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WEAK QUALITATIVE PROBABILITY ON FINITE SETS

BY PETER C. FISHBURN

Research Analysis Corporation

1. Introduction. Recent works on intuitive and subjective probability [3, 9, 12, 13, 16, 23, 24] give axioms for a binary relation $\precsim$ ("is not more probable than") on an algebra that imply the existence of a probability measure P on the algebra that strictly agrees $[A \precsim B \Leftrightarrow P(A) \leqq P(B)]$ with $\precsim$. Kraft, Pratt, and Seidenberg [9] were the first to present necessary and sufficient conditions for strict agreement when the set S of states is finite. Scott [16] rephrases these conditions.

This paper examines several finite-S axiomatizations that result in partial rather than strict agreement. They take $\prec$ ("is less probable than") as primitive. In all cases $\prec$ is asymmetric so that at most one of $A \prec B$ and $B \prec A$ holds for any $A, B \subseteq S$.

In the next section we shall consider the case where P almost agrees with $\prec$; $A \prec B \Rightarrow P(A) < P(B)$. Adams [1] gives necessary and sufficient conditions for this case. We shall also consider slightly stronger sufficient conditions that seem natural in the context of qualitative probability.

Section 3 presents even stronger conditions that yield a P and a $\sigma \geqq 0$ such that $A \prec B \Leftrightarrow P(A) + \sigma(A) < P(B)$. In connection with this we shall present a theorem similar to Stelzer's [19] that gives necessary and sufficient conditions for a P and a real number $0 \leqq \epsilon < 1$ such that $A \prec B \Leftrightarrow P(A) + \epsilon < P(B)$.

All our theorems are proved using a theorem of the alternative from linear algebra [2, 6, 21] whose broad applicability to relation–representation problems has been noted elsewhere [1, 5, 16, 22]. This theorem is in fact very efficient for uncovering conditions for numerical representation in linear systems, and it has been used in this way for the theorems of this paper. It is presented in Section 4 where proofs of two theorems of Section 3 are given.

Throughout, we define $A \sim B \Leftrightarrow$ (not $(A \prec B)$, not $(B \prec A)$). Our main divergence from the strict-agreement axioms [9, 16] is that we shall not assume that $\sim$ is transitive. This adds a dimension of reality to the theory of qualitative probability, and is an attempt to formalize the vagueness in judgment that Savage [14, 15] and others [7, 8, 18] have recognized. Now $A \sim B$ might have one of several interpretations, including the notion that A and B are equally probable, that there is not a definite feeling that A is less probable than B or vice versa, or that A and B are incomparable [7, 8]. Whatever the interpretation, an insistence that $\sim$ be transitive seems questionable. For example, suppose A, B, and C are the events "it will rain here within the next 48 hours," "it will rain here within the next 49 hours," and "a Republican will be elected President in 1980." Then $A \prec B$, $A \sim C$, $B \sim C$ might well apply for an individual.

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2. Almost-agreeing measures. Throughout we shall assume that S is finite. $\mathcal{S}$ is the set of all subsets of S. With $A \varepsilon \mathcal{S}$, $A(s) = 1$ means that $s \varepsilon A$; $A(s) = 0$ otherwise, and

$$(1) \qquad \sum_{j=1}^m A_j \geqq \sum_{j=1}^m B_j \Leftrightarrow \sum_{j=1}^m A_j(s) \geqq \sum_{j=1}^m B_j(s) \quad \text{for each} \quad s \varepsilon S.$$

$\sum_1^m A_j = \sum_1^m B_j$ if equality holds for each s on the right in (1).

Our first theorem is a particular application of Adam's [1] Theorem 1.3. His proof uses the theorem of the alternative.

