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ASYMPTOTIC NORMALITY OF SIMPLE LINEAR RANK STATISTICS UNDER ALTERNATIVES II

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0. Summary. This is a straightforward continuation of Hájek (1968). We provide a further extension of the Chernoff-Savage (1958) limit theorem. The requirements concerning the scores-generating function are relaxed to a minimum: we assume that this function is a difference of two non-decreasing and square integrable functions. Thus, in contradistinction to Hájek (1968), we dropped the assumption of absolute continuity. The main results are accumulated in Section 2 without proofs. The proofs are given in Sections 4 through 7. Section 3 contains auxiliary results.

1. Introduction. Basic tools used in this paper are the same as in Hájek (1968), namely the variance inequality (Theorem 3.1) and the Projection Lemma 4.1. The main trouble to overcome was the treatment of the scores-generating function, which has just one jump and is constant otherwise (Theorem 1). The solution of this seemingly simple problem took three Sections—3, 4 and 5. The remaining theorems were then obtained relatively easily by combining Theorem 1 with the results of Hájek (1968).

Since we have loosened the conditions concerning the scores-generating function, we had to introduce additional smoothness requirements concerning the distribution functions F_{Ni} of individual observations. Without it the theorems of Hájek (1968) do not hold for discrete scores-generating functions, as is there illustrated by a counter example.

In framing the theorems we had to balance two requirements: the generality of conditions and the readiness for applications. To satisfy both we presented six variants of conditions under which the conclusion of Theorem 1 holds.

If the scores-generating function may be discrete, the assumptions become more complex. For this reason we had to abandon the ϵ-form used in Hájek (1968), and to switch to limiting theorems concerning sequences. To simplify the notation, the sequences are indexed by the sample size, though, in principle, the sample size could be made a function of a new index as well.

Pyke and Shorack (1968) obtained results similar to ours for the two-sample and c-sample problems and for a slightly less general scores-generating function, and they provide a simplified expression for the asymptotic expectation, which is not attempted in the present paper. Their approach, completely different from ours, is based on convergence properties of certain stochastic processes and has some common points with an earlier paper by Govindarajulu–LeCam–Raghavachari (1967).

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2. Main results. For every $N \geqq 1$, let $R_{N1}, \cdots, R_{NN}$ denote the ranks of independent random variables $X_{N1}, \cdots, X_{NN}$. Choose some scores $a_N(1), \cdots, a_N(N)$ and regression constants $c_{N1}, \cdots, c_{NN}$, and put

$$S_N = \sum_{i=1}^N c_{Ni} a_N(R_{Ni}). \tag{2.1}$$

We inquire under what conditions the linear rank statistics S_N are asymptotically normal.

We shall assume that the c_{Ni}'s satisfy either the *Noether* condition

$$\lim_{N\to\infty} \max_{1\leqq i\leqq N} (c_{Ni} - \bar{c}_N)^2 / \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 = 0, \tag{2.2}$$

or, more stringently, the *boundedness* condition

$$\limsup_{N\to\infty} N \max_{1\leqq i\leqq N} (c_{Ni} - \bar{c}_N)^2 / \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 < \infty. \tag{2.3}$$

The scores will be generated by a function $\varphi(t)$, $0 < t < 1$, which is representable as a difference of two nondecreasing, square integrable functions:

$$\varphi(t) = \varphi_1(t) - \varphi_2(t), \tag{2.4}$$

φ_i nondecreasing, $\int_0^1 \varphi_i^2\, dt < \infty$. The scores are obtained either by interpolation,

$$a_N(i) = \varphi(i(N+1)^{-1}), \qquad 1 \leqq i \leqq N, \tag{2.5}$$

or by a procedure satisfying

$$\sum_{i=1}^N |a_N(i) - \varphi(i(N+1)^{-1})| = O(1), \tag{2.6}$$

or, more loosely,

$$\sum_{i=1}^N |a_N(i) - \varphi(i(N+1)^{-1})| = o(N^{\frac{1}{2}}). \tag{2.7}$$

The distribution functions of $X_{N1}, \cdots, X_{NN}$, say $F_{N1}, \cdots, F_{NN}$, will be assumed *continuous*. We shall put

$$H_N(x) = N^{-1} \sum_{i=1}^N F_{Ni}(x), \qquad -\infty < x < \infty, \tag{2.8}$$

$$H_N^{-1}(t) = \inf\{x : H_N(x) > t\}, \qquad 0 < t < 1, \tag{2.9}$$

and

$$L_{Ni}(t) = F_{Ni}(H_N^{-1}(t)), \qquad 0 < t < 1. \tag{2.10}$$

It is easy to show that

$$N^{-1} \sum_{i=1}^N L_{Ni}(t) = t, \qquad 0 < t < 1, N \geqq 1. \tag{2.11}$$

In the sequel, v will denote a jump point of the scores-generating function φ or, more generally, a point of some set containing the singular set of φ. We shall need a sort of uniform differentiability of the functions L_{Ni} at this point (these points), which will appear as a combination of two requirements. First, for such a $v \varepsilon (0, 1)$ and for every $K > 0$, $K' > 0$

$$\max_{1\leqq i\leqq N, KN^{-\frac{1}{2}} \leqq |t-v| \leqq K'N^{-\frac{1}{2}}\log^{\frac{1}{2}} N} |(L_{Ni}(t) - L_{Ni}(v))/(t - v)| = O(1). \tag{2.12}$$

Second, we shall assume that the L_{Ni}'s are uniformly approximately linear in the vicinity of the point $v \varepsilon (0, 1)$ in the following sense: there exist numbers $l_{Ni}(v)$ such that for every $K > 0$

(2.13) $$\max_{1 \leqq i \leqq N, |t-v| \leqq KN^{-\frac{1}{2}}} |L_{Ni}(t) - L_{Ni}(v) - (t - v)l_{Ni}(v)| = o(N^{-\frac{1}{2}}).$$

If (2.13) is satisfied for some l_{Ni}'s, then setting $t = N^{-\frac{1}{2}} + v$ in (2.13) we see that it is also satisfied for $l^*_{Ni}(v) = N^{\frac{1}{2}}[L_{Ni}(v + N^{-\frac{1}{2}}) - L_{Ni}(v)]$. Consequently, in view of (2.11), we may assume

(2.14) $$N^{-1} \sum_{i=1}^{N} l_{Ni}(v) = 1$$

in (2.13) without any loss of generality. For the same reason, we may assume that $l_{Ni}(v)$ as a function of v is measurable. Further it is important to note that (2.13) neither implies nor is implied by the existence of the derivative $L'_{Ni}(v)$ at the point v. Consequently we generally cannot put $l_{Ni}(v) = L'_{Ni}(v)$, even if the latter number exists. Of course, if the derivative exists *uniformly* with respect to i and N, then (2.12) and (2.13) are satisfied with $l_{Ni}(v) = L'_{Ni}(v)$. The somewhat artificial looking conditions (2.12) and (2.13) result from our method of proving Theorem 1 and from our recognition that the assumption of uniform differentiability would be too restrictive (see Theorem 4 for example).

Other conditions concerning the L_{Ni}'s used in the sequel are as follows:

(2.15) $$\liminf_{N\to\infty} N^{-1} \sum_{i=1}^{N} L_{Ni}(v)[1 - L_{Ni}(v)] > 0,$$

(2.16) $$\liminf_{N\to\infty} \min_{1 \leqq i \leqq N} L_{Ni}(v)[1 - L_{Ni}(v)] > 0.$$

Obviously, (2.16) entails (2.15).

For applications, it is necessary to replace (2.12), (2.13) and (2.15) by some more feasible conditions. For example, we may assume that the distribution functions F_{Ni} possess densities f_{Ni} such that for some open interval $(\alpha, \beta) \subset (-\infty, \infty)$ the following holds:

(2.17)

(a) $f_{Ni}(x) = 0$ for $x < \alpha (x > \beta)$, if $\alpha(\beta)$ is finite;
(b) $f_{Ni}(x)$ are continuous on every compact subinterval of (α, β), uniformly in (x, N, i);
(c) for every compact interval $C \subset (\alpha, \beta)$ there exists an $\epsilon > 0$ such that for all $N \geqq 1$, N^{-1} card $\{i: \inf_{x \varepsilon C} f_{Ni}(x) > \epsilon\} > \epsilon$;
(d) $\alpha < \liminf_{N\to\infty} H_N^{-1}(t) \leqq \limsup_{N\to\infty} H_N^{-1}(t) < \beta$ for all $t \varepsilon (0, 1)$; card$A$ stands for the number of elements of the set A.

In particular, (2.17) is satisfied for

(2.18)

(a) $f_{Ni}(x) = f(x - d_{Ni})$;
(b) $f(x)$ is uniformly continuous and positive on $(-\infty, \infty)$;
(c) for every $\epsilon > 0$ there exists a compact interval C such that for all $N \geqq 1$, N^{-1} card $\{i: |d_{Ni}| \not\varepsilon C\} < \epsilon$.

Obviously, (2.18) is entailed by

(2.19) (a) $f_{Ni}(x) = f(x - d_{Ni})$;
(b) $f(x)$ is uniformly continuous and positive on $(-\infty, \infty)$;
(c) $\sup_{i,N} |d_{Ni}| < \infty$.

The last group of conditions concerns the nondegeneration of $\text{Var}\, S_N$. The milder form is

$$\liminf_{N\to\infty} \text{Var}\, S_N / \textstyle\sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 > 0, \tag{2.20}$$

the stricter form is

$$\liminf_{N\to\infty} \text{Var}\, S_N / N \max_{1\leqq i\leqq N} (c_{Ni} - \bar{c}_N)^2 > 0. \tag{2.21}$$

Obviously (2.21) entails (2.20). Note that

$$\text{Var}\, S_N = (N-1)^{-1} \textstyle\sum_{i=1}^N (a_N(i) - \bar{a}_N)^2 \cdot \sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2$$
$$\text{if} \quad F_{N1} = F_{N2} = \cdots = F_{NN},$$

which entails $\text{Var}\, S_N = O(\sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2)$ under (2.4) and (2.5) or (2.6) or (2.7). If the regression constants c_{Ni} are bounded in sense of (2.3), then, obviously, (2.20) and (2.21) are equivalent.

Alternatively we may assume that (2.20) and (2.21) hold with $\text{Var}\, S_N$ replaced by some approximative variance σ_N^2:

$$\liminf_{N\to\infty} \sigma_N^2 / \textstyle\sum_{i=1}^N (c_{Ni} - \bar{c}_N)^2 > 0, \tag{2.22}$$

$$\liminf_{N\to\infty} \sigma_N^2 / N \max_{1\leqq i\leqq N} (c_{Ni} - \bar{c}_N)^2 > 0. \tag{2.23}$$

THEOREM 1. *Consider statistics* (2.1) *with scores satisfying* (2.6), *where*

$$\begin{aligned} \varphi(t) &= 0 \qquad 0 < t < v, \\ &= 1 \qquad v \leqq t < 1. \end{aligned} \tag{2.24}$$

Put

$$\tilde{c}_N(v) = N^{-1} \textstyle\sum_{i=1}^N c_{Ni} l_{Ni}(v) \tag{2.25}$$

and

$$\sigma_N^2 = \textstyle\sum_{i=1}^N (c_{Ni} - \tilde{c}_N(v))^2 L_{Ni}(v)[1 - L_{Ni}(v)]. \tag{2.26}$$

Then S_N is asymptotically normal with parameters (ES_N, $\text{Var}\, S_N$) *and also* (ES_N, σ_N^2), *if any of the following sets of conditions is satisfied:*

C_1 : (2.2), (2.12), (2.13), (2.15), (2.20) *or* (2.22)
C_2 : (2.2), (2.12), (2.13), (2.16)
C_3 : (2.2), (2.17), (2.20) *or* (2.22)
C_4 : (2.2), (2.18), (2.20) *or* (2.22)
C_5 : (2.2), (2.19)
C_6 : (2.3), (2.12), (2.13), (2.20) *or* (2.22).

