}
$$

Now, by (C2) and (4.33), the first term on the right-hand side is bounded by $2C_0(\kappa - 1)(\kappa + 1)^{-2}$, and thus tends to zero as $N \to \infty$. Further, by (4.35) and (3.8), $S_{n,x}(\cdot)$ is a constant $S_{n,x}(x) = S_{n,x,k}$ whenever $x \in B_{x,k}$, so that by (4.37) we have

$$
\begin{aligned}
\bar{W}_{n,\kappa} &= \textstyle\sum_k W_{n,\kappa}(k), \quad \text{where} \\
W_{n,\kappa}(k) &= n^{-1} \textstyle\sum_{r=1}^{n} w_{\kappa,k}(\theta_r, S_{r,\kappa,\kappa}) - \varphi_{\kappa,k}(\tau_n), \\
w_{\kappa,k}(\tau, \alpha) &= \textstyle\int_\Theta \sum_{i=1}^{m} l(\theta, a^{(i)}) \alpha^{(i)} \nu_\theta(B_{\kappa,k})\, d\tau(\theta),
\end{aligned}
$$

and

$$
\varphi_{\kappa,k}(\tau) = \min_{a \in A} \{w_{\kappa,k}(\tau, a)\}.
$$

By (C2) we have

$$
|W_{n,\kappa}(k)| \leqq 2C_0 \nu_{\tau_n}(B_{\kappa,k}), \tag{4.39}
$$

and following the proof of Theorem 1, from the expression (4.12) and with p_n defined by (4.32) we conclude that there is a finite constant C and a number η, $0 < \eta < 1$ such that

$$
E|W_{n,\kappa}(k)| \leqq C n^{-\eta}. \tag{4.40}
$$

Let $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{reg}}$ and $\epsilon > 0$ and let again $B(\epsilon, \nu_{\tau_n})$ be an ϵ-carrier of ν_{τ_n} from our definition of regularity. Then by (2.7a) there is a subset of positive integers $\Lambda(\epsilon, \kappa)$ such that $B(\epsilon, \nu_{\tau_n}) \subset \bigcup_{k \in \Lambda(\epsilon,\kappa)} B_{\kappa,k}$ and such that $\Lambda(\epsilon, \kappa)$ does not contain more than $M^\kappa K(\epsilon)$ elements. Hence by (4.39) and (4.40) $\sum_k E|W_{n,\kappa}(k)| < CK(\epsilon) M^\kappa n^{-\eta} + 2C_0\epsilon$, and since for N_κ given by (4.33), $N \to \infty \Rightarrow \kappa \to \infty \Rightarrow M^\kappa N_\kappa^{-\eta} \to 0$, we have $|\bar{W}_{N_\kappa,\kappa}| \to 0$ uniformly in $\boldsymbol{\theta} \in \boldsymbol{\Theta}_{\text{reg}}$, and similarly for the next two terms in (4.38).

To show that the last two terms in (4.38) also converge to zero, let $\boldsymbol{\theta} = \{\theta_n\}$ be an arbitrary sequence of $\theta \in \Theta$ and let μ be a (σ-finite) measure dominating the countable family $\{\nu_{\theta n}\}$. We define for every $x \in \Xi$, $\theta \in \boldsymbol{\theta}$, $\kappa = 1, 2, \cdots$ the elementary function

$$
f_\theta^{(\kappa)}(x) = \textstyle\sum_k \nu_\theta(B_{\kappa,k})[\mu(B_{\kappa,k})]^{-1} I_{B_{\kappa,k}}(x),
$$

the summation being over those k's for which $\mu(B_{\kappa,k}) > 0$. We have now for every τ_n of the sequence $\boldsymbol{\theta}$

$$
\rho_\kappa(\tau_n) = \textstyle\int_\Xi \min_{a \in A} \int_\Theta l(\theta, a) f_\theta^{(\kappa)}(x)\, d\tau_n(\theta)\, d\mu(x),
$$

and denoting by $f_\theta(\theta \varepsilon \boldsymbol{\theta})$ the Radon-Nikodym derivative of ν_θ with respect to the measure μ, we have also

$$\rho(\tau_n) = \int_\Xi \min_{a \varepsilon A} \int_\Theta l(\theta, a) f_\theta(x)\, d\tau_n(\theta)\, d\mu(x),$$

