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A MONOTONICITY PROPERTY OF THE DISTRIBUTION OF THE STUDENTIZED SMALLEST CHI-SQUARE[1]

BY KHURSHEED ALAM[2]

Indiana University

1. Main theorem. Let $X_1, \cdots, X_k$ be k i.i.d. random variables, each having a gamma distribution with m degrees of freedom. The random variable

$$X = \min(X_1/X_k, \cdots, X_{k-1}/X_k), \tag{1.1}$$

is called the Studentized smallest chi-square. Its cumulative distribution function (cdf) is given by

$$G_m(x) = 1 - \int_0^\infty (1-F_m(xy))^{k-1}\, dF_m(y) \tag{1.2}$$

where $F_m(y) = \{\Gamma(m)\}^{-1}\int_0^y x^{m-1} e^{-x}\, dx$ denotes the incomplete gamma function. Clearly, $G_m(1) = (k-1)/k$. A monotonicity property of the cdf of X, which has some applications, is given by the following theorem.

THEOREM 1.1. *For $m > 1$, $G_m(x)$ is increasing (decreasing) in m for $x > (<) 1$.*

PROOF. Let Y denote a random variable with cdf $F_m(y)$. For $m > 1$ let

$$X = F_m(cY) \tag{1.3}$$

where $c > 0$ is a constant. The probability density function of X is given by

$$\begin{aligned} g_m(x) &= (f_m(F_m^{-1}(x)/c))/(cf_m(F_m^{-1}(x))) \\ &= c^{-m}\exp((c-1)F_m^{-1}(x)/c), \qquad 0 < x < 1, \end{aligned} \tag{1.4}$$

where $f_m(x) = x^{m-1}e^{-x}/\Gamma(m)$ and $F_m^{-1}(x)$ denotes the inverse function of $F_m(x)$. For $r > 0$ let

$$\begin{aligned} A(x) &= f_m(F_m^{-1}(x)) - f_{m+r}(F_{m+r}^{-1}(x)) \\ &= F_{m-1}(F_m^{-1}(x)) - F_{m+r-1}(F_{m+r}^{-1}(x)) \end{aligned} \tag{1.5}$$

and

$$B(x) = \log g_{m+r}(x) - \log g_m(x). \tag{1.6}$$

Then

$$\begin{aligned} dB(x)/dx &= (c-1)c^{-1}(1/f_{m+r}(F_{m+r}^{-1}(x)) - 1/f_m(F_m^{-1}(x))) \\ &= (c-1)c^{-1}A(x)/(f_m(F_m^{-1}(x))f_{m+r}(F_{m+r}^{-1}(x))). \end{aligned} \tag{1.7}$$

It is shown below that $A(x)$ is nonnegative. Therefore, from (1.7) we have that $B(x)$ is nondecreasing (nonincreasing) in x for $c > (<) 1$. This result will be used in the sequel.

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[2] Now at Clemson University.

Let

(1.8) $$u = (m-1)F_{m+r}^{-1}(x) - (m+r-1)F_m^{-1}(x),$$ then

$$du/dx = ((m-1)f_m(F_m^{-1}(x)) - (m+r-1)f_{m+r}(F_{m+r}^{-1}(x))) \div (f_m(F_m^{-1}(x))f_{m+r}(F_{m+r}^{-1}(x)))$$

(1.9) $$= ((m-1)A(x) - rf_{m+r}(F_{m+r}^{-1}(x)))/(f_m(F_m^{-1}(x))f_{m+r}(F_{m+r}^{-1}(x))) < 0$$

for $A(x) < 0$, as $m > 1$ and $r > 0$. Also,

(1.10) $$dA(x)/dx = u/(F_{m+r}^{-1}(x)F_m^{-1}(x)).$$

To show that $A(x)$ is nonnegative for $0 \leqq x \leqq 1$, suppose that the contrary is true. As $A(x) = 0$ for $x = 0$ and 1 we have that for some value of $x = \xi$, say, where $0 < \xi < 1$,

(1.11) $$A(x) < 0$$ and

(1.12) $$dA(x)/dx < 0.$$

From (1.9) and (1.10) we have

(1.13) $$u < 0$$ and

(1.14) $$du/dx < 0$$

for $x = \xi$. Suppose that $A(x) < 0$ for $\xi \leqq x \leqq \xi + h < 1$. Then for $\xi \leqq x \leqq \xi + h$ we have from (1.9) that $du/dx < 0$, from (1.13) that $u < 0$ from (1.10) that $dA/dx < 0$. It follows that $A(x)$ is decreasing in x for $\xi \leqq x < 1$ which contradicts the relation $A(x) = 0$ for $x = 1$. Therefore, $A(x) \geqq 0$ for $0 \leqq x \leqq 1$.