PROOF. Asymptotic normality of S_N will be shown by first showing that S_N is asymptotically equivalent to its projection $\hat{S}_N$ onto a space of linear statistics (see Sections 3 and 4) and then showing that $\hat{S}_N$ is asymptotically equivalent to a sum of independent random variables $\sum_{i=1}^{N} Z_{Ni}$ (see Section 5) to which the Lindeberg central limit theorem applies. The components of σ_N^2 in (2.26) are the variances of the Z_{Ni}'s. Moreover, for this choice of σ_N^2, conditions (2.20) and (2.22) will be shown to be equivalent.

EXAMPLE. If we employ conditions (2.2) and (2.19) we can see that in the two-sample location case the median test statistic is asymptotically normal if (a) the underlying density is uniformly continuous and positive on $(-\infty, \infty)$, (b) the difference of the location parameters remains bounded, (c) both the sample sizes converge to infinity.

On combining Theorem 1 with Theorem 2.3 of Hájek (1968), we obtain a powerful result for the case of bounded regression constants (see (2.3)). Before formulating it, let us recall that every nondecreasing function may be decomposed into an absolutely continuous part and a singular part. (The singular part includes both the jump and singular continuous components.) We shall have $\varphi = \varphi_1 - \varphi_2$, where each φ_i is nondecreasing. Let us denote their absolutely continuous and singular parts by φ_i^{ac} and φ_i^{s}, respectively:

$$\varphi_i(t) = \varphi_i^{ac}(t) + \varphi_i^{s}(t), \quad i = 1, 2, \quad 0 < t < 1. \tag{2.27}$$

Then put

$$\varphi_{ac} = \varphi_1^{ac} - \varphi_2^{ac}, \quad \varphi_s = \varphi_1^{s} - \varphi_2^{s}. \tag{2.28}$$

To every nondecreasing φ there corresponds uniquely a measure ν such that $\nu\{(a, b)\} = \varphi(b) - \varphi(a)$, if a and b are continuity points of φ. We shall understand by $\int_A d\varphi$ and $\int h\, d\varphi$ the expressions $\nu(A)$ and $\int h\, d\nu$, respectively.

THEOREM 2. *Consider statistics* (2.1) *with scores satisfying* (2.7), *where* φ *fulfills* (2.4). *Denote by* A *the set of values* $v \in (0, 1)$ *for which at least one of the conditions* (2.12), (2.13) *and* (2.15) *is not satisfied, and assume that there is measurable set* $B \supset A$ *such that*

$$\int_B (d\varphi_1^{s}(v) + d\varphi_2^{s}(v)) = 0. \tag{2.29}$$

Put

$$\sigma_N^2 = \sum_{i=1}^{N} \operatorname{Var}\left[\int_{\frac{1}{2}}^{H_N(X_{Ni})} (\tilde{c}_N(v) - c_{Ni})\, d\varphi(v)\right] \tag{2.30}$$

where

$$\tilde{c}_N(v) = N^{-1} \sum_{j=1}^{N} c_{Nj} l_{Nj}(v), \tag{2.31}$$

if $v \not\in B$, *with the* l_{Nj}*'s chosen so as to be measurable and satisfy* (2.14), *and*

$$\tilde{c}_N(v) = N^{-1} \sum_{j=1}^{N} c_{Nj} L'_{Nj}(v), \tag{2.32}$$

if $v \in B$ *but the derivative* $L'_{Nj}(v)$ *exists.*

Then S_N *is asymptotically normal with parameters* $(ES_N, \operatorname{Var} S_N)$ *and also* (ES_N, σ_N^2) *provided either* (2.21) *or* (2.23) *holds, with* σ_N^2 *given by* (2.30).

PROOF. See Section 6.

REMARK 1. Since the L_{Nj} are absolutely continuous, and since (2.29) holds, $\tilde{c}_N(v)$ is defined almost everywhere with respect to the measure induced by $\varphi_1 + \varphi_2$. Consequently, the integral (2.30) is well-defined.

REMARK 2. Now (2.11), entailing $N^{-1} \sum_{i=1}^{N} L'_{Ni}(t) = 1$, and (2.14) imply $\tilde{c}_N(v)$ always is a weighted average of $c_{N1}, \cdots, c_{NN}$. Consequently,

$$|\tilde{c}_N(v) - c_{Ni}| \leqq 2 \max_{1 \leqq j \leqq N} |c_{Nj} - \bar{c}_N|, \tag{2.33}$$

where $\bar{c}_N = N^{-1} \sum_{j=1}^{N} c_{Nj}$.

REMARK 3. If (2.17) or (2.18) or (2.19) holds, then we may put

$$l_{Ni}(v) = L'_{Ni}(v) = f_{Ni}(H_N^{-1}(v))/h_N(H_N^{-1}(v));$$

(see the part $(C_3) \Rightarrow (C_1)$ of the proof of Theorem 1 in Section 5). Consequently,

$$\tilde{c}_N(v) = \sum_{j=1}^{N} c_{Nj} f_{Nj}(H_N^{-1}(v)) / \sum_{j=1}^{N} f_{Nj}(H_N^{-1}(v)), \qquad 0 < v < 1. \tag{2.34}$$

REMARK 4. As is shown at the end of Section 6, the conditions of Theorem 2 entail that the regression constants c_{Ni} are bounded in the sense of (2.3).

The following theorem, based on Theorem 1 and on Theorem 2.1 of Hájek (1968), combines unbounded c_{Ni} with a class of bounded scores-generating functions.

THEOREM 3. *Consider statistics* (2.1) *with scores satisfying* (2.6), *and assume that* $\varphi = \varphi_1 + \varphi_2$, *where* φ_1 *is constant but for a finite number of jumps and* φ_2 *has a bounded second derivative. Assume that* (2.2) *holds. Further assume that* (2.17), *or* (2.18), *or* (2.19) *is true.*

Then S_N *is asymptotically normal with parameters* $(ES_N, \operatorname{Var} S_N)$ *and also* (ES_N, σ_N^2), *provided* (2.20) *or* (2.22) *holds, with* σ_N^2 *given by* (2.30) *and* $\tilde{c}_N$ *given by* (2.34).

PROOF. The proof follows easily from the proofs of our Theorem 1 and Theorem 2.1 of Hájek (1968), since S_N may be represented as a sum of a statistic considered in the latter theorem and a linear combination of statistics considered in the former one. We approximate each component statistic by a sum of independent random variables (denoted by Z_{Ni} in Hájek (1968) and in our Section 5) and then we bound the variance of the difference by a multiple of the sum of the bounds for residual variances of individual terms. It will be clear that (2.30) is the right formula for σ_N^2 if you note that (2.30) reduces to (2.26) when φ is the unit step function.

In conclusion, we formulate two theorems concerning the important two-sample case, in which

$$\begin{aligned} F_{Ni}(x) &= F(x) \qquad 1 \leqq i \leqq m_N, \\ &= G(x) \qquad m_N < i \leqq N; \end{aligned} \tag{2.35}$$

and

$$\begin{aligned} c_{Ni} &= 1 \qquad 1 \leqq i \leqq m_N, \\ &= 0 \qquad m_N < i \leqq N. \end{aligned} \tag{2.36}$$

Then, obviously,

$$S_N = \sum_{i=1}^{m_N} a_N(R_{Ni}). \tag{2.37}$$

THEOREM 4. *Consider the statistics S_N under the two-sample case* (2.35) *and* (2.36). *Assume that φ satisfies* (2.4), *the scores $a_N(i)$ satisfy* (2.7) *and that for some fixed λ_0*

$$m_N/N = \lambda_0 + O(N^{-\frac{1}{2}}). \qquad (0 < \lambda_0 < 1) \tag{2.38}$$

Put $H_0 = \lambda_0 F + (1 - \lambda_0)G$, $L_0 = FH_0^{-1}$ and $M_0 = GH_0^{-1}$, where FH_0^{-1} denotes the composition of functions F and H_0^{-1} and GH_0^{-1} has a similar meaning. Denote by B the set of values $t \in (0, 1)$ for which the derivatives $L_0'(t)$, $M_0'(t)$ do not exist. Assume that

$$\int_B (d\varphi_1^s + d\varphi_2^s) = 0, \tag{2.39}$$

where φ_i^s denotes the singular part of φ_i. Put

$$\tau_0^2 = 2\lambda_0(1 - \lambda_0) \int_0^1 \int_0^w \{(1 - \lambda_0)M_0'(v)M_0'(w)L_0(v)[1 - L_0(w)] + \lambda_0 L_0'(v)L_0'(w)M_0(v)[1 - M_0(w)]\} \, d\varphi(v) \, d\varphi(w), \tag{2.40}$$

and postulate that $\tau_0^2 > 0$.

Then S_N is asymptotically normal with parameters (ES_N, $\operatorname{Var} S_N$) *and also* (ES_N, $N\tau_0^2$).

PROOF. See Section 7.

REMARK 5. The expression τ_0^2 is a λ_0^2-multiple of expression (4.4) in Pyke and Shorack (1968). The statistic T^* considered there is linearly connected with S_N: $S_N = \lambda_0 N^{\frac{1}{2}} T^* +$ constant.

REMARK 6. Since L_0 and M_0 both are absolutely continuous, their derivative exists a.e. with respect to the Lebesgue measure, and, in turn, with respect to the measure generated by $\varphi_1^{ac} + \varphi_2^{ac}$. Consequently, (2.39) entails

$$\int_B (d\varphi_1 + d\varphi_2) = 0 \tag{2.41}$$

and (2.40) is well-defined.

THEOREM 5. *Consider the two-sample case* (2.35) *and* (2.36). *Assume that φ satisfies* (2.4) *and $a_N(i)$ satisfy* (2.7). *Furthermore assume that*

$$0 < \liminf_{N\to\infty} m_N/N \leqq \limsup_{N\to\infty} m_N/N < 1 \tag{2.42}$$

and that the densities $f = F'$ and $g = G'$ exist and $f(x) + g(x) > 0$ for $x \in (\alpha, \beta)$, whereas $f(x) + g(x) = 0$ for $x \notin [\alpha, \beta]$. Let f and g be continuous on (α, β). Put $L_N = FH_N^{-1}$ and $M_N = GH_N^{-1}$, and

$$\tau_N^2 = 2\lambda_N(1 - \lambda_N) \int_0^1 \int_0^w \{(1 - \lambda_N)M_N'(v)M_N'(w)L_N(v)[1 - L_N(w)] + \lambda_N L_N'(v)L_N'(w)M_N(v)[1 - M_N(w)]\} \, d\varphi(v) \, d\varphi(w) \tag{2.43}$$

where $\lambda_N = m_N/N$. Assume that

$$\liminf \tau_N^2 > 0. \tag{2.44}$$

Then S_N *is asymptotically normal with parameters* $(ES_N, \operatorname{Var} S_N)$ *and also* $(ES_N, N\tau_N^2)$.

PROOF. See Section 7.

