The Annals of Mathematical Statistics
1970, Vol. 41, No. 1, 237–250

THE ACCURACY OF INFINITELY DIVISIBLE APPROXIMATIONS TO SUMS OF INDEPENDENT VARIABLES WITH APPLICATION TO STABLE LAWS

BY VIROOL BOONYASOMBUT AND JESSE M. SHAPIRO

Chulalongkorn University (Bangkok); Ohio State University and Augsburg College

1. Introduction and summary. Let $\{F_n\}$ be a sequence of distribution functions defined on the real line, and suppose $\{F_n(x)\}$ converges to some limiting distribution function $F(x)$. It is of interest to investigate the error involved in using $F(x)$ as an approximation to $F_n(x)$, that is to investigate the rate of convergence of $\{F_n\}$ to F. This leads to the problem of finding bounds on $M_n = \sup_{-\infty<x<\infty} |F_n(x)-F(x)|$. In particular, this problem has been studied by several authors for cases where $F_n(x)$ represents the distribution function of a certain sum of independent random variables.

For cases involving the classical forms of the central limit theorem Berry [1] and Esseen [3] have obtained certain bounds on M_n which have been reinvestigated and improved by many authors (c.f. [4] Chapter XVI).

Let (X_{nk}), $k = 1, 2, \cdots, k_n$; $n = 1, 2, \cdots$ be a system of random variables such that for each n, $X_{n1}, \cdots, X_{nk}$ are independent (we say the system is independent within each row). In [6], under suitable conditions, bounds have been obtained on M_n for the case where $F_n(x)$ is the distribution function of $S_n = X_{n1}, +\cdots+X_{nk}$ and $F(x)$ is an infinitely divisible distribution. A basic assumption made in [6] was that both X_{nk} and $F(x)$ have finite variances.

The purpose of this study is to extend the results of [6] to include the case where neither $F(x)$ nor X_{nk} need have finite variance. Our main theorem (Theorem 1) gives a bound on M_n under a mild assumption on X_{nk} and a certain assumption on the derivative of the infinity divisible distribution $F(x)$. It is shown in Section 4, that if $F(x)$ satisfies an additional condition which is considerably weaker than that having finite variance, then the bound obtained tends to zero as n becomes infinite under necessary and sufficient conditions that $\{F_n(x)\}$ converge to $F(x)$.

In Section 5 our general results are applied to the case of convergence of distribution functions of normed sums of independent identically distributed random variables to an arbitrary stable law with exponent α, $0 < \alpha < 2$.

2. Notation and preliminaries. Let $F(x)$ be an infinitely divisible distribution with characteristic function $\varphi(t)$. According to the Lévy-Khintchine formula we have

$$\log \varphi(t) = i\gamma t + \int_{-\infty}^{\infty} \left(e^{itu} - 1 - \frac{i+u}{1+u^2}\right)\frac{1+u^2}{u^2}\, dG(u) \tag{2.1}$$

Received April 18, 1968.

where γ is a constant and $G(u)$ is a bounded nondecreasing function. The logarithm of $\varphi(t)$ can also be represented by Lévy's formula:

$$\log \varphi(t) = i\gamma t - \tfrac{1}{2}(t^2\sigma^2) + \int_{-\infty}^{0-} \left(e^{itx} - 1 - \frac{itx}{i+x^2}\right) dM(x) + \int_{0+}^{\infty} \left(e^{itx} - 1 - \frac{itx}{i+x^2}\right) dN(x), \tag{2.2}$$

where the connection between σ^2, M, N and G is given by

$$M(x) = \int_{-\infty}^{x} \frac{1+u^2}{u^2}\, dG(u) \qquad \text{for} \quad x < 0,$$

$$N(x) = -\int_{x}^{\infty} \frac{1+u^2}{u^2}\, dG(u) \qquad \text{for} \quad x > 0, \tag{2.3}$$

$$\sigma^2 = G(0+) - G(0-).$$

If an infinitely divisible distribution function F has finite variance, Kolmogorov's formula yields

$$\log \varphi(t) = i\mu t + \textstyle\int_{-\infty}^{\infty} (i^{itv} - 1 - itv)v^{-2}\, dK(v), \tag{2.4}$$

where μ is the mean of F, and K is a bounded nondecreasing function with $K(-\infty) = 0$ and $K(+\infty)$ equal to the variance of F. The relationship between (2.4) and (2.1) is given by