THEOREM 1. *There is a probability measure P on $\mathcal{S}$ such that*

$$(2) \qquad A < B \Rightarrow P(A) < P(B), \qquad \text{for all} \quad A, B \varepsilon \mathcal{S},$$

if and only if, for all $A_1, A_2, \cdots, B_1, B_2, \cdots$ in $\mathcal{S}$ and $m \geqq 1$,

A1. $(\sum_1^m A_j \geqq \sum_1^m B_j,\ A_j < B_j$ *for each* $j < m) \Rightarrow$ *not* $(A_m < B_m)$.

Among other things, A1 implies that $<$ is irreflexive and asymmetric, that $A \subseteq B \Rightarrow (A < B \text{ or } A \sim B)$, and that $(A_1 < A_2, A_2 < A_3, \cdots, A_{m-1} < A_m) \Rightarrow (A_1 < A_m \text{ or } A_1 \sim A_m)$. A1 does not imply that $\varnothing < S$, or that $<$ is transitive, or that $A < C$ when $A \subseteq B$ and $B < C$.

To illustrate this last assertion suppose that $S = \{r, s, t\}$ and $< = \{\{r, s\} < \{t\}\}$. Then P almost agrees with $<$ whenever $P(r) + P(s) < P(t)$. But $(A \subseteq B, B < C) \Rightarrow A < C$ fails since $\{r\} \sim \{t\}$.

In qualitative probability it seems natural to suppose indeed that $(A \subseteq B, B < C) \Rightarrow A < C$, and this is reflected in the next theorem.

THEOREM 2. *There is a probability measure P on $\mathcal{S}$ that satisfies* (2) *if, for all $A, B, C, A_1, \cdots, B_1, \cdots$ in $\mathcal{S}$ and $m \geqq 1$,*

B1. $(\sum_1^m A_j = \sum_1^m B_j,\ A_j < B_j$ *for each* $j < m) \Rightarrow$ *not* $(A_m < B_m)$,

B2. $(A \subseteq B, B < C) \Rightarrow A < C$.

Although B1 is clearly necessary for (2), B2 is not. B1 is of course implied by A1, but the converse if false since B1 is not sufficient for (2). With $S = \{r, s, t\}$ and $< = \{\{r, s\} < \{t\}, \{r, t\} < \{s\}\}$, B1 holds, and any measure Q that satisfies (2) must have $Q(r)$ negative. Although B1 and B2 do not imply the companion of B2, $(A < B, B \subseteq C) \Rightarrow A < C$, this could be added to B1 and B2 with little (if any) loss in applicability.

Theorem 2 is easily proved using Theorem 1. Suppose A1 fails with $\sum_1^m A_j \geqq \sum_1^m B_j$ and $A_j < B_j$ for all j. It follows from (1) that there are $C_j \varepsilon \mathcal{S}$ such that $C_j \subseteq A_j$ for all j and $\sum_1^m C_j = \sum_1^m B_j$. Then, if B2 holds, $C_j < B_j$ for all j, which contradicts B1. Hence not A1 $\Rightarrow$ not (B1, B2), so that (B1, B2) $\Rightarrow$ A1.

3. More precise but still imperfect judgment. Our next set of conditions strengthens B1 and B2 and adds a new condition (C3). In strengthening B1 we shall use a binary relation $<^*$ on $\mathcal{S}$ defined from $<$ as follows:

$$(3) \qquad A \leq^* B \Leftrightarrow [C < A \Rightarrow C < B, \qquad \text{for all} \quad C \varepsilon \mathcal{S}],$$

$$(4) \qquad A \prec^* B \Leftrightarrow A \preceq^* B \quad \text{and not} \quad (B \preceq^* A).$$

In words, $A \prec^* B \Leftrightarrow$ any event less probable than A is also less probable than B and there is some event less probable than B that is not less probable than A.

The following theorem is proved in the next section.