REMARK 7. There are many further possible variants of the above theorems. For example, if we assume that $\varphi(t)$ is continuous at $t = \frac{1}{2}$ and the densities f_{Ni} are *symmetric* with respect to a fixed point x_0, then (2.17) may be relaxed as follows: There exists a $0 < \alpha \leqq +\infty$ such that

(2.45)
(a) $f_{Ni} = 0$ for $x < x_0 - \alpha$, if α is finite;
(b) $f_{Ni}(x)$ are continuous on every compact subinterval of $(x_0 - \alpha, x_0)$, uniformly in (x, N, i);
(c) for any compact interval $C \subset (x_0 - \alpha, x_0)$ there exists an $\epsilon > 0$ such that for all, $N \geqq 1$, N^{-1} card $\{i: \inf_{x \varepsilon C} f_{Ni}(x) > \epsilon\} > \epsilon$;
(d) $x_0 - \alpha < \liminf_{N\to\infty} H_N^{-1}(t) \leqq \limsup_{N\to\infty} H_N^{-1}(t) < x_0$ for every $t \,\varepsilon\, (0, \frac{1}{2})$.

It may be shown that (2.45) implies the satisfaction of (2.12) and (2.13) with $l_{Ni}(v) = L'_{Ni}(v) = f_{Ni}(H_N^{-1}(v))/h_N(H_N^{-1}(v))$ for all $v \neq \frac{1}{2}$, $0 < v < 1$ (see Section 5, the proof of $C_3 \Rightarrow C_1$). If φ is continuous at $v = \frac{1}{2}$, then the singular parts of φ_1 and φ_2 give measure 0 to this single point, and, consequently, Theorems 1, 2, 3 are applicable with (2.12), (2.13) and (2.15) replaced by (2.45). Condition (2.45) is satisfied, for example, if $f_{Ni}(x) = e^{-d_{Ni}}f(xe^{-d_{Ni}})$, where $f(x)$ is continuous and positive on $(-\infty, \infty)$, unimodal, symmetric with respect to the origin and the d_{Ni}'s satisfy part (c) of (2.18).

REMARK 8. The (first) subscript N of the symbols introduced in this Section as well as below, will be omitted in the following. Further, $K_1, K_2, \cdots$ will denote positive constants, independent of N and numbered in order of appearance.

3. Lemmas concerning the Poisson binomial distribution.

Denote by

$$B(k; p_1, \cdots, p_N)$$

the probability of k successes in N independent trials with respective probabilities of success $p_1, p_2, \cdots, p_N$. Let $B_{i_1,\cdots,i_s}^{i_1',\cdots,i_r'}(k; p_1, \cdots, p_N)$ denote the above probability, where the parameters $p_{i_1}, \cdots, p_{i_s}$ were replaced by zeros and $p_{i_1'}, \cdots, p_{i_r'}$ by ones. Let B^* stand for either of the symbols $B, B^i, B_i, B^{ij}, B_j^i, B_{ij}$, $1 \leqq i, j \leqq N$, $i \neq j$. Let us put $q_i = 1 - p_i$. Further, let us abbreviate $B(k; p_1, \cdots, p_N)$ as $B(k)$, if $p_1, \cdots, p_N$ are fixed.

LEMMA 1. *For* $A > 4$ *we have*

$$\sum B^*(k; p_1, \cdots, p_N) < \max\,[\exp(-A^2/16 \textstyle\sum_{i=1}^N p_i q_i), \exp(-A/8)], \tag{3.1}$$

where the summation on the left extends either over $k > \sum_{i=1}^N p_i + A$ *or over* $k < \sum_{i=1}^N p_i - A$.

PROOF. Let us first prove that

$$\textstyle\sum_{k>k_0+2} B^*(k) \leqq \sum_{k>k_0} B(k) \leqq \sum_{k>k_0-2} B^*(k). \tag{3.2}$$

The above definitions entail

$$(3.3)\quad \begin{aligned} B(k) &= p_ip_jB^{ij}(k) + p_iq_jB_j^{\,i}(k) + q_ip_jB_i^{\,j}(k) \\ &\quad + q_iq_jB_{ij}(k) \\ &= p_ip_jB^{ij}(k) + p_iq_jB^{ij}(k+1) + q_ip_jB^{ij}(k+1) \\ &\quad + q_iq_jB^{ij}(k+2). \end{aligned}$$

Obviously (3.3) yields (3.2) for $B^* = B^{ij}$. The proof of (3.2) for the remaining variants of B^* is similar.

Now, we shall use Theorem 18.1.A, proposition (i) in Loève [5]. A random variable Y with distribution $B(\cdot\,; p_1, \cdots, p_N)$, centered by its mean $\sum_{i=1}^N p_i$, satisfies the conditions of the quoted theorem with $s = (\sum_{i=1}^N p_iq_i)^{\frac{1}{2}}$ and $c = s^{-1}$, in Loève's notation. If we put, moreover, $\epsilon s = A$, we get

$$(3.4)\quad \begin{aligned} 0 < A \leqq \textstyle\sum_{i=1}^N p_iq_i &\Rightarrow \textstyle\sum_{k>\sum_{i=1}^N p_i + A} B(k) < \exp\left(-\frac{1}{4}A^2 \sum_{i=1}^N p_iq_i\right), \\ A \geqq \textstyle\sum_{i=1}^N p_iq_i &\Rightarrow \textstyle\sum_{k>\sum_{i=1}^N p_i + A} B(k) < \exp\left(-\frac{1}{4}A\right). \end{aligned}$$

As the conditions of the theorem remain satisfied also if the random variable Y is multiplied by -1, the relation (3.4) holds true also when the summation extends over $k < \sum_{i=1}^N p_i - A$. Finally, suppose that $A > 4$. Then $A - 2 > A/2$ and (3.2) entails

$$\textstyle\sum_{k>\sum_{i=1}^N p_i + A} B^*(k) \leqq \sum_{k>\sum_{i=1}^N p_i + \frac{1}{2}A} B(k).$$

Combining this with (3.4), we obtain (3.1) for $k > \sum_{i=1}^N p_i + A$. The case $k < \sum_{i=1}^N p_i - A$ may be treated similarly.

In what follows, $\phi(x; \mu, \sigma^2)$ and $\Phi(x; \mu, \sigma^2)$ will denote the normal density and the normal distribution function, respectively, with parameters (μ, σ^2). Further, we shall write $\phi(x)$ and $\Phi(x)$ instead of $\phi(x; 0, 1)$ and $\Phi(x; 0, 1)$.

LEMMA 2. *For all x, μ, h_1 and all sufficiently small $|h_2|/\sigma^2$ we have*

$$(3.5)\quad |\phi(x; \mu + h_1, \sigma^2 + h_2) - \phi(x; \mu, \sigma^2)| \leqq |h_1|/4\sigma^2 + |h_2|/5\sigma^3,$$

$$(3.6)\quad |\Phi(x; \mu + h_1, \sigma^2 + h_2) - \Phi(x; \mu, \sigma^2)| \leqq 2|h_1|/5\sigma + |h_2|/8\sigma^2.$$

PROOF. The proof follows easily from Taylor's formula.

In the following lemma, k_0, k_1 may depend on N.

LEMMA 3. *Suppose that*

$$(3.7)\quad \lim_{N\to\infty} \textstyle\sum_{i=1}^N p_iq_i/\log N = +\infty,$$

$$(3.8)\quad |k_0 - k_1| \leqq K_1 \quad \textit{for all} \quad N \geqq 1.$$

Then we have, for N sufficiently large,

$$(3.9)\quad |B^*(k_0; p_1, \cdots, p_N) - \phi(k_1; \textstyle\sum_{i=1}^N p_i, \sum_{i=1}^N p_iq_i)| \leqq K_2(\sum_{i=1}^N p_iq_i)^{-1},$$

$$(3.10)\quad \begin{aligned} &|\textstyle\sum_{k\leqq k_0} B^*(k; p_1, \cdots, p_N) - \Phi(k_1; \sum_{i=1}^N p_i, \sum_{i=1}^N p_iq_i)| \\ &\qquad\qquad \leqq K_3(\textstyle\sum_{i=1}^N p_iq_i)^{-\frac{1}{2}}. \end{aligned}$$

PROOF. Specializing Theorem 1 in Petrov (1957), we get, under assumption (3.7),

$$(3.11)\qquad |B(k) - \phi(k; \textstyle\sum_{i=1}^N p_i, \sum_{i=1}^N p_i q_i)| \leqq K_4 \Lambda (\sum_{i=1}^N p_i q_i)^{-\frac{1}{2}},$$

where Λ is the Ljapunov ratio, i.e.,

$$(3.12)\qquad \Lambda = \textstyle\sum_{i=1}^N (p_i^3 q_i + p_i q_i^3)(\sum_{i=1}^N p_i q_i)^{-3/2} < (\sum_{i=1}^N p_i q_i)^{-\frac{1}{2}}.$$

Recalling the definition of $B^*(k)$, let $\phi^*(k)$ denote either $\phi(k; \sum_{i=1}^N p_i, \sum_{i=1}^N p_i q_i)$ or the value obtained from ϕ by replacement of one or two numbers p_i (and, correspondingly, q_i) by 0 or 1. Thus we have

$$(3.13)\qquad \phi^*(k) = \phi(k; \textstyle\sum_{i=1}^N p_i + h_1, \sum_{i=1}^N p_i q_i + h_2),$$

where $|h_1| \leqq 2$, $-\frac{1}{2} \leqq h_2 \leqq 0$. From (3.11) we get for N sufficiently large

$$(3.14)\qquad |B^*(k) - \phi^*(k)| < K_5(\textstyle\sum_{i=1}^N p_i q_i - \frac{1}{2})^{-1} \leqq 2K_5(\sum_{i=1}^N p_i q_i)^{-1};$$

and (3.13), (3.5) and (3.8) entail

$$(3.15)\qquad |\phi^*(k_0) - \phi(k_1; \textstyle\sum_{i=1}^N p_i, \sum_{i=1}^N p_i q_i)|$$
$$\leqq \tfrac{1}{4}(K_1 + 2)(\textstyle\sum_{i=1}^N p_i q_i)^{-1} + (1/10)(\sum_{i=1}^N p_i q_i)^{-3/2} < K_6(\sum_{i=1}^N p_i q_i)^{-1}.$$

Setting $k = k_0$ in (3.14) and combining it with (3.15) we obtain (3.9).

As for the distribution functions, we make use of the Berry-Esseen theorem (Theorem 20.3.B in [5]) instead of Petrov's theorem. So we get

$$(3.16)\qquad |\textstyle\sum_{k \leqq k_0} B^*(k) - \Phi(k_0; \sum_{i=1}^N p_i + h_1, \sum_{i=1}^N p_i q_i + h_2)|$$
$$\leqq K_7(\textstyle\sum_{i=1}^N p_i q_i)^{-\frac{1}{2}};$$

and, in view of (3.6),

$$(3.17)\qquad |\Phi(k_0; \textstyle\sum_{i=1}^N p_i + h_1, \sum_{i=1}^N p_i q_i + h_2)$$
$$- \Phi(k_1; \textstyle\sum_{i=1}^N p_i, \sum_{i=1}^N p_i q_i)| < K_8(\sum_{i=1}^N p_i q_i)^{-\frac{1}{2}}.$$

Inequalities (3.16) and (3.17) yield (3.10).

4. Proof of Theorem 1: upper bound for $E(S - \hat{S})^2$.

LEMMA 4. *The functions $L_i(t)$, defined by* (2.10), *satisfy the relation*

$$(4.1)\qquad |L_i(t) - L_i(s)| \leqq N|t - s|, \qquad 0 < s, \quad t < 1, \quad 1 \leqq i \leqq N, \quad N \geqq 1;$$

(*hence they are absolutely continuous*).