$$\mu = \gamma + \textstyle\int_{-\infty}^{\infty} u\, dG(u) \qquad \text{and} \tag{2.5}$$

$$K(v) = \textstyle\int_{-\infty}^{v} (1+u^2)\, dG(u).$$

A system of random variables (X_{nk}) as considered in Section 1 is said to be infinitesimal if for any $\varepsilon > 0$ $\lim_{n\to\infty} \max_{1 \leqq k \leqq k_n} P(|X_{nk}| > \varepsilon) = 0$. Given such a system, for any $a > 0$ let X_{nk}^a be defined by

$$X_{nk}^a = X_{nk} \qquad \text{if} \quad -a < x_{nk} \leqq a,$$

$$= 0 \qquad \text{otherwise;}$$

and let $F_{nk}^a(x)$, $\varphi_{nk}^a(t)$, $\mu_{nk}(a)$, $\sigma_{nk}^2(a)$ denote respectively, the distribution function, characteristic function, mean and variance of X_{nk}^a. Let $S_n = X_{n1} + \cdots + X_{nk_n}$ and $S_n{}^a = X_{n1}^a + \cdots + X_{nk_n}^a$. Let $F_n(x)$ and $\varphi_n(t)$ denote the distribution function and characteristic function of S_n and let $F_n{}^a(x)$, $\varphi_n{}^a(t)$, $\mu_n(a)$ and $\sigma_n{}^2(a)$ denote respectively the distribution function, characteristic function, mean, and variance of $S_n{}^a$.

Let $F(x)$ be an infinitely divisible distribution with corresponding $G(u)$ and γ given by (2.1). For each $a > 0$ such that $\pm a$ are continuity points of $G(u)$ we define

$$G^a(u) = 0, \qquad u \leqq -a \tag{2.6}$$

$$= G(u) - G(-a), \qquad -a < u \leqq a$$

$$= G(a) - G(-a), \qquad u > a$$

and

$$\gamma^a = \gamma - \int_{|u|>a} u^{-1}\, dG(u).$$

The nondecreasing function G^a and the constant γ^a determine a unique infinitely divisible distribution $F^a(x)$ through the formula (2.1). In [8] it is shown that if $F_n(x) \to F(x)$ then for any $a > 0$ (with $\pm a$ continuity points of G) $F_n{}^a(x) \to F^a(x)$ and that $\lim_{n\to\infty} \int_{-\infty}^{\infty} x^k\, dF_n{}^a(x) = \int_{-\infty}^{\infty} x^k\, dF^a(x)$ for any positive integer k (the result of [7] shows that limit is finite). We let $\varphi^a(t)$, $\mu(a)$, and $\sigma^2(a)$ denote the characteristic function mean, and variance of $F^a(x)$. From (2.4) we have

$$\log \varphi^a(t) = i\mu(a)t + \int_{-\infty}^{\infty} (e^{itv} - 1 - itv)v^{-2}\, dK^a(v) \tag{2.7}$$

where

$$\mu(a) = \gamma^a + \int_{-\infty}^{\infty} u\, dG^a(u) \qquad \text{and}$$

$$K^a(v) = \int_{-\infty}^{v} (1+u^2)\, dG^a(v).$$

Let

$$K_n{}^a(v) = \sum_{k=1}^{k_n} \int_{-\infty}^{v} x^2\, dF_{nk}^a(x + \mu_{nk}(a)).$$

For any $A > 0$ such that $-A$ and A are continuity points of $G(u)$, and hence continuity points of $K^a(v)$, let $0 < \delta \leqq 2A$ and define

$$m = m(A, \delta) = [2A/\delta] + 1.$$

Let $-A = x_0 < x_1 < \cdots < x_m = A$ be such that x_i is a continuity point of $K^a(v)$ and such that $\max_{1 \leqq i \leqq m}(x_i - x_{i-1}) < \delta$.
Define

$$\begin{aligned} E^a(n, t, m(A, \delta)) = {} & \tfrac{5}{4}\delta|t|^3(\sigma_n{}^2(a) + \sigma^2(a)) + \tfrac{1}{2}t^2 \textstyle\sum_{i=0}^{m} |K_n{}^a(x_i) - K^a(x_i)| \\ & + 2A^{-1}|t|(K_n{}^a(+\infty) - K_n{}^a(A) + K^a(+\infty) - K^a(A) \\ & + K_n{}^a(-A) + K^a(-A)). \end{aligned} \tag{2.8}$$