THEOREM 3. *There is a probability measure P on $\mathcal{S}$ and a non-negative real valued function σ on $\mathcal{S}$ such that*

$$(5) \qquad A \prec B \Leftrightarrow P(A) + \sigma(A) < P(B), \qquad \text{for all} \quad A, B \,\varepsilon\, \mathcal{S},$$

if, for all $A, B, C, D, A_1, \cdots, B_1, \cdots$ in $\mathcal{S}$ and $m \geqq 1$,

C1. $(\sum_1^m A_j = \sum_1^m B_j, A_j \prec^* B_j$ *for each* $j < m) \Rightarrow$ *not* $(A_m \prec^* B_m)$,

C2. $(A \subseteq B, B \prec C) \Rightarrow A \prec C$; $(A \prec B, B \subseteq C) \Rightarrow A \prec C$,

C3. $(A \prec B, C \prec D) \Rightarrow A \prec D$ *or* $C \prec B$.

C4. *not* $(A \prec A)$.

In preceding cases, C4 was implied by A1 or B1, but it needs to be stated explicitly in the present case, since it is not implied by C1, C2, and C3. For example, with $S = \{s\}$, C1, C2, and C3 are seen to hold when $\prec \; = \{\emptyset \prec \emptyset, \{s\} \prec \{s\}, \emptyset \prec \{s\}$, and $\{s\} \prec \emptyset\}$.

Asymmetry and transitivity for $\prec$ follow from C3 and C4, so that $\prec$ is a strict partial order. When this is the case, $A \prec B \Rightarrow A \prec^* B$ so that C1 $\Rightarrow$ B1.

C1, C3, and C4 are necessary for (5) but C2 is not. To show the necessity of C1 suppose $\sum_1^m A_j = \sum_1^m B_j$ and $A_j \prec^* B_j$ for all j. Then, for each j, there is a $C_j \,\varepsilon\, \mathcal{S}$ such that $(C_j \prec B_j$, not $(C_j \prec A_j))$ by (3) and (4). Hence, by (5), $P(C_j) + \sigma(C_j) < P(B_j)$ and $P(A_j) \leqq P(C_j) + \sigma(C_j)$, so that $P(A_j) < P(B_j)$. Summing over j and using $\sum A_j = \sum B_j$ we get $0 < 0$. Hence C1 is necessary for (5).

To show that C2 is not necessary for (5) let $S = \{r, s\}$ with $P(r) = .4$, $P(s) = .6$, $\sigma(\emptyset) = .7$, $\sigma(r) = .1$, $\sigma(s) = .1$, $\sigma(S) = 0$, and define $\prec$ according to (5). Then $\{r\} \prec \{s\}$ since $P(r) + \sigma(r) < P(s)$, but not $(\emptyset \prec \{s\})$ since $\sigma(\emptyset) > P(s)$.

As noted above, the axioms of Theorem 3 imply those of Theorem 2. To show that the converse is false let $S = \{r, s, t\}$ with $\{r\} \prec \{s\}$, $\{t\} \prec \{r, s\}$, and $\prec$ applying elsewhere only when it can be deduced from these two with the aid of B2. Then, with $P(r) < P(s)$ and $0 < P(t) < P(r) + P(s)$, (2) and B1 and B2 hold, but both C3 and the second half of C2 fail.

Finally, an objection to C3, which is a counterpart of an axiom used in preference theory [4, 16, 17, 20], is in order. Let A and B be the rain events used before (within 48 hours, within 49 hours), and let C and D be similar to each other but rather different than A and B: for example, C = "it will snow in Chicago within 48 hours after noon on January 1, 1978," D = "it will snow in Chicago within 49 hours after noon on January 1, 1978." Then, with $A \prec B$ and $C \prec D$, it might happen that $A \sim D$ and $B \sim C$.

We now consider a theorem that is very similar to a theorem proved by Stelzer [19]. Because it differs slightly from his and its proof further illustrates the use of the theorem of the alternative, a proof will be given in the next section.