where ρ_κ and ρ are, of course, the functionals defined by (4.36) and (2.1) respectively. Next let $\mathfrak{X}_\kappa$ be the minimum σ-field over the partition $\mathfrak{B}_\kappa$. Since by (2.7a, b)

$$\mathfrak{X}_1 \subset \mathfrak{X}_2 \subset \mathfrak{X}_3 \subset \cdots \subset \mathfrak{X}, \qquad \{f_\theta^{(\kappa)} : \kappa = 1, 2, \cdots\}$$

is for every $\theta \varepsilon \boldsymbol{\theta}$ a martingale sequence on the measure space $(\Xi, \mathfrak{X}, \mu)$ closed on the right by f_θ. Moreover it is easily seen that f_θ is the nearest $\mathfrak{X}$-measurable and μ-integrable function closing the sequence on the right. Hence, by the Martingale closure theorem ([3], page 394), we have for every $\theta \varepsilon \boldsymbol{\theta}$ $\lim_{\kappa \to \infty} f_\theta^{(\kappa)} = f_\theta$ a.e. μ. Finally, since ρ_κ and ρ are uniformly bounded, we have for every τ_n of the sequence $\boldsymbol{\theta}$, $\rho_\kappa(\tau_n) \to \rho(\tau_n)$ as $\kappa \to \infty$, and since the sequence $\boldsymbol{\theta}$ was arbitrary we conclude that

$$\lim_{\kappa \to \infty} \rho_\kappa(\tau) = \rho(\tau) \tag{4.41}$$

for every $\tau \varepsilon T_0$. However, $\rho, \rho_1, \rho_2, \cdots$ are continuous functionals on the compact T_0 so that the convergence in (4.41) is uniform in $\tau \varepsilon T_0$. Hence the last two terms in (4.38) both converge to zero uniformly in $\boldsymbol{\theta} \varepsilon \boldsymbol{\Theta}$ and the theorem is proved.

5. Concluding remarks. To the end, let us add a few remarks. First, the proof of Theorem 2 indicates that this theorem could have been proved under more general assumptions about the sequences $\{p_n\}$ and $\{N_\kappa\}$. In particular, we could have assumed that the sequence $\{p_n\}$ satisfies (4.3) of Theorem 1, and require the sequence $\{N_\kappa\}$ to increase fast enough so that the terms in (4.38) influenced by this sequence vanish. We have not done so mainly because the condition on this sequence would be rather involved.

Second, a question arises as to what happens if the sequence of parameters is not regular according to our definition. The following simple example illustrates that in this case the decision procedure $\mathfrak{D}^*$ may yield an arbitrary sequence of decisions and thus lose completely the desired property. Let $\Theta = \Xi = \{n : n = 1, 2, \cdots\}$ and $\nu_\theta(\{x\}) = 1$ if $\theta = x$. Then (discrete case) the argument of s^* in (3.9) is always zero so that $D_n = U_n V_n + (1 - U_n)\alpha_n$, where $\alpha_n \varepsilon \mathfrak{A}$ is arbitrary. On the other hand, if also $A = \{a^{(1)}, a^{(2)}\}$, $L(\theta, a^{(1)}) = 1$ for θ even, $L(\theta, a^{(2)}) = 1$ for θ odd, and $L(\theta, a) = 0$ otherwise, we have at the same time $\rho(\cdot) = 0$.

Next, let us mention that apart from rather different assumptions on the statistician's information, this paper is related to other works in the area (see [5], [6] and [7] for extensive references) so far as the result and basic principle of the proof are concerned. The main difference, however, consists in the interpretation as discussed in Section 1. In this, we attempted to carry on the idea of

"experience theory viewpoint" originated by A. Špaček [8] rather than to consider a population of decision problems as a compound problem—the idea brought about by H. Robbins [4]. A corresponding result for a game situation under assumptions similar to ours has been obtained recently by A. Baños [2] and may be interpreted in the same spirit. A special case of the problem studied in this paper, namely when there are no samples ("games against Nature"), has been investigated by the author [9], and a previous attempt has been made in [10] to generalize it; the result obtained in the latter is, however, incorrect.

Finally, it is evident that the decision procedure of Section 3 is not the only one for which the convergence can be established. More sophisticated and probably more efficient procedures (as for the rate of convergence) may be designed, which may be useful in possible applications. The purpose of this paper was to show that even a procedure based on the basic rules of learning from experience yields the desired result, and thus to justify the attitude toward the Bayesian model that has been discussed in Section 1.

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