E^a is used in obtaining the desired bounds which will involve $g^a(n, m(A, \delta), r)$ defined by

$$\begin{aligned} g^a(n, m(A, \delta), r) = {} & [\tfrac{1}{3}\sigma_n{}^2(a) \max_{1 \leqq k \leqq k_n} \sigma_{nk}^2(a)]^{1/5} + [\tfrac{5}{6}\delta(\sigma_n{}^2(a) + \sigma^2(a)]^{1/4} \\ & + [\tfrac{1}{2} \textstyle\sum_{i=0}^{m} |K_n{}^a(x_i) - K^a(x_i)|]^{1/3} \\ & + \{4A^{-1}[K_n{}^a(+\infty) - K_n{}^a(A) + K^a(+\infty) - K^a(A) \\ & + K_n{}^a(-A) + K^a(-A)] + 2|\mu_n(a) - \mu(a)|\}^{1/2} \\ & + [8r^{-1} \textstyle\int_{|u|>a} |u|^r\, dG(u)]^{1/(1+r)} \end{aligned} \tag{2.9}$$

where r is a positive real number.

3. The general bound.

We require two lemmas.

LEMMA 1. *With $F_n(x)$ and $F_n{}^a(x)$ defined as in Section 2, it follows that*

$$|F_n(x)-F_n{}^a(x)| \leqq \sum_{k=1}^{k_n} \{F_{nk}(-a)+1-F_{nk}(a)\}.$$

PROOF. We have that

$$\begin{aligned} F_n(x) = P(S_n \leqq x) &= P(S_n \leqq x,\ X_{nk} \in (-a, a] \text{ for all } k) \\ &\quad + P(S_n \leqq x \text{ and } X_{nk} \notin (-a, a] \text{ for some } k) \\ &\leqq P(S_n{}^a \leqq x) + P(X_{nk} \notin (-a, a] \text{ for some } k) \\ &\leqq F_n{}^a(x) + \textstyle\sum_{k=1}^{k_n} P(X_{nk} \notin (-a, a]). \end{aligned}$$

Thus $F_n(x)-F_n{}^a(x) \leqq \sum_{k=1}^{k_n} [F_{nk}(-a)+1-F_{nk}(a)]$. A similar argument, starting with $F_n{}^a(x)$ proves the lemma.

LEMMA 2. *Let $F(x)$ be an infinitely divisible distribution with characteristic function $\varphi(t)$ and with corresponding $G(u)$ given by (2.1). Then for any real numbers a and r with $a > 1$, ($\pm a$ continuity points of G) and $0 < r \leqq 1$ we have $|\varphi^a(t)-\varphi(t)| \leqq 4|t|^r \int_{|u|>a} |u|^r \, dG(u)$ where $\varphi^a(t)$ is given by (2.7).*

PROOF. Since $\log \varphi^a(t)$ is given by formula (2.1) using $G^a(u)$ and γ^a given by (2.6), we have using Lemma 1 of [6] that

$$\begin{aligned} |\varphi^a(t)-\varphi(t)| &\leqq |\log \varphi^a(t) - \log \varphi(t)| \\ &= \left| -it \int_{|u|>a} u^{-1}\, dG(u) - \int_{|u|>a} \left(e^{itu} - 1 - \frac{itu}{1+u^2}\right) \frac{1+u^2}{u^2}\, dG(u) \right| \\ &= \left| \textstyle\int_{|u|>a} (e^{itu}-1) u^{-2}(1+u^2)\, dG(u) \right|. \end{aligned}$$

Now for $|u| > a > 1$ we have

$$\left|(e^{itu}-1)\frac{1+u^2}{u^2}\right| \leqq 2|e^{itu}-1| = 4|\sin \tfrac{1}{2}tu|.$$

Furthermore, for $|\frac{1}{2}tu| \geqq 1$ we have $4|\sin \frac{1}{2}tu| \leqq 4 \leqq 4|\frac{1}{2}tu|^r \leqq 4|tu|^r$, and for $|\frac{1}{2}tu| \leqq 1$ and $0 < r \leqq 1$ we have $4|\sin \frac{1}{2}tu| \leqq 4|\frac{1}{2}tu| \leqq 4|\frac{1}{2}tu|^r \leqq 4|tu|^r$.

It follows that $|\varphi^a(t)-\varphi(t)| \leqq 4|t|^r \int_{|u|>a} |u|^r dG(u)$, which proves the lemma.