THEOREM 4. *There is a probability measure P on $\mathcal{S}$ and a real number ϵ, with $0 \leqq \epsilon < 1$ such that*

$$A \prec B \Leftrightarrow P(A) + \epsilon < P(B), \qquad \text{for all} \quad A, B \in \mathcal{S}, \tag{6}$$

if and only if, for all A, B, C, $A_1, \cdots, B_1, \cdots$ in $\mathcal{S}$ and all $m \geqq 1$,

D1. $(\sum_1^{2m} A_j = \sum_1^{2m} B_j$, *not* $(B_j \prec A_j)$ *for* $j = 1, \cdots, m$ *and* $A_j \prec B_j$ *for* $j = m + 1, \cdots, 2m - 1) \Rightarrow$ *not* $(A_{2m} \prec B_{2m})$,

D2. $(A \subseteq B, B \prec C) \Rightarrow A \prec C$; $(A \prec B, B \subseteq C) \Rightarrow A \prec C$,

D3. *not* $(\varnothing \prec \varnothing)$,

D4. $\varnothing \prec S$.

If (6) holds but D1 fails then $\sum_1^m P(A_j) \leqq \sum_1^m P(B_j) + m\epsilon$ and $\sum_{m+1}^{2m} P(A_j) + m\epsilon < \sum_{m+1}^{2m} P(B_j)$, which on adding and using $\sum_1^{2m} A_j = \sum_1^{2m} B_j$ yields $0 < 0$. Hence D1 is necessary for (6). D2, D3, and D4 are also clearly necessary. D4, which is required by (6) and $\epsilon < 1$, is not implied by preceding axioms.

With $(A_1, A_2) = (B_1, B_2) = (\varnothing, A)$ for D1, D1 and D3 imply not $(A \prec A)$, which is C4. $D2(=C2)$ and D3 imply not $(A \prec \varnothing)$, for if $A \prec \varnothing$ then $\varnothing \prec \varnothing$ by D2. D1 and D3 imply that $\prec$ is asymmetric on using $(A_1, \cdots, A_4) = (\varnothing, \varnothing, A, B)$ and $(B_1, \cdots, B_4) = (\varnothing, \varnothing, B, A)$ in D1. Suppose $A \prec B$ and $B \prec C$, and not $(A \prec C)$. Then with $((A_j)) = (\varnothing, C, A, B)$ and $((B_j)) = (\varnothing, A, B, C)$, D1 is contradicted when D3 holds. Hence D1 and D3 imply that $\prec$ is transitive.

Stelzer's interest in (6) stemmed in part from a similar (but nonadditive) model in preference theory that is based on Luce's [12] notion of a semiorder [16, 17, 20]. The semiorder axioms are irreflexivity, C3, and $(A \prec B, B \prec C) \Rightarrow (A \prec D$ or $D \prec C)$. The latter two follow readily from D1. Suppose C3 fails with $A \prec B$, $C \prec D$, not $(A \prec D)$, not $(C \prec B)$. Then D1 fails with $((A_j)) = (D, B, A, C)$ and $((B_j)) = (A, C, B, D)$. Suppose the other semiorder axiom fails with $A \prec B$, $B \prec C$, not $(A \prec D)$, not $(D \prec C)$. Then D1 fails with $((A_j)) = (D, C, A, B)$ and $((B_j)) = (A, D, B, C)$. Hence D1 and D3 imply that $\prec$ on $\mathcal{S}$ is a semiorder.

We have noted prior to Theorem 4 a case where C3 might not hold. The other semiorder axiom might very well fail in the situation where A, B, and C refer respectively to rain within 48, 49, and 50 hours, and D = "it will snow in Chicago within 49 hours after noon on January 1, 1978." It would not seem alarming if, for some individual, $A \prec B \prec C$ and $D \sim A$, $D \sim B$, $D \sim C$.

To conclude this section we state a theorem that is equivalent to those proved in [9, 16].