PROOF. It follows easily from the definitions (2.8)–(2.10) and from the continuity of the F_i's.

Throughout this section, we shall assume that the scores $a(i)$ are defined by (2.5) with the function φ given by (2.24), and that the conditions (2.12), (2.13) and (2.15) are fulfilled.

First we shall prove two auxiliary propositions.

LEMMA 5. *Let x, y be real numbers; let us denote as V the integer part of* $(N + 1)v$.

Then to each $K_9 > 2$ *there exists a* $K_{10} > 1$ *such that*

$$(4.2)\quad v - H(x) > K_9 N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N \Rightarrow P(R_i > V \mid X_i = x, X_j = y) < N^{-K_{10}}$$

and

$$(4.3)\quad v - H(x) < -K_9 N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N \Rightarrow P(R_i \leqq V \mid X_i = x, X_j = y) < N^{-K_{10}}$$

hold for $N > N_0(K_9)$.

The relations (4.2), (4.3) *remain true even when the condition* $X_j = y$ *is omitted.*

PROOF. If x satisfies the left side of (4.2), then

$$\begin{aligned} V &= \textstyle\sum_{m=1}^N F_m(x) + (V - Nv) + N(v - H(x)) \\ &> \textstyle\sum_{m=1}^N F_m(x) - 1 + K_9 N^{\frac{1}{2}} \lg^{\frac{1}{2}} N > \sum_{m=1}^N F_m(x) + K_{11} N^{\frac{1}{2}} \lg^{\frac{1}{2}} N \end{aligned}$$

for each $2 < K_{11} < K_9$ and $N > N_0(K_{11})$.

From Lemma 1 we get (with B^* being equal to B^{ij} or $B_j{}^i$ for $x \geqq y$ and $x < y$, respectively)

$$\begin{aligned} & P(R_i > V \mid X_i = x, X_j = y) = \textstyle\sum_{k > V} B^*(k; F_1(x), \cdots, F_N(x)) \\ & \leqq \textstyle\sum_{k > \sum_{m=1}^N F_m(x) + K_{11} N^{\frac{1}{2}} \lg^{\frac{1}{2}} N} B^*(k; F_1(x), \cdots, F_N(x)) \\ (4.4)\quad & < \max[\exp(-K_{11}^2 N \lg N / 16 \textstyle\sum_{m=1}^N F_m(x)(1 - F_m(x))), \\ & \quad \exp(-K_{11} N^{\frac{1}{2}} \lg^{\frac{1}{2}} N/8)] \\ & < \exp(-\tfrac{1}{4} K_{11}^2 \lg N) = N^{-K_{10}}, \text{ where } K_{10} = \tfrac{1}{4} K_{11}^2 \text{ and } N > N_0(K_{11}). \end{aligned}$$

The proof of (4.3) is quite similar. The last assertion of the lemma follows by integration with respect to $dF_j(y)$.

In the sequel, we shall denote $D^2 = N^{-1} \sum_{m=1}^N L_m(v)(1 - L_m(v))$ and again, $V = [(N + 1)v]$.

LEMMA 6. *Suppose that* $|v - H(x)| \leqq K_{12} N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N$. *Then, for sufficiently large* N, *we have*

$$(4.5)\quad |\textstyle\sum_{m=1}^N F_m(x)(1 - F_m(x)) - ND^2| \leqq K_{13} N^{\frac{1}{2}} \lg^{\frac{1}{2}} N,$$

$$\begin{aligned} (4.6)\quad & |\phi(V; \textstyle\sum_{m=1}^N F_m(x), \sum_{m=1}^N F_m(x)(1 - F_m(x))) \\ & \qquad - \phi(Nv; \textstyle\sum_{m=1}^N F_m(x), ND^2)| < K_{14} N^{-1} \lg^{\frac{1}{2}} N, \end{aligned}$$

$$\begin{aligned} (4.7)\quad & |\Phi(V; \textstyle\sum_{m=1}^N F_m(x), \sum_{m=1}^N F_m(x)(1 - F_m(x))) \\ & \qquad - \Phi(Nv; \textstyle\sum_{m=1}^N F_m(x), ND^2)| < K_{15} N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N. \end{aligned}$$

PROOF. Let us put $H(x) = t$. For $K_{16} N^{-\frac{1}{2}} \leqq |t - v| \leqq K_{12} N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N$ we have, according to (2.12),

$$\begin{aligned} & \textstyle\sum_{m=1}^N F_m(x)(1 - F_m(x)) \\ & \quad = \textstyle\sum_{m=1}^N L_m(t)(1 - L_m(t)) \\ & \quad = \textstyle\sum_{m=1}^N L_m(v)(1 - L_m(v)) + O(N|t - v|) = ND^2 + O(N^{\frac{1}{2}} \lg^{\frac{1}{2}} N), \end{aligned}$$

and, according to (2.13), we also have the same result for $|t - v| \leqq K_{16}N^{-\frac{1}{2}}$. Thus we get (4.5). Now, putting

$$|h_1| = |V - Nv| \leqq 1,$$

$$|h_2| = |\sum_{m=1}^{N} F_m(x)(1 - F_m(x)) - ND^2| \leqq K_{13}N^{\frac{1}{2}}\lg^{\frac{1}{2}} N, \quad \sigma^2 = ND^2 \geqq K_{17}N$$

(according to (2.15)), we obtain (4.6) and (4.7) from Lemma 2.

The purpose of this section is to derive an upper bound for the residual variance $E(S - \hat{S})^2$, where

$$\hat{S} = \sum_{i=1}^{N} E(S \mid X_i) - (N - 1)ES. \tag{4.8}$$

Combining the formulas 4.8 and 4.18 in [4], we get

$$\begin{aligned} &E(S - \hat{S})^2 \\ &\leqq \sum_{i=1}^{N} (c_i - \bar{c})^2 E[a(R_i) - E(a(R_i)| X_i)]^2 \\ &+ \sum\sum_{i \neq j} (c_i - \bar{c})(c_j - \bar{c})\{E[\mathrm{Cov}\,(a(R_i), a(R_j)| X_i, X_j)] \\ &+ E[(E(a(R_i)| X_i, X_j) - E(a(R_i)| X_i))(E(a(R_j)| X_i, X_j) \\ &- E(a(R_j)| X_j))] - \sum_{m \neq i,j} \mathrm{Cov}\,(E(a(R_i)| X_m), E(a(R_j)| X_m))\}. \end{aligned} \tag{4.9}$$

Let us investigate each term on the right-hand side separately.

LEMMA 7. *For $N \to +\infty$ we have*

$$E[\mathrm{Cov}\,(a(R_i), a(R_j)| X_i, X_j)] = N^{-1}D^2 l_i(v)l_j(v) + o(N^{-1})$$

uniformly with respect to $1 \leqq i, j \leqq N$.

PROOF. We have

$$\begin{aligned} &\mathrm{Cov}\,(a(R_i), a(R_j)| X_i = x, X_j = y) \\ &= E(a(R_i)a(R_j)| X_i = x, X_j = y) - E(a(R_i)| X_i = x, X_j = y) \\ &\quad \cdot E(a(R_j)| X_i = x, X_j = y) \\ &= P(R_i > V \mid X_i = x, X_j = y)P(R_j \leqq V \mid X_j = y, X_i = x) \\ &\qquad\qquad \text{for} \quad x < y, \\ &= P(R_j > V \mid X_j = y, X_i = x)P(R_i \leqq V \mid X_i = x, X_j = y) \\ &\qquad\qquad \text{for} \quad x > y. \end{aligned} \tag{4.10}$$

Let $K_9 > 2$. Denote

$$I = \{(x, y): |H(x) - v| \leqq K_9 N^{-\frac{1}{2}}\lg^{\frac{1}{2}} N, \ |H(y) - v| \leqq K_9 N^{-\frac{1}{2}}\lg^{\frac{1}{2}} N\}.$$

In view of (4.10) and Lemma 5, there exists a $K_{10} > 1$ such that

$$0 \leqq \mathrm{Cov}\,(a(R_i), a(R_j)| X_i = x, X_j = y) < N^{-K_{10}} \tag{4.11}$$

for all $N > N_0(K_{10})$ and all $(x, y) \not\in I$. Further, in view of Lemma 3, (2.15) and

(4.5), for $(x, y) \varepsilon I$ and $x < y$, expression (4.10) may be continued as follows:

$$\begin{aligned}\operatorname{Cov}\,&(a(R_i), a(R_j) | X_i = x, X_j = y)\\ &= \textstyle\sum_{k>V} B_j^i(k; F_1(x) \cdots, F_N(x))\\ &\quad \cdot \textstyle\sum_{l \leqq V} B^{ij}(l; F_1(y), \cdots, F_N(y))\\ &= [1 - \Phi(V; \textstyle\sum_{m=1}^N F_m(x), \sum_{m=1}^N F_m(x)(1 - F_m(x)))]\\ &\quad \cdot \Phi(V; \textstyle\sum_{m=1}^N F_m(y), \sum_{m=1}^N F_m(y)(1 - F_m(y))) + \vartheta_1 N^{-\frac{1}{2}},\end{aligned}$$

with $|\vartheta_1| \leqq K_{18}$. According to (4.7), we may further write

$$\text{(4.12)}\quad \operatorname{Cov}[a(R_i), a(R_j) | X_i = x, X_j = y]$$
$$= \Phi((H(x) - v)/(DN^{-\frac{1}{2}}))(1 - \Phi((H(y) - v)/(DN^{-\frac{1}{2}}))) + \vartheta_2 N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N$$

with $|\vartheta_2| \leqq K_{19}$. For $(x, y) \varepsilon I$, $x > y$, the positions of x and y on the right-hand side of (4.12) should be interchanged.

Result (4.12) for the conditional covariance remains true even when we enlarge the square I to the square

$$I' = \{(x, y): \max(|H(x) - v|, |H(y) - v|) \leqq K_9 K_{20}^{-1} DN^{-\frac{1}{2}} \lg^{\frac{1}{2}} N\}$$

where K_{20} is such that $D \geqq K_{20}$ for all N. Writing now $K_9 K_{20}^{-1} = K_{21}$, $(H(x) - v)/DN^{-\frac{1}{2}} = p$, $(H(y) - v)/DN^{-\frac{1}{2}} = q$, and

$$I'' = \{(p, q): \max(|p|, |q|) \leqq K_{21} \lg^{\frac{1}{2}} N\},$$

we have

$$\begin{aligned}E[\operatorname{Cov}\,&(a(R_i), a(R_j) | X_i, X_j)]\\ &= \textstyle\int\int_{I'' \cap \{p<q\}} \Phi(p)(1 - \Phi(q))\, d_p L_i(v + DN^{-\frac{1}{2}}p)\, d_q L_j(v + DN^{-\frac{1}{2}}q)\\ &\quad + \textstyle\int\int_{I'' \cap \{p>q\}} \Phi(q)(1 - \Phi(p))\, d_p L_i(v + DN^{-\frac{1}{2}}p)\, d_q L_j(v + DN^{-\frac{1}{2}}q)\\ &\quad + N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N \textstyle\int\int_{I''} \vartheta_3\, d_p L_i(v + DN^{-\frac{1}{2}}p)\, d_q L_j(v + DN^{-\frac{1}{2}}q)\\ &\quad + \vartheta_4 N^{-K_{10}},\end{aligned} \tag{4.13}$$