We are now in a position to state and prove the theorem giving a general bound mentioned in Section 1. We use the notation developed in Section 2.

THEOREM 1. *Let $F(x)$ be an infinitely divisible distribution function with corresponding $G(u)$ given by the Levy-Khintchine formula (2.1). Let (X_{nk}) $k = 1, \cdots k_n$; $n = 1, 2, \cdots$ be a system of random variables independent within each row. Let $F_n(x)$ be the distribution function of the sum $S_n = X_{n1}, + \cdots + X_{nk_1}$ and suppose that*

$$dF(x)/dx = F'(x) \quad \textit{exists and} \quad |F'(x)| < B \quad \textit{for all} \quad x.$$

Assume that $\sigma_{nk}^2(a) \leqq 1$ *for all* n *and* k *(as will be seen in the proof of Theorem* 2, *this assumption is quite weak). Then it follows that for* $0 < r \leqq 1$ *and* $a \geqq 1$,

$$M_n = \sup_{-\infty < x < \infty} |F_n(x) - F(x)| \leqq k(B) g^a(n, m(A,\delta), r) + \sum_{k=1}^{k_n} \{F_{nk}(-a) + 1 - F_{nk}(a)\}$$

where $k(B)$ *is a constant depending only on* B *and where* $g^a(n, m(A, \delta), r)$ *is given by* (2.9).

PROOF. We have

$$|F_n(x) - F(x)| \leqq |F_n(x) - F_n^a(x)| + |F_n^a(x) - F(x)|. \tag{3.1}$$

Letting $\varphi(t)$ be the characteristic function of $F(x)$, from Lemma 2 it follows that

$$\begin{aligned} |\varphi_n^a(t) - \varphi(t)| &\leqq |\varphi_n^a(t) - \varphi^a(t)| + |\varphi^a(t) - \varphi(t)| \\ &\leqq |\varphi_n^a(t) - \varphi^a(t)| + 4|t|^r \textstyle\int_{|u|>a} |u|^r \, dG(u). \end{aligned}$$

If we let $T_n = [g^a(n, m(A, \delta), r)]^{-1}$ and restrict $|t| \leqq T_n$, by an argument analogous to the proof of Theorem 3 of [6], it follows that

$$\begin{aligned} |\varphi_n^a(t) - \varphi^a(t)| \leqq {} & \tfrac{5}{8} t^4 \max_{1 \leqq k \leqq k_n} \sigma_{nk}^2(a) \sigma_n^{\,2}(a) \\ & + |\mu_n(a) - \mu(a)|\,|t| + E^a(n, t, m(A, \delta)) \end{aligned}$$

where E^a is given by (2.8). Thus

$$\begin{aligned} |\varphi_n^a(t) - \varphi(t)| &\leqq \tfrac{5}{8} t^4 \max_{1 \leqq k \leqq k_n} \sigma_{nk}^2(a) \sigma^2(a) \\ &\quad + |\mu_n(a) - \mu(a)|\,|t| + E^a(n, t, m(A, \delta)) + 4|t|^r \textstyle\int_{|u|>a} |u|^r \, dG(u) \\ &\equiv h^a(t, n, m(A, \delta), r). \end{aligned}$$

From this it follows that

$$\int_{-T_n}^{T_n} |t^{-1}(\varphi_n^a(t) - \varphi(t))| \, dt \leqq 2 \int_0^{T_n} t^{-1} [h^a(t, n, m(A, \delta), r)] \, dt \leqq g^a(n, m(A, \delta), r).$$

Now applying a Theorem of Esseen ([3] Theorem 2a, page 32), we have

$$M_n^a \equiv \sup_{-\infty < x < \infty} |F_n^a(x) - F(x)| \leqq (2\pi)^{-1} p g^a(n, m(A, \delta), r) + c(p) \cdot B/T_n$$

where $p > 1$ and $c(p)$ is a constant depending only on $p (p > 1$ is arbitrary). Thus

$$M_n^a \leqq (p/(2\pi) + c(p)B) g^a(n, m(A, \delta), r).$$

Applying Lemma 1 and (3.1) we have the proof of the Theorem.

4. Behavior of the estimate. In this section we examine, under suitable conditions, the behavior of the bound

$$D(n, A, \delta, a, r) = k(B) g^a(n, m(A, \delta), r) + \sum_{k=1}^{k_n} \{F_{nk}(-a) + 1 - F_{nk}(a)\} \tag{4.1}$$

given in Theorem 1. Several lemmas will be needed.