THEOREM 5. *There is a probability measure P on $\mathcal{S}$ such that*

$$A \prec B \Leftrightarrow P(A) < P(B), \qquad \text{for all} \quad A, B \in \mathcal{S},$$

if and only if, for all $s \in S$, all $A_1, \cdots, B_1, \cdots$ in $\mathcal{S}$ and all $m \geqq 2$,

E1. $(\sum_1^m A_j = \sum_1^m B_j, A_j \prec B_j$ *or* $A_j \sim B_j$ *for each* $j < m) \Rightarrow$ *not* $(A_m \prec B_m)$,

E2. *not* $(\{s\} \prec \varnothing)$,

E3. $\varnothing \prec S$.

4. Proofs for Theorems 3 and 4. A form of the theorem of the alternative

[1, 2, 6, 21] that is applicable to each of the theorems stated above is:

THEOREM OF THE ALTERNATIVE. *Suppose $a^1, \cdots, a^M$ are N-dimensional real vectors and $1 \leqq K \leqq M$. Then either there is an N-dimensional real vector ρ such that*

$$0 < \rho \cdot a^k \quad \text{for} \quad k = 1, \cdots, K,$$
$$0 = \rho \cdot a^k \quad \text{for} \quad k = K+1, \cdots, M,$$

or there are nonnegative numbers $r_1, \cdots, r_K$ at least one of which is positive and numbers $r_{K+1}, \cdots, r_M$ such that

$$\textstyle\sum_{k=1}^{M} r_k a_i^k = 0 \quad \text{for} \quad i = 1, \cdots, N.$$

Throughout the proofs of Theorems 3 and 4 we shall let n be the number of states in S with $S = \{s_1, \cdots, s_n\}$. Without further elaboration we note here that, in both theorems, the axioms imply that $\prec$ is a strict partial order, that $\varnothing \prec A$ or $\varnothing \sim A$ for each $A \in \mathcal{S}$, and that exactly one of $A \prec B$, $B \prec A$, $A \sim B$ holds for each $A, B \in \mathcal{S}$.

In both cases we shall argue that only the $0 < \rho \cdot a^k$ part (and not the $0 = \rho \cdot a^k$ part) of the first alternative needs to be used. For Theorem 4, which is proved first, $N = n + 1$, the extra dimension arising from ϵ. For Theorem 3, $N = n + 2^n$, where 2^n is the size of $\mathcal{S}$, the number of argument for σ.

SUFFICIENCY PROOF OF THEOREM 4. Throughout, D1 through D4 are assumed to hold.

Suppose (6) holds with $0 \leqq \epsilon < 1$, with

(7) $$P(A) + \epsilon < P(B) \qquad \text{for all } A, B \text{ such that } A \prec B,$$

(8) $$P(A) \leqq P(B) + \epsilon \quad \text{and} \quad P(B) \leqq P(A) + \epsilon \qquad \text{for all } A, B \text{ such that } A \sim B.$$

Now if any $\leqq$ in (8) is $=$, we can make it $<$ by increasing ϵ slightly without disturbing $<$ in (7) or $\epsilon < 1$. And if $\epsilon = 0$ the same slight increase can be made. Hence, if there is a (P, ϵ) solution then there is such a solution with $<$ holding in (8) and $\epsilon > 0$. If, in the latter solution, $P(s) = 0$ for one or more $s \in S$, we can alter P by making each such $P(s) > 0$ (but definitely $< \epsilon$) and reducing larger $P(r)$ by a compensating amount without disturbing any of the $<$ in (7) or in (8) modified. Hence (6) holds with $0 \leqq \epsilon < 1$, if and only if there is a (P, ϵ) such that $0 < \epsilon < 1$, $P(s) > 0$ for all s, and (7) and (8) hold with $<$ throughout.

These changes have been made to facilitate application of the Theorem of The Alternative. For its use we take $\rho = (p(s_1), \cdots, p(s_n), \epsilon)$ in $n + 1$ dimensions. For the theorem, (7) and (8) give rise to

(7*) $$0 < \rho \cdot a^k \quad \text{or} \quad \textstyle\sum_A p(s) + \epsilon < \sum_B p(s) \qquad (\text{for all } A \prec B)$$

(8*) $$\left.\begin{array}{l} 0 < \rho \cdot a^k \quad \text{or} \quad \sum_B p(s) < \sum_A p(s) + \epsilon \\ 0 < \rho \cdot a^k \quad \text{or} \quad \sum_A p(s) < \sum_B p(s) + \epsilon \end{array}\right\} \qquad (\text{for all } A \sim B)$$

where k runs through successive, nonrepeating positive integers, each a^k is an $n + 1$ dimensional vector of -1's, 0's, and 1's, and (8^*) includes all $A \sim A$ cases. Since we know that if there is a solution then there is one in which $P(s) > 0$ for all s, we add the following to (7^*) and (8^*):

$$(9^*) \qquad 0 < \rho \cdot a^k \quad \text{or} \quad 0 < p(s) \qquad (\text{for all } s \text{ such that } \varnothing \sim \{s\}).$$

If (7^*) through (9^*) has a solution then it must be true that $0 < \epsilon$ on using $A \sim A$ in (8^*), and that $p(s) > 0$ for all s in S by (7^*) and (9^*) and the fact that $\varnothing \prec \{s\}$ or $\varnothing \sim \{s\}$ for each s. Normalizing the system by dividing everything through by $\sum_S p(s)$, so that the sum of the new p's equals one, we have that the new ϵ (i.e., the old ϵ divided by $\sum_S p(s)$) must be less than one by (7^*) since $\varnothing \prec S$ as in D4.

It remains to show that $(7^*) - (9^*)$ has a ρ solution. Suppose that there is no ρ solution. Then, by the Theorem of The Alternative, there are nonnegative numbers $r_1, r_2, \cdots, r_k, \cdots, r_K$ (where K is the total number of statements in (7^*) through (9^*)), at least one of which is positive, such that

$$(10) \qquad \textstyle\sum_{k=1}^{K} r_k a_i^k = 0 \qquad \text{for} \quad i = 1, \cdots, n+1.$$

Because of the finiteness of the system and the rationality of the a_i^k, rational and hence integer r_k satisfy (10). We can then view r_k as the number of times the kth inequality in $\{0 < \rho \cdot a^k : k = 1, \cdots, K\}$ comes into play in (10). Now since $i = n + 1$ in (10) refers to ϵ, and $a_{n+1}^k = -1$ when k applies to (7^*), and $a_{n+1}^k = 1$ when k applies to (8^*), the sum of the r_k for (7^*) is the same as the sum of the r_k for (8^*). Let this common sum equal m. Let the m $A \sim B$ statements (with possible repeats when $r_k > 1$) for (8^*) be $A_1 \sim B_1, \cdots, A_m \sim B_m$ with the convention that the ordering of pairs designates which half of (8^*) is used: that is, $A_j \sim B_j$ refers to $\sum_{A_j} p(s) < \sum_{B_j} p(s) + \epsilon$ in (8^*). Let the m $A \prec B$ statements for (7^*) be $A_{m+1} \prec B_{m+1}, \cdots, A_{2m} \prec B_{2m}$. Finally, let t be the sum of the r_k that apply to (9^*), with the corresponding statements $\varnothing \sim \{s^1\}, \cdots, \varnothing \sim \{s^t\}$ where each $s^j \varepsilon S$ but s^j may refer to any one of the s_i.

It is easily seen that $m \geqq 1$, for if $m = 0$ then $t > 0$ and (10) would necessarily fail for any i whose s_i was involved in a (9^*) statement for which $r_k > 0$. Hence, by the failure of a solution to (7^*)–(9^*) we obtain $m \geqq 1$ and

$$A_1 \sim B_1, \cdots, A_m \sim B_m, \qquad A_{m+1} \prec B_{m+1}, \cdots, A_{2m} \prec B_{2m},$$
$$\varnothing \sim \{s^1\}, \cdots, \varnothing \sim \{s^t\}$$

where, by (10) for each $i < n + 1$, we have

$$(11) \qquad \textstyle\sum_{j=1}^{2m} A_j(s_i) = \sum_{j=1}^{2m} B_j(s_i) + \sum_{j=1}^{t} \{s^j\}(s_i), \quad i = 1, \cdots, n,$$

so that $\sum A_j = \sum B_j + \sum \{s^j\}$. If $t = 0$ we have an obvious violation of D1. To show a violation of D1 when $t > 0$ we need to get rid of the $\{s^j\}$.