with $|\vartheta_3| \leqq K_{22}$, $|\vartheta_4| \leqq 1$. The last two terms are $o(N^{-1})$ uniformly in i, j, as is easily seen from (2.12) and $K_{10} > 1$. The evaluation of the remaining two integrals is essentially the same, so let us calculate only the first of them in detail. We shall divide the domain I'' into two parts, J and $I'' - J$, where $J = \{(p, q): \max(|p|, |q|) \leqq K_{23}\}$. In J we shall make use of the expansion

$$\text{(4.14)}\quad L_i(v + DN^{-\frac{1}{2}}p) = L_i(v) + l_i(v) DN^{-\frac{1}{2}}p + \Lambda_i(p), \qquad 1 \leqq i \leqq N,$$

where the functions $\Lambda_i(p)$ are absolutely continuous and are of order $o(N^{-\frac{1}{2}})$ uniformly in the interval $[-K_{23}, K_{23}]$ and with respect to $1 \leqq i \leqq N$. This follows from (2.13). Let us first treat the following integral:

$$\begin{aligned}\int\int_{J\cap\{p<q\}} \Phi(p)(1-\Phi(q))\,d_pL_i\,d_qL_j \\ &= N^{-1}D^2 l_i(v)l_j(v)\int\int_{J\cap\{p<q\}}\Phi(p)(1-\Phi(q))\,dp\,dq \\ &+ N^{-\frac{1}{2}}Dl_i(v)\int\int_{J\cap\{p<q\}}\Phi(p)(1-\Phi(q))\,dp\,d\Lambda_j(q) \\ &+ N^{-\frac{1}{2}}Dl_j(v)\int\int_{J\cap\{p<q\}}\Phi(p)(1-\Phi(q))\,d\Lambda_i(p)\,dq \\ &+ \int\int_{J\cap\{p<q\}}\Phi(p)(1-\Phi(q))\,d\Lambda_i(p)\,d\Lambda_j(q).\end{aligned} \tag{4.15}$$

If K_{23} is sufficiently large, the integral $\int\int_{J\cap\{p<q\}}\Phi(p)(1-\Phi(q))\,dp\,dq$ is arbitrarily close to

$$\begin{aligned}\int\int_{\{p<q\}}\Phi(p)(1-\Phi(q))\,dp\,dq \\ &= \int_{-\infty}^{+\infty}(1-\Phi(q)(\int_{-\infty}^{q}\Phi(p)\,dp)\,dq \\ &= \int_{-\infty}^{+\infty}(1-\Phi(q))(q\Phi(q)+\phi(q))\,dq = \tfrac{1}{2}.\end{aligned} \tag{4.16}$$

If K_{23} is fixed, then the remaining terms on the right-hand side of (4.15) are $o(N^{-1})$, as may be shown by integration by parts, and making use of (4.14). Moreover, the numbers $l_i(v)$ considered as functions of (i, N), $1 \leqq i \leqq N < \infty$, are bounded, as appears by setting $t = v + N^{-\frac{1}{2}}$ in (2.12) and (2.13). Consequently,

$$\int\int_{J\cap\{p<q\}}\Phi(p)[1-\Phi(q)]\,d_pL_i\,d_qL_j = \tfrac{1}{2}N^{-1}D^2l_i(v)l_j(v) + o(N^{-1}). \tag{4.17}$$

The same holds for the integral extending over the region $J \cap \{q < p\}$.

The proof of Lemma 7 will be completed, if we show that

$$\begin{aligned}\int\int_{(I''-J)\cap\{p<q\}}\Phi(p)(1-\Phi(q))\,d_pL_i\,d_qL_j \\ + \int\int_{(I''-J)\cap\{p>q\}}\Phi(q)(1-\Phi(p))\,d_pL_i\,d_qL_j = o(N^{-1}).\end{aligned} \tag{4.18}$$

The left-hand side of (4.18) may be decomposed into four integrals extending over areas similar to one which is shaded on Figure 1. The integral corresponding to the shaded area equals

$$\int_{-K_{21}\lg^{\frac{1}{2}}N}^{-K_{23}}\Phi(p)\left(\int_{p}^{-p}(1-\Phi(q))\,d_qL_j\right)d_pL_i. \tag{4.19}$$

In (4.19), where $p < 0$, we have

$$\begin{aligned}\int_p^{-p}(1-\Phi(q))\,d_qL_j &\leqq L_j(v+DN^{-\frac{1}{2}}|p|) - L_j(v-DN^{-\frac{1}{2}}|p|) \\ &\leqq K_{24}N^{-\frac{1}{2}}|p|,\end{aligned} \tag{4.20}$$

owing to (2.12) and to the inequality $K_{23} \leqq |p| \leqq K_{21}\lg^{\frac{1}{2}} N$. Thus the expression (4.19) is less or equal to

$$\begin{aligned}K_{24}N^{-\frac{1}{2}}\int_{-K_{21}\lg^{\frac{1}{2}}N}^{-K_{23}}|p|\Phi(p)\,d_pL_i \\ &\leqq K_{24}N^{-\frac{1}{2}}\{-[(L_i(v+DN^{-\frac{1}{2}}p)-L_i(v))p\Phi(p)]_{-K_{21}\lg^{\frac{1}{2}}N}^{-K_{23}} \\ &+ \int_{-K_{21}\lg^{\frac{1}{2}}N}^{-K_{23}}(L_i(v+DN^{-\frac{1}{2}}p)-L_i(v))(\Phi(p)+p\phi(p))\,dp\} \\ &< K_{25}N^{-1}\int_{-\infty}^{-K_{23}}p^2\phi(p)\,dp < \epsilon N^{-1}\end{aligned}$$

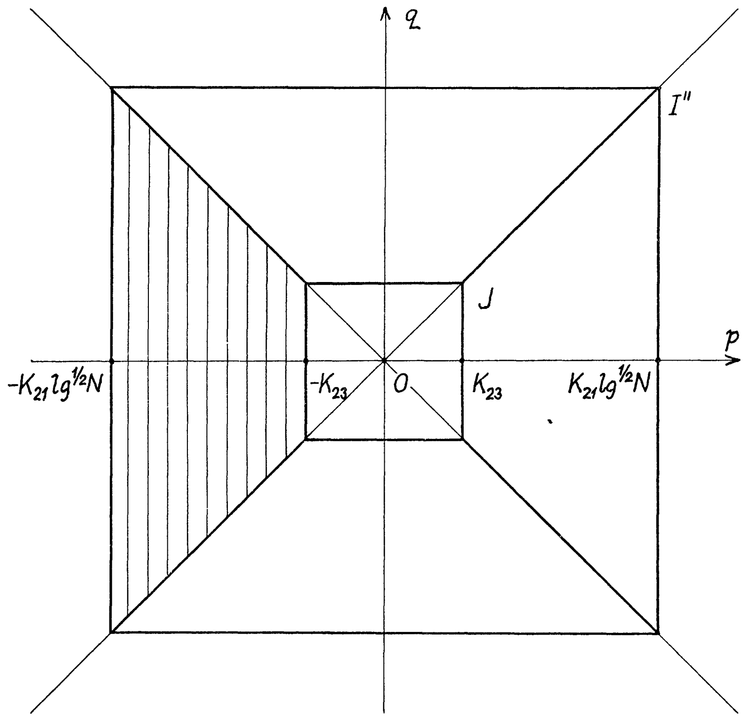

FIG. 1

with $\epsilon > 0$ arbitrarily small for K_{23} sufficiently large. As the same bound is valid for the three other integrals, too, the proof of Lemma 7 is finished.

LEMMA 8. *For $N \to \infty$ we have*

$$\sum_{1 \leqq m \leqq N, m \neq i,j} \operatorname{Cov}\,(E(a(R_i) \mid X_m), E(a(R_j) \mid X_m)) = N^{-1}D^2 l_i(v) l_j(v) + o(N^{-1}),$$

uniformly with respect to $1 \leqq i, j \leqq N$.

PROOF. According to Lemma 3.2 in Hájek (1968) we have

$$(4.21)\quad E(a(R_i) \mid X_i = x, X_m = z) - E(a(R_i) \mid X_i = x) = [u(x - z) - F_m(x)]P(R_i = V + 1 \mid X_i = x, X_m = x - 1), \quad i \neq m.$$

(We have used the notation introduced in Lemma 5 and the unit step function $u(x) = 1, x \geqq 0; = 0, x < 0$.) Lemma 5 shows that

$$P(R_i = V + 1 \mid X_i = x, X_m = x - 1) < N^{-K_{10}}$$

for some $K_{10} > 1$ and for all $|H(x) - v| \geqq K_9 N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N$, $N > N_0(K_{10})$. On the other hand Lemmas 3 and 6 imply

$$P(R_i = V + 1 \mid X_i = x, X_m = x - 1)$$

$$\begin{aligned} (4.22) \qquad &= B^{ij}(V + 1 \mid F_1(x), \cdots, F_N(x)) \\ &= \phi(V; \textstyle\sum_{n=1}^N F_n(x), \sum_{n=1}^N F_n(x)(1 - F_n(x))) \\ &\quad + \vartheta_5(\textstyle\sum_{n=1}^N F_n(x)(1 - F_n(x)))^{-1} \\ &= \phi(Nv; NH(x), ND^2) + \vartheta_6 N^{-1} \lg^{\frac{1}{2}} N \end{aligned}$$

for some $|\vartheta_5| \leqq K_{26}$, $|\vartheta_6| \leqq K_{27}$ and all $|H(x) - v| \leqq K_9 N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N$.

Now, we proceed similarly as in the proof of the preceding lemma. The result (4.22) remains true even in the larger interval

$$|H(x) - v| \leqq K_9 K_{20}^{-1} D N^{-\frac{1}{2}} \lg^{\frac{1}{2}} N \qquad \text{(where } K_{20} \leqq D,\ N \geqq 1\text{)}.$$

Writing again $K_{21} = K_9 K_{20}^{-1}$, $(H(x) - v)/DN^{-\frac{1}{2}} = p$, $(H(z) - v)/DN^{-\frac{1}{2}} = q$, and integrating (4.21) with respect to $dF_i(x)$, we get after some calculations,

$$\begin{aligned} &E(a(R_i) \mid X_m = z) - E(a(R_i)) \\ (4.23) \qquad &= D^{-1}N^{-\frac{1}{2}} \int_{|p| \leqq K_{21}\lg^{\frac{1}{2}}N} [u(p - q) - L_m(v)]\phi(p)\, d_p L_i(v + DN^{-\frac{1}{2}}p) \\ &\quad + o(N^{-1}), \end{aligned}$$

uniformly with respect to $-\infty < z < +\infty$.

Let us again divide the domain of integration into $\{|p| \leqq K_{23}\}$ and $\{K_{23} \leqq |p| \leqq K_{21} \lg^{\frac{1}{2}} N\}$ and in the first domain let us use the expansion (4.14). We get

$$\begin{aligned} &\int_{-K_{23}}^{K_{23}} [u(p - q) - L_m(v)]\phi(p)\, d_p L_i \\ &\qquad = DN^{-\frac{1}{2}} l_i(v) \int_{-K_{23}}^{K_{23}} [u(p - q) - L_m(v)]\phi(p)\, dp \\ &\qquad\quad + \int_{-K_{23}}^{K_{23}} [u(p - q) - L_m(v)]\phi(p)\, d\Lambda_i(p), \end{aligned}$$

where the integral $\int_{-K_{23}}^{K_{23}} [\cdots]\phi(p)\, dp$ is arbitrarily close (uniformly with respect to z) to the expression $1 - \Phi(q) - L_m(v)$ for sufficiently large K_{23}, whereas the integral $\int_{-K_{23}}^{K_{23}} [\cdots]\phi(p)\, d\Lambda_i(p)$ is of the order of magnitude $o(N^{-\frac{1}{2}})$, as can be easily shown by integration by parts. Further we have

$$\begin{aligned} &\left|\int_{-K_{21}\lg^{\frac{1}{2}}N}^{-K_{23}} [\cdots]\phi(p)\, d_p L_i\right| \\ &\qquad \leqq [(L_i(v + DN^{-\frac{1}{2}}p) - L_i(v))\phi(p)]_{-K_{21}\lg^{\frac{1}{2}}N}^{-K_{23}} \\ &\qquad\quad - \int_{-K_{21}\lg^{\frac{1}{2}}N}^{-K_{23}} (L_i(v + DN^{-\frac{1}{2}}p) - L_i(v))\phi'(p)\, dp \\ &\qquad \leqq K_{28} N^{-\frac{1}{2}} \int_{-\infty}^{-K_{23}} p^2 \phi(p)\, dp < \epsilon N^{-\frac{1}{2}} \qquad \text{for } K_{23} \text{ large}; \end{aligned}$$

the same for $\int_{K_{23}}^{K_{21}\lg^{\frac{1}{2}}N} [\cdots]\phi(p)\, d_p L_i$. Altogether, we have

$$\begin{aligned} (4.24) \qquad &E(a(R_i) \mid X_m = z) - E(a(R_i)) \\ &\qquad = N^{-1} l_i(v)\{1 - \Phi(q) - L_m(v)\} + o(N^{-1}), \end{aligned}$$

uniformly in $-\infty < z < +\infty$. Hence,

$$\begin{aligned}
&\operatorname{Cov}\,(E(a(R_i)\,|\,X_m), E(a(R_j)\,|\,X_m)) \\
(4.25)\quad &= \int_{-\infty}^{+\infty} (E(a(R_i)\,|\,X_m = z) - E(a(R_i))) \\
&\quad \cdot (E(a(R_j)\,|\,X_m = z) - E(a(R_j)))\, dF_m(z) \\
&= N^{-2} l_i(v) l_j(v) \int_{-\infty}^{+\infty} \{1 - \Phi(q) - L_m(v)\}^2\, d_q L_m(v + DN^{-\frac{1}{2}} q) + o(N^{-2}).
\end{aligned}$$

As each of the integrals $\int_{-\infty}^{+\infty} (1 - \Phi(q))\, d_q L_m$ and $\int_{-\infty}^{+\infty} (1 - \Phi(q))^2\, d_q L_m$ evidently equals $L_m(v) + o(1)$, we have

$$\int_{-\infty}^{+\infty} (1 - \Phi(q) - L_m(v))^2\, d_q L_m = L_m(v)(1 - L_m(v)) + o(1).$$

Inserting this into (4.25) and summing over $1 \leqq m \leqq N$, $m \neq i, j$, we finally get the assertion of the lemma. (Observe that deleting the ith and jth terms can diminish the sum $ND^2 = \sum_{m=1}^{N} L_m(v)(1 - L_m(v))$ at most by $\frac{1}{2}$.)

LEMMA 9. *For $N \to \infty$ we have*

$$E[(E(a(R_i)\,|\,X_i, X_j) - E(a(R_i)\,|\,X_i))(E(a(R_j)\,|\,X_i, X_j) - E(a(R_j)\,|\,X_j))] = o(N^{-1}),$$

uniformly in $1 \leqq i, j \leqq N$.

LEMMA 10. *For $N \to \infty$ we have*

$$E(a(R_i) - E(a(R_i)\,|\,X_i))^2 = o(1),$$

uniformly in $1 \leqq i \leqq N$.

Proofs of Lemmas 9 and 10 will be omitted; they are similar to the proofs of two preceding lemmas, but substantially simpler.

LEMMA 11. *For $N \to \infty$ we have*

$$E(S - \hat{S})^2 = o(\textstyle\sum_{i=1}^{N} (c_i - \bar{c})^2).$$

PROOF. It follows by inserting the results of Lemmas 7–10 into the inequality (4.9) and making use of the inequality

$$\sum_{i=1}^{N} \sum_{j=1}^{N} |c_i - \bar{c}||c_j - \bar{c}| \leqq N \sum_{i=1}^{N} (c_i - \bar{c})^2.$$

5. Proof of Theorem 1: completion. In this section, we shall at first assume, that the scores $a(i)$ are defined by (2.5) with the function φ given by (2.24), and that the assumptions (2.2), (2.12), (2.13), (2.15) and (2.20) or (2.22) (with σ^2 given by (2.26)) are satisfied.

From the definitions of S and $\hat{S}$ and from the obvious relation

$$\sum_{j=1}^{N} [E(a(R_j)\,|\,X_i) - E(a(R_j))] = 0$$

it follows $\hat{S} - E\hat{S} = \sum_{i=1}^{N} Y_i$, where

$$Y_i = \sum_{j=1}^{N} (c_j - c_i)[E(a(R_j)\,|\,X_i) - E(a(R_j))], \tag{5.1}$$

Y_i are independent, $EY_i = 0$, $1 \leqq i \leqq N$, $\operatorname{Var} \hat{S} = \sum_{i=1}^{N} \operatorname{Var} Y_i$.

Further denote

$$Z_i = (\bar{c}(v) - c_i)[u(v - H(X_i)) - L_i(v)], \qquad 1 \leqq i \leqq N, \tag{5.2}$$

where $\bar{c}(v)$ is given by (2.25). Random variables Z_i are independent, with zero expectations, and such that $\sum_{i=1}^N \operatorname{Var} Z_i = \sigma^2$ (defined by (2.26)).

According to (4.24) we can write, using (2.12), (2.13) and (2.15),

$$Y_i = (\bar{c}(v) - c_i)[\Phi((v - H(X_i))/DN^{-\frac{1}{2}}) - L_i(v)] + N^{-1} \sum_{j=1}^N (c_j - c_i)\eta_j, \tag{5.3}$$

where η_j are random variables such that $|\eta_j| \leqq \epsilon_N$, $1 \leqq j \leqq N$, for some sequence of constants $\epsilon_N \to 0$. Consequently, as in the close of the proof of Lemma 8, we get

$$E(Y_i - Z_i)^2 = o(N^{-1} \sum_{j=1}^N (c_j - \bar{c})^2), \tag{5.4}$$

uniformly in $i \leqq i \leqq N$.

Now, the inequality $\operatorname{Var}(S - \sum_{i=1}^N Z_i) \leqq 2 \sum_{i=1}^N E(Y_i - Z_i)^2 + 2E(S - \hat{S})^2$ together with (5.4) and with Lemma 11 yield the following.

LEMMA 12.

$$\operatorname{Var}(S - \sum_{i=1}^N Z_i) = o(\sum_{i=1}^N (c_i - \bar{c})^2).$$

LEMMA 13. (2.20) *holds if and only if* (2.22) *holds with* σ^2 *given by* (2.26). *In this case* $\lim_{N\to\infty} \operatorname{Var} S/\sigma^2 = 1$.

PROOF. From Minkowski's inequality we obtain

$$((\operatorname{Var} U_1/\operatorname{Var} U_2)^{\frac{1}{2}} - 1)^2 \leqq \operatorname{Var}(U_1 - U_2)/\operatorname{Var} U_2. \tag{5.5}$$

If (2.20) is satisfied, we put $U_1 = \sum_{i=1}^N Z_i$, $U_2 = S$ in (5.5); if (2.22) is satisfied, we put $U_1 = S$, $U_2 = \sum_{i=1}^N Z_i$. In both cases, Lemma 12 entails $\operatorname{Var} S/\sigma^2 \to 1$ and, consequently, the validity of the entire Lemma 13.

LEMMA 14. *The random variables* $\sum_{i=1}^N Z_i$ *are asymptotically normal with parameters* $(0, \sigma^2)$.

PROOF. From (5.2) it follows $|Z_i| \leqq 2 \max_{1 \leqq j \leqq N} |c_j - \bar{c}|$, $1 \leqq i \leqq N$; (2.2) and (2.22) then imply that $\max_{1 \leqq i \leqq N} |Z_i|/\sigma = o(1)$, but this means that the Lindeberg condition for asymptotic normality is trivially satisfied.

Now, the assertion of Theorem 1, under conditions listed at the beginning of this Section, is a consequence of the following (already proved) propositions:

$$\mathfrak{L}(\sum_{i=1}^N Z_i/\sigma) \to \mathfrak{N}(0,1), \qquad [\operatorname{Var}(S - \sum_{i=1}^N Z_i)]/\sigma^2 \to 0, \qquad (\operatorname{Var} S)^{\frac{1}{2}}/\sigma \to 1.$$

(Namely, we can write

$$(S - ES)/(\operatorname{Var} S)^{\frac{1}{2}} = (\sum_{i=1}^N Z_i/\sigma + (S - ES - \sum_{i=1}^N Z_i)/\sigma)\sigma/(\operatorname{Var} S)^{\frac{1}{2}} \tag{5.6}$$

and apply the Theorem 20.6 in Cramér [2].)

Let us relax the condition (2.5) to (2.6) and denote the corresponding statistics as S and S^*. Then we have from (2.6) and (2.2)

$$\operatorname{Var}(S - S^*) \leqq \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 \cdot (\sum_{j=1}^N |a(i) - \varphi(i/(N+1))|)^2 = o(\sum_{i=1}^N (c_i - \bar{c})^2).$$

Hence, with the help of (2.22), (5.5), (5.6), the asymptotic normality of S^* easily follows.

Thus we have proved Theorem 1 under conditions C_1. If remains to show that this set of conditions is entailed by the other sets.

$C_2 \Rightarrow C_1$ will follow from (2.16) $\Rightarrow$ [(2.15), (2.22)]. However that (2.16) $\Rightarrow$ (2.15) is obvious, and (2.16) $\Rightarrow$ (2.22) follows from (2.26) and from

$$\sum_{i=1}^{N} (c_i - \tilde{c}(v))^2 \geqq \sum_{i=1}^{N} (c_i - \bar{c})^2.$$

$C_3 \Rightarrow C_1$ will follow from (2.17) $\Rightarrow$ [(2.12), (2.13), (2.15)]. If we put $h(x) = N^{-1} \sum_{i=1}^{N} f_i(x)$, we obtain from (c) of (2.17) that on every compact $C \subset (\alpha, \beta)$

$$\inf_{x \varepsilon C} h(x) > \epsilon^2. \tag{5.7}$$

Further, from (d) of (2.17) it follows that for every $0 < v_1 < v < v_2 < 1$ there exists a compact $C \subset (\alpha, \beta)$ such that

$$H^{-1}(t) \,\varepsilon\, C \quad \text{for all} \quad v_1 < t < v_2. \tag{5.8}$$

Putting (5.7) and (5.8) together, and noting that $[h(H^{-1}(t))]^{-1}$ represents the derivative of $H^{-1}(t)$, we obtain for all $N \geqq 1$

$$\sup_{v_1 < t < v_2} |dH^{-1}(t)/dt| < \epsilon^{-2}.$$

Consequently, $H^{-1}(t)$ is uniformly continuous in a neighborhood of v. The same is true about

$$dL_i(t)/dt = f_i(H^{-1}(t))/h(H^{-1}(t))$$

since all three involved functions, f_i, H^{-1} and the reciprocal of h, are uniformly continuous in a neighborhood of v; for f_i, consult (2.17 b) and for the reciprocal of h, see (2.17 b) and (5.7). From this fact (2.12) and (2.13), with $l_i = L_i'$ easily follow.

Now it remains to prove (2.17) $\Rightarrow$ (2.15). Part (d) of (2.17) entails the existence of numbers x_1 and x_2 such that $\alpha < x_1 < x_2 < \beta$ and such that for all sufficiently large N, $x_1 \leqq H^{-1}(v) \leqq x_2$. Consequently

$$L_i(v)[1 - L_i(v)] \geqq F_i(x_1)[1 - F_i(x_2)]. \tag{5.9}$$

Further, (c) of (2.17) entails that for fixed y_1, y_2 such that $\alpha < y_1 < x_1 < x_2 < y_2 < \beta$ there exists an $\epsilon > 0$ such that

$$N^{-1} \text{ card } \{i: \inf_{y_1 \leqq x \leqq y_2} f_i(x) > \epsilon\} > \epsilon;$$

which implies

$$N^{-1} \text{ card } \{i: F_i(x_1) > \epsilon(x_1 - y_1),\ 1 - F_i(x_2) > \epsilon(y_2 - x_2)\} > \epsilon.$$

This combined with (5.9) gives

$$N^{-1} \sum_{i=1}^{N} L_i(v)[1 - L_i(v)] > \epsilon^3(x_1 - y_1)(y_2 - x_2).$$

Note that C_1 concerns one point v, whereas C_3 ensures the satisfaction of C_1 for any $0 < v < 1$.

$C_4 \Rightarrow C_3$ follows from (2.18) $\Rightarrow$ (2.17), which is easily shown.

$C_5 \Rightarrow C_4$ follows from (2.19) $\Rightarrow$ (2.18) and (2.19) $\Rightarrow$ (2.16) $\Rightarrow$ (2.22). The first implication is obvious, and (2.16) $\Rightarrow$ (2.22) has been proved by proving $C_2 \Rightarrow C_1$; (2.19) $\Rightarrow$ (2.16) follows from

$$L_i(v)[1 - L_i(v)] \geqq F(F^{-1}(v) - 2K)[1 - F(F^{-1}(v) + 2K)] > 0,$$

where F corresponds to f and $K \geqq \sup_{i,N} |d_i|$.

$C_6 \Rightarrow C_1$ is entailed by (2.3) $\Rightarrow$ (2.2) and [(2.3), (2.22)] $\Rightarrow$ (2.15). The former implication is obvious, in the other we utilize the fact that

$$\sigma^2 \leqq 2 \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 \sum_{i=1}^N L_i(v)[1 - L_i(v)].$$

This concludes the proof of Theorem 1.

Let us show that, under conditions of Theorem 1, σ^2 and $\operatorname{Var} S$ are of order $\sum_{i=1}^N (c_i - \bar{c})^2$. From (2.26) it follows that

$$\sigma^2 \leqq \sum_{i=1}^N (c_i - \tilde{c}(v))^2 = \sum_{i=1}^N (c_i - \bar{c})^2 + N(\bar{c} - \tilde{c}(v))^2.$$

Now, since the numbers $l_i(v)$ are for fixed v uniformly bounded in (N, i) (see below (4.16)), we have

$$N(\bar{c} - \tilde{c}(v))^2 = N^{-1}(\sum_{i=1}^N (\bar{c} - c_i)l_i(v))^2 \leqq K_v^2 \sum_{i=1}^N (\bar{c} - c_i)^2$$

for some K_v. Thus $\sigma^2 = O(\sum_{i=1}^N (c_i - \bar{c})^2)$ and, hence in view of Theorem 1 (see also Lemma 12),

$$\operatorname{Var} S = O(\sum_{i=1}^N (c_i - \bar{c})^2). \tag{5.10}$$

6. Proof of Theorem 2. Assuming first that (2.5) is satisfied, put

$$S_\varphi = \sum_{i=1}^N c_i \varphi(R_i/(N + 1)).$$

Consulting Hájek (1968), namely (3.7) and (5.34), we can see that, for φ *nondecreasing*,

$$\operatorname{Var} S_\varphi \leqq 42\, N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 \int_0^1 (\varphi(t) - \bar{\varphi})^2\, dt. \tag{6.1}$$

Further put

$$T_v = \sum_{i=1}^N [u(v - H(X_i)) - L_i(v)](\tilde{c}(v) - c_i) \tag{6.2}$$

where $\tilde{c}(v)$ is defined by (2.31) if $v \not\in B$ and by (2.32), if $v \in B$. We have, obviously, in view of (2.11),

$$\begin{aligned} \operatorname{Var} T_v &= \sum_{i=1}^N (\tilde{c}(v) - c_i)^2 L_i(v)[1 - L_i(v)] \\ &\leqq 4N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 v(1 - v). \end{aligned} \tag{6.3}$$

Note that T_v equals $\sum_{i=1}^N Z_i$ from Section 5. Consequently, by Lemma 12,

$$E(S_v - ES_v - T_v)^2 = o(\sum_{i=1}^N (c_i - \bar{c})^2). \tag{6.4}$$

If φ is nondecreasing and square integrable, then $\int_0^1 v(1 - v)\, d\varphi < +\infty$ and

$$T_\varphi = \int_0^1 T_v\, d\varphi(v) \tag{6.5}$$

is well-defined. Similarly as in proving (5.37) in Hájek (1968), we can show that

$$\text{Var } T_\varphi \leqq 4N \max_{1\leqq i\leqq N} (c_i - \bar{c})^2 \int_0^1 \varphi^2(t)\, dt. \tag{6.6}$$

If φ is absolutely continuous and the φ-measure vanishes on the complement of B, then T_φ of (6.5) may be equivalently expressed by

$$T_\varphi = \sum_{i=1}^N N^{-1} \sum_{j=1}^N (c_j - c_i) \int [u(x - X_i) - F_i(x)]\varphi'(H(x))\, d_{Fj}(x). \tag{6.7}$$

This follows from the obvious formula

$$\int [u(x - X_i) - F_i(x)]\varphi'(H(x))\, dF_j(x) = \int_0^1 [u(v - H(X_i)) - L_i(v)]L_j'(v)\, d\varphi(v).$$

Now for every $\alpha > 0$, φ may be decomposed as follows

$$\varphi(t) = \psi(t) - \lambda(t) + \gamma(t) - \eta(t) + g(t) - h(t), \tag{6.8}$$

where all functions on the right side are nondecreasing and square integrable, and

(a) ψ and λ are absolutely continuous such that

$$\int_B (d\psi + d\lambda) = \int_0^1 (d\psi + d\lambda);$$

(b) $\gamma(t)$ and $\eta(t)$ are bounded such that

$$\int_B (d\gamma + d\eta) = 0;$$

(c) $\int_0^1 (g^2(t) + h^2(t))\, dt < \alpha$, and $\int_B (dg + dh) = 0$.

Now, if ψ has a bounded second derivative, Theorem 4.2 of Hájek (1968) entails

$$\lim_{N\to\infty} [E(S_\psi - ES_\psi - T_\psi)^2]/[N \max_{1\leqq i\leqq N} (c_i - \bar{c})^2] = 0. \tag{6.9}$$

However, the bounds (5.33) and (5.37) of Hájek (1968) show that (6.9) holds for an arbitrary nondecreasing, absolutely continuous and square integrable ψ.

Next, we can easily see that

$$S_\gamma - ES_\gamma - T_\gamma = \int_0^1 (S_v - ES_v - T_v)\, d\gamma(v). \tag{6.10}$$

Consequently,

$$E(S_\gamma - ES_\gamma - T_\gamma)^2 \leqq [\gamma(1-) - \gamma(0+)] \int_0^1 E(S_v - ES_v - T_v)^2\, d\gamma. \tag{6.11}$$

Taking into account (6.4), (6.3) and (6.1), which entails $\text{Var } S_v \leqq 42N \max_{1\leqq i\leqq N} (c_i - \bar{c})^2 v(1 - v)$, we easily see that (6.9) holds also for ψ replaced by γ or η. Finally (6.1) and (6.6) entail

$$\begin{aligned} E(S_g - ES_g - T_g)^2 &\leqq 2 \text{ Var } S_g + 2 \text{ Var } T_g \\ &\leqq 92\, N \max_{1\leqq i\leqq N} (c_i - \bar{c})^2 \int_0^1 g^2(t)\, dt. \end{aligned} \tag{6.12}$$

Thus

$$E(S_g - S_h - ES_g + ES_h - T_g + T_h)^2 \leqq \alpha\, 184N \max_{1\leqq i\leqq N} (c_i - \bar{c})^2. \tag{6.13}$$

Now noting that (6.9) holds for ψ replaced by λ, γ, η and that α in (6.13) may be arbitrarily small, we conclude that

$$[E(S_\varphi - ES_\varphi - T_\varphi)^2]/[N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2] \to 0. \tag{6.14}$$

By (6.14) the problem of asymptotic normality of S_φ with natural parameters is reduced to the same problem for T_φ (see (5.6) and the text that follows). Assumption (2.23) entails

$$\liminf_{N\to\infty} (\operatorname{Var} T_\varphi)/[N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2] > 0 \tag{6.15}$$

since $\sigma_N^2 = \operatorname{Var} T_\varphi$. Also (2.21) in connection with (6.14) entails (6.15). Now $T_\varphi = \sum_{i=1}^N Y_i$, where

$$Y_i = \int_0^1 [u(t - H(X_i)) - L_i(t)](\tilde{c}(t) - c_i)\, d\varphi(t). \tag{6.16}$$

If φ were bounded, we would have

$$|Y_i| \leqq 2 \max_{1 \leqq j \leqq N} |c_j - \bar{c}| \text{ (a variation of } \varphi). \tag{6.17}$$

Therefore, in view of (6.15), $\max |Z_i|/\operatorname{Var} T_\varphi \to 0$, and the central limit theorem trivially applies.

If φ is not bounded, we write $\varphi(t) = b(t) + c(t)$, where $b(t)$ is bounded, and $\int c^2(t)\, dt < \epsilon$. Then, by (6.6)

$$[E(T_\varphi - T_b)^2]/[N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2] = [\operatorname{Var} T_c]/[N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2] \leqq 4\epsilon$$

and

$$((\operatorname{Var} T_b/\operatorname{Var} T_\varphi)^{\frac{1}{2}} - 1)^2 \leqq \operatorname{Var} T_c/\operatorname{Var} T_\varphi \leqq [4\epsilon N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2]/\operatorname{Var} T_\varphi .$$

Now, for large N, if ϵ is sufficiently small, $\operatorname{Var} T_b/\operatorname{Var} T_\varphi$ will be as close to 1 as we want (see (6.15)) and the difference between $\mathcal{L}(T_b(\operatorname{Var} T_b)^{-\frac{1}{2}})$ and $\mathcal{L}(T_\varphi(\operatorname{Var} T_b)^{-\frac{1}{2}})$ will also be small. Consequently, since $\mathcal{L}(T_b(\operatorname{Var} T_b)^{-\frac{1}{2}}) \to \mathfrak{N}(0, 1)$, we must also have $\mathcal{L}(T_\varphi(\operatorname{Var} T_\varphi)^{-\frac{1}{2}}) \to \mathfrak{N}(0, 1)$. From (6.14) and (6.15) we also have $\mathcal{L}((S_\varphi - ES_\varphi)(\operatorname{Var} S_\varphi)^{-\frac{1}{2}}) \to \mathfrak{N}(0, 1)$.

The proof will be completed, if we show that (2.5) may be relaxed to (2.7). Putting $S_\varphi = \sum_{i=1}^N c_i\varphi(R_i/(N + 1))$ and $S_\varphi^* = \sum_{i=1}^N c_i a_N(R_i)$, we have from (2.7)

$$\begin{aligned} E(S_\varphi - S_\varphi^* - ES_\varphi + ES_\varphi^*)^2 &\leqq \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2 [\textstyle\sum_{i=1}^N |a_N(i) - \varphi(i/(N + 1))|]^2 \\ &= o(N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2). \end{aligned}$$

This concludes the proof.

In addition we shall show that under conditions of Theorem 2 the regression constants have to be bounded in the sense of (2.3). We shall show that

$$\lim_{N\to\infty} [\textstyle\sum_{i=1}^N (c_i - \bar{c})^2]/[N \max_{1 \leqq i \leqq N} (c_i - \bar{c})^2] = 0 \tag{6.18}$$

entails

(6.19) $$\lim_{N\to\infty} [\mathrm{Var}\, S_\varphi]/[N \max_{1\leqq i\leqq N} (c_i - \bar{c})^2] = 0$$

contradicting thus our assumption (2.21). Since $\mathrm{Var}\, S_\varphi/\sigma_N^2 \to 1$, (6.19) contradicts also (2.23).

If γ is nondecreasing and bounded, then

$$\mathrm{Var}\, S_\gamma \leqq [\gamma(1-) - \gamma(0+)] \int_0^1 \mathrm{Var}\, S_v \, d\gamma(v)$$

where

$$\mathrm{Var}\, S_v \leqq 42N \max_{1\leqq i\leqq N} (c_i - \bar{c})^2 v(1 - v).$$

If $\int_B d\gamma = 0$, then we can see from (5.10) that (6.19) is satisfied for φ replaced by γ.

If ψ has a bounded second derivative then we may compile from results of Hájek (1968), namely from Theorem 4.2 and (5.17) and (2.14), that $\mathrm{Var}\, S_\psi = O(\sum_{i=1}^N (c_i - \bar{c})^2)$, so that (6.18) entails (6.19), if we replace φ by ψ. Finally for any nondecreasing φ, we have (6.1). Now under assumptions of Theorem 2 we may decompose φ as $\varphi = \psi + \gamma - \eta + \varphi_1 - \varphi_2$, where all components except ψ are nondecreasing, ψ has a bounded second derivative, γ and η are bounded and such that $\int_B (d\gamma + d\eta) = 0$, and φ_1 and φ_2 are arbitrarily small in mean square. However, that completes the proof of our assertion that the c_{Ni}'s have to be bounded in the sense of (2.3), if the conditions of Theorem 2 are satisfied.

7. Proof of Theorems 4 and 5. Let us first suppose, that the assumptions of Theorem 4 are satisfied, but let us use in the proof also the symbols L, M, λ, introduced in Theorem 5. Thus

$$L(t) = FH^{-1}(t) = FH_0^{-1}(H_0H^{-1}(t)),$$
$$H_0 = H + (\lambda_0 - \lambda)(F - G),$$
$$H_0H^{-1}(t) = t + (\lambda_0 - \lambda)(L(t) - M(t)) = t' \quad \text{(say)};$$

i.e., for each $0 < t < 1$ there exists a $0 < t' < 1$ such that

(7.1) $$L(t) = L_0(t') \quad \text{and} \quad |t - t'| \leqq K_{29}N^{-\frac{1}{2}}.$$

Further, from the identity $\lambda L(t) + (1 - \lambda)M(t) = t$, $0 < t < 1$, and from the fact that λ's are bounded away from zero as well as from one, it follows that the functions L, M (and also L_0, M_0) satisfy the Lipschitz condition for some constant K_{30}, uniformly in $N \geqq 1$. Hence also the validity of (2.12) follows for every $0 < v < 1$.

For each $v \in B$ let us set

(7.2) $$l_i(v) = \lambda_0\lambda^{-1}L_0'(v), \qquad 1 \leqq i \leqq m,$$
$$= (1 - \lambda_0)(1 - \lambda)^{-1}M_0'(v), \qquad m < i \leqq N.$$

We shall show, that (2.13) holds true, i.e., that for arbitrary $\epsilon > 0$ and $K > 0$

$$(7.3) \qquad |(L(t) - L(v))/(t-v) - \lambda_0\lambda^{-1}L_0'(v)| < \epsilon N^{-\frac{1}{2}}|t-v|^{-1}$$

for all $N > N_0(\epsilon, K)$ and all $|t - v| \leqq KN^{-\frac{1}{2}}$ (and similarly for $M(t)$). For $|t - v| < \epsilon/2K_{30}N^{\frac{1}{2}}$, the inequality (7.3) is evident, because its right-hand side is larger than $2K_{30}$, whereas its left-hand side is less than $2K_{30}$ (provided that N is sufficiently large). So it suffices to prove (7.3) in the domain

$$(7.4) \qquad \epsilon/2K_{30}N^{\frac{1}{2}} \leqq |t-v| \leqq KN^{-\frac{1}{2}};$$

as the right-hand side of (7.3) is $\geqq \epsilon/K$ in this domain, it actually suffices to prove

$$(7.5) \qquad |(L(t) - L(v))/(t-v) - \lambda_0\lambda^{-1}L_0'(v)| < \epsilon/K$$

in the domain (7.4). We observe that

$$(7.6) \qquad \begin{aligned} &(L(t) - L(v))/(t-v) \\ &\quad = [(L_0(t') - L_0(v))/(t'-v)](t'-v)/(t-v) \\ &\qquad - [(L_0(v') - L_0(v))/(v'-v)](v'-v)/(t-v), \end{aligned}$$

where t' and v' are defined by (7.1); and note that

$$|(t'-v)/(t-v)| \leqq 1 + 2K_{29}K_{30}/\epsilon$$

and

$$|(v'-v)/(t-v)| \leqq 2K_{29}K_{30}/\epsilon$$

for all t from the domain (7.4). Furthermore,

$$|(t'-v')/(t-v) - 1| \leqq K_{29}K_{30}N^{-\frac{1}{2}}$$

for all t. Finally, from (2.38) and from the existence of $L_0'(v)$ it follows that both the differences

$$|(L_0(t') - L_0(v))/(t'-v) - \lambda_0\lambda^{-1}L_0'(v)|$$

and

$$|(L_0(v') - L_0(v))/(v'-v) - \lambda_0\lambda^{-1}L_0'(v)|$$

are arbitrarily small (say $<\epsilon/3K(1 + 2K_{29}K_{30}\epsilon^{-1})$) for t from (7.4) and N large. Utilizing all these inequalities in (7.6), we obtain (7.5). Thus we have proved (2.13).

From (7.2) it immediately follows that (2.14) is fulfilled. (2.15) can also be easily proved. First, obviously, it cannot be that $L_0'(v) = 0$ and $M_0'(v) = 0$ simultaneously. So suppose $L_0'(v) > 0$ without loss of generality; hence $L_0(v)(1 - L_0(v))$ is positive, say equal to a $\Delta > 0$. But $L(v) \to L_0(v)$ for $N \to \infty$ hence $L(v)(1 - L(v)) > \frac{1}{2}\Delta$ for large N; from the boundedness of λ's away from 0 and 1, (2.15) then follows.

Now, evaluating (2.30) in our special case, we get

$$(7.7)\quad \sigma^2 = N\lambda \operatorname{Var}\left(\int_{\frac{1}{2}}^{H(X_1)} (1-\lambda_0)M_0'(v)\,d\varphi(v)\right) + N(1-\lambda)\operatorname{Var}\left(\int_{\frac{1}{2}}^{H(X_N)} \lambda_0 L_0'(v)\,d\varphi(v)\right).$$

It remains to pass to the limit (for $N \to \infty$). Let us denote

$$q(t) = \int_{\frac{1}{2}}^{t} M_0'(v)\,d\varphi(v);$$

with this notation,

$$(7.8)\quad \operatorname{Var}\int_{\frac{1}{2}}^{H(X_1)} M_0'(v)\,d\varphi(v) = \int_0^1 q^2(t)L'(t)\,dt - \left(\int_0^1 q(t)L'(t)\,dt\right)^2.$$

We may confine ourselves to a nondecreasing φ. The Lipschitz condition for M_0 entails the inequality

$$|q(t)| \leqq A\,|\varphi(t)| + B \quad \text{or} \quad q^2(t) \leqq C\varphi^2(t) + D$$

(with some constants A, B, C, D) and its uniform (in N) validity for the L's entails further that the integrals

$$\left(\int_0^{\epsilon} + \int_{1-\epsilon}^{1}\right) q^2(t)L'(t)\,dt$$

can be made arbitrarily small for all N and a suitable $\epsilon > 0$. Now, the integration by parts followed by the limit passage ($N \to \infty$) yields

$$\int_{\epsilon}^{1-\epsilon} q^2L'\,dt \to \int_{\epsilon}^{1-\epsilon} q^2L_0'\,dt,$$

and the limit is, in turn, arbitrarily close to $\int_0^1 q^2L_0'\,dt$. As the same argument applies to the integral $\int_0^1 q(t)L'(t)\,dt$, as well as to the second term on the right-hand side of (7.7), we get finally

$$(7.9)\quad \sigma^2 \approx N\lambda_0(1-\lambda_0)^2 \operatorname{Var}\int_{\frac{1}{2}}^{H_0(X_1)} M_0'\,d\varphi + N\lambda_0^2(1-\lambda_0)\operatorname{Var}\int_{\frac{1}{2}}^{H_0(X_N)} L_0'\,d\varphi$$

(in the sense that the ratio of both sides tends to 1). But the right-hand side of (7.9) is exactly $N\tau_0^2$; and the postulate $\tau_0^2 > 0$ implies (2.23). An application of Theorem 2 then completes the proof of Theorem 4.

As to the proof of Theorem 5, it is easy to show that the assumptions (2.12)–(2.14) are satisfied with

$$l_i(v) = L'(v) \qquad 1 \leqq i \leqq m,$$
$$= M'(v) \qquad m < i \leqq N, \qquad 0 < v < 1;$$

and that (2.30) gives exactly $N\tau^2$. It remains to prove (2.15). Since f and g are continuous and $f(x) + g(x) > 0$ on (α, β) there is a point y such that $f(y)g(y) > 0$. Let

$$\epsilon = \min\left[v, \int_{\alpha}^{y} f(u)\,du, \int_{y}^{\beta} f(u)\,du, \int_{\alpha}^{y} g(u)\,du, \int_{y}^{\beta} g(u)\,du\right].$$

Now we shall show that

$$[G(H^{-1}(v)) < \epsilon \quad \text{or} \quad G(H^{-1}(v)) > 1-\epsilon] \Rightarrow [\epsilon < F(H^{-1}(v)) < 1-\epsilon].$$

Assume that $G(H^{-1}(v)) < \epsilon$, for example. Then

$$mN^{-1}F(H^{-1}(v)) + nN^{-1}G(H^{-1}(v)) = v$$

entails $F(H^{-1}(v)) > \epsilon$. Further $H^{-1}(v) < y$, and

$$1 - F(H^{-1}(v)) = \int_{H^{-1}(v)}^{\beta} f(u)\, du > \int_{y}^{\beta} f(u)\, du \geqq \epsilon.$$

Thus $F(H^{-1}(v)) < 1 - \epsilon$. Similarly we may treat the case $G(H^{-1}(v)) > 1 - \epsilon$.

Consequently,

$$N^{-1} \sum_{i=1}^{N} L_i(v)[1 - L_i(v)] \geqq N^{-1} \min(m, n)\epsilon(1 - \epsilon)$$

and (2.15) is entailed by (2.42).

Thus, again, an application of Theorem 2 completes the proof.

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