LEMMA 3. *Let* $\{Q_n(a)\}$ $n = 0, 1, 2, \cdots$ *be defined for* $a > 0$ *and be such that*

(i) *for each* n, $Q_n(a) \geqq Q_n(b) \geqq 0$ *for* $a < b$.

(ii) $\lim_{a\to+\infty} Q_0(a) = 0$.

(iii) $\lim_{n\to\infty} Q_n(a) = Q_0(a)$ *at every continuity point of* $Q_0(a)$.

Then for any *nondecreasing sequence* $\{a_n\}$ *such that each* a_n *is a continuity point of* $Q_0(a)$, *and such that* $\lim_{n\to\infty} a_n = +\infty$ *we have* $\lim_{n\to\infty} Q_n(a_n) = 0$.

PROOF. Given $\varepsilon > 0$, let k be such that $Q_0(a_k) < \varepsilon/2$. Let n_k be such that $n > n_k$ implies $Q_n(a_k) < Q_0(a_k)+\varepsilon/2 < \varepsilon$. Now for $n > n_k$ we have $n > k$ so that by (i), $Q_n(a_n) \leqq Q_n(a_k) < \varepsilon$, which proves the lemma.

LEMMA 4. *Let* (X_{nk}), $F_n(x)$, $F(x)$ *and* $G(u)$ *be as in Theorem* 1. *If*

(i) *the system* (X_{nk}) *is infinitesimal*,

(ii) $\lim_{n\to\infty} F_n(x) = F(x)$ *at all continuity points of* $F(x)$,

then for any nondecreasing sequence $\{a_n\}$ *such that* $-a_n$ *and* a_n *are continuity points of* G *and* $\lim_{n\to\infty} a_n = \infty$ *we have*

$$\lim_{n\to\infty} \sum_{k=1}^{k_n} \{F_{nk}(-a_n)+1-F_{nk}(a_n)\} = 0.$$

PROOF. Let $M(x)$ and $N(x)$ be given by (2.2) or (2.3) and let

$$Q_n(a) = \sum_{k=1}^{k_n} \{F_{nk}(-a)+1-F_{nk}(a)\} \quad \text{and} \quad Q_0(a) = M(-a)-N(a).$$

By Theorem 1, page 116 of [5] we have $\lim_{n\to\infty} Q_n(a) = Q_0(a)$ at continuity points of $Q_0(a)$ so that by Lemma 3, $\lim_{n\to\infty} Q_n(a_n) = 0$.

LEMMA 5. *Let* $g(n, a, \delta)$ *be nonnegative and be such that to each* a *there corresponds a sequence* $\{\delta_n(a)\}$ *of positive real numbers such that* $\lim_{n\to\infty} g(n, a, \delta_n(a)) = 0$. *Then there exists a nondecreasing sequence* $\{a_n\}$ *such that* $\lim_{n\to\infty} a_n = \infty$ *and a sequence* $\{\delta_n\}$ *such that*

$$\lim_{n\to\infty} g(n, a_n, \delta_n) = 0.$$

PROOF. Let $\{\varepsilon_k\}$ be a sequence such that $\varepsilon_k \downarrow 0$ and let $\{\bar{a}_n\}$ be such that $\bar{a}_n \uparrow \infty$. By hypothesis we can choose $\{n_k\}$ such that $n_k > n_{k-1}$ and such that $g(n, \bar{a}_k, \delta_n(\bar{a}_k)) < \varepsilon_k$ for $n > n_k$.
If we define

$$\begin{aligned} \delta_n &= \delta_1(\bar{a}_1) \quad \text{for} \quad n \leqq n_1, \\ &= \delta_n(\bar{a}_k) \quad \text{for} \quad n_k < n \leqq n_{k+1}, \qquad k = 1, 2, \cdots \end{aligned}$$

and

$$\begin{aligned} a_n &= \bar{a}_1 \quad \text{for} \quad n \leqq n_1 \\ &= \bar{a}_k \quad \text{for} \quad n_k < n \leqq n_{k+1}, \qquad k = 1, 2, \cdots \end{aligned}$$

it is not difficult to see that $\{a_n\}$ and $\{\delta_n\}$ satisfy the conclusion of the lemma.

We now show that under suitable conditions we can choose sequences $\{