Suppose then that $t > 0$. Using $\sum A_j = \sum B_j + \sum \{s^j\}$, we can get rid of one $\{s^j\}$ at a time by reducing an A_j that contains the s_i corresponding to $\{s^j\}$ to

$A_j - \{s_i\}$. Consider the first reduction with $\{s_i\} = \{s^1\}$. Suppose first that $s_i \varepsilon A_j$ for $m < j \leqq 2m$. Then, by D2 and $A_j \prec B_j$, $A_j - \{s_i\} \prec B_j$. Suppose next that $s_i \varepsilon A_j$ for $j \leqq m$, with $A_j \sim B_j$. We want not $(B_j \prec A_j - \{s_i\})$. To the contrary suppose $B_j \prec A_j - \{s_i\}$. Then, by the other half of D2, $B_j \prec A_j$, which is false. Hence not $(B_j \prec A_j - \{s_i\})$. Continuing this for each successive $\{s^j\}$ it is clear that we arrive at

$$\text{not } (B_j \prec A_j') \qquad j = 1, \cdots, m$$
$$A_j' \prec B_j \qquad j = m + 1, \cdots, 2m$$
$$\sum A_j' = \sum B_j,$$

which violates D1.

We have thus obtained a contradiction to the supposition that there is no ρ solution to (7^*)–(9^*) when D1 through D4 hold, and the proof is complete.

Proof of Theorem 3. Throughout, C1 through C4 are assumed to hold. With $\preceq^*$ and $\prec^*$ defined on $\mathcal{S}$ as in (3) and (4), suppose not $(A \preceq^* B)$ and not $(B \preceq^* A)$. Then there are $C, D \,\varepsilon\, \mathcal{S}$ such that $\{C \prec A, \text{not } (C \prec B), D \prec B, \text{not } (D \prec A)\}$, which contradicts C3. Hence $\preceq^*$ on $\mathcal{S}$ is connected and not $(A \preceq^* B) \Rightarrow B \prec^* A$. Suppose $A \sim B$ and $B \prec C$. Then, if $C \preceq^* A$, (3) implies $B \prec A$, which contradicts $A \sim B$. Hence

$$(A \sim B, B \prec C) \Rightarrow A \prec^* C. \tag{12}$$

We shall use this later in the proof.

Suppose (5) holds so that

(13) $\quad P(A) + \sigma(A) < P(B) \qquad$ for all A, B such that $A \prec B$,

(14) $\quad P(A) \leqq P(B) + \sigma(B)$ and $P(B) \leqq P(A) + \sigma(A)$

for all A, B such that $A \sim B$.

Since $A \sim A$, (14) gives $\sigma(A) \geqq 0$ for all $A \,\varepsilon\, \mathcal{S}$. By an analysis like that used in the proof of Theorem 4 it is clear that if (13) and (14) hold then they hold also for some (P, σ) for which $\sigma(A) > 0$ for all $A \,\varepsilon\, \mathcal{S}$, $P(s) > 0$ for all $s \,\varepsilon\, S$, and every $\leqq$ in (14) is $<$. We shall use this fact in applying the Theorem of The Alternative.

For that theorem we take

$$\rho = (p(s_1), \cdots, p(s_n), \sigma(\varnothing), \ \sigma(\{s_1\}), \cdots, \sigma(\{s_n\}), \ \sigma(\{s_1, s_2\}), \cdots, \sigma(S))$$

in $n + 2^n$ dimensions. With the changes noted above, (13) and (14) give rise to

(13*) $\quad 0 < \rho \cdot a^k \quad$ or $\quad \sum_A p(s) + \sigma(A) < \sum_B p(s)$

(for all $A \prec B$)

(14*) $\quad 0 < \rho \cdot a^k \quad$ or $\quad \sum_A p(s) < \sum_B p(s) + \sigma(B)$

(for all $A \sim B$)

$\quad 0 < \rho \cdot a^k \quad$ or $\quad \sum_B p(s) < \sum_A p(s) + \sigma(A)$

to which we add

$$(15^*) \qquad 0 < \rho \cdot a^k \quad \text{or} \quad 0 < p(s) \qquad \text{(for all } s \text{ such that } \varnothing \sim \{s\}).$$

In (13^*)–(15^*) k runs through successive, nonrepeating positive integers and each a^k is an $n + 2^n$ dimensional vector of -1's, 0's, and 1's. (13^*) and (15^*) give $p(s) > 0$ for all s, and (14^*) implies $0 < \sigma(A)$ for all $A \varepsilon \mathcal{S}$. If (13^*)–(15^*) has a ρ solution, normalization leads to a corresponding solution for (5).

It remains to show that (13^*)–(15^*) has a ρ solution. To the contary, suppose there is no ρ solution. It then follows from the Theorem of The Alternative and the rationality of the a_i^k that there are nonnegative integers $r_1, \cdots, r_k, \cdots, r_K$ (where K is the total number of statements in (13^*)–(15^*)), at least one of which is positive, such that

$$(16) \qquad \textstyle\sum_{k=1}^K r_k a_i^k = 0 \qquad i = 1, \cdots, n, n+1, \cdots, n+2^n.$$

From (13^*) and (14^*) and $i > n$ in (16) it follows that the number of statements from (13^*) out of the total of $\sum r_k$ that have a given $\sigma(A)$ on the left of $\sum_A p(s) + \sigma(A) < \sum_B p(s)$ is equal to the number from (14^*) that have the same $\sigma(A)$ on the right of $\sum_B p(s) < \sum_A p(s) + \sigma(A)$. Along with (16) for $i \leqq n$ this means that (16) implies that, for some $m \geqq 1$, there are $A_1, \cdots, A_m, B_1, \cdots, B_m, A_{m+1}, \cdots, A_{2m}, B_{m+1}, \cdots, B_{2m}$, and $\{s^1\}, \cdots, \{s^t\}$ (the latter for the positive r^k, if any, for k in (15^*)) such that $A_j \prec B_j$ for $j \leqq m$, $A_j \sim B_j$ for $m < j \leqq 2m$, $A_j = B_{j+m}$ for $j = 1, \cdots, m$, $\varnothing \sim \{s^j\}$ for $j = 1, \cdots, t$, and

$$\textstyle\sum_1^{2m} A_j = \sum_1^{2m} B_j + \sum_1^t \{s^j\},$$

where each s^j is one of the s_i. Using (12) as derived at the beginning of this proof it follows that

$$A_{j+m} \prec^* B_j \qquad j = 1, \cdots, m$$
$$\varnothing \sim \{s^j\} \qquad j = 1, \cdots, t$$
$$\textstyle\sum_{j=1}^m A_{j+m} = \sum_{j=1}^m B_j + \sum_{j=1}^t \{s^j\}.$$

Using the last expression we can eliminate the $\{s^j\}$ by taking them out of the A_{j+m}. Letting A'_{j+m} denote the reduced A_{j+m} we then have $\sum_1^m A'_{j+m} = \sum_1^m B_j$. Using both parts of C2 it is easily seen that $(s \varepsilon A, A \prec^* B) \Rightarrow (A - \{s\}) \prec^* B$. Hence $A'_{j+m} \prec^* B_j$ for $j = 1, \cdots, m$, and we have arrived at a contradiction of C1. Thus, (13^*)–(15^*) has a ρ solution.

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