A real-valued random variable X with probability density function $p_\theta(x)$ depending on a real parameter θ is said to have monotone likelihood ratio (m.l.r.) property if $p_{\theta_1}(x_1)p_{\theta_2}(x_2) \geqq p_{\theta_1}(x_2)p_{\theta_2}(x_1)$ for $x_1 < x_2$ and $\theta_1 < \theta_2$. The m.l.r. property implies that

(1.15) $$E_{\theta_1}\psi(X) \leqq (\geqq) E_{\theta_2}\psi(X)$$

for all monotone nondecreasing (nonincreasing) function $\psi(x)$. Strict inequality holds in (1.15) if $\psi(x)$ is strictly monotone.

From (1.7) and the result shown above that $A(x) \geqq 0$ we see that the distribution of X, given by (1.3), has m.l.r. property for $c > 1$ and in the opposite direction for $c < 1$. From (1.15) it follows that $G_m(c) = 1 - E(1-x)^{k-1}$ is increasing (decreasing) in m for $c > (<) 1$. □

2. Applications. Consider a multinomial population with K cells and the associated ordered probabilities $p_{[1]} \leqq \cdots \leqq p_{[k]}$ where $\sum_{i=1}^k p_{[i]} = 1$. Cacoullos and Sobel [1] have considered the sequential procedure for selecting the "best" cell, that is, the cell corresponding to $p_{[k]}$: Take observations one at a time from the population until any one cell has n counts in it and select that cell as the best cell.

For $(p_{[k]}/p_{[k-1]}) \geqq \theta > 1$ the probability of a correct selection (Pcs) is minimized for $p_{[i]} = 1/(\theta+k-1)$, $i = 1, \cdots, k-1$; $p_{[k]} = \theta/(\theta+k-1)$ and the minimum value of the Pcs is given by (see (4.5) of [1])

$$\text{(2.1)} \qquad \min \text{Pcs} = \frac{\Gamma(kN)}{(\Gamma(N))^k} \int_{\theta^{-1}}^{\infty} \cdots \int_{\theta^{-1}}^{\infty} \frac{(y_1 \cdots y_{k-1})^{N-1} \, dy_1 \cdots dy_{k-1}}{(1+\sum_1^{k-1} y_i)^{kN}} .$$

The multiple integral on the right-hand side of (2.1) can be shown to be equal to

$$\text{(2.2)} \qquad \Pr\{X_i \geqq \theta^{-1} X_k; \quad i = 1, \cdots, k-1\} = 1 - G_N(\theta^{-1})$$

where $X_1, \cdots, X_k$ denote k i.i.d. random variables, each having a gamma distribution with N degrees of freedom. From Theorem 1.1 it follows that the minimum value of the Pcs, given by (2.1) is increasing in N. Therefore, given θ and p^*, the smallest value of N for which Pcs $\geqq p^*$ when $(p_{[k]}/p_{[k-1]}) \geqq \theta$ is uniquely determined.

Similar application of Theorem 1.1 arises in a problem of selecting a subset of k given normal populations which includes the population with the smallest variance. This problem has been considered by Gupta and Sobel [2].

REFERENCES

[1] CACOULLOS, T. and SOBEL, M. (1966). An inverse sampling procedure for selecting the most probable event in a multinomial distribution. In *Proceedings of International Symposium on Multivariate Analysis.* Dayton, Aerospace Research Laboratories, 423–455.

[2] GUPTA, S. S. and SOBEL, M. (1961). On selecting a subset containing the population with the smallest variance. *Biometrika* **49** 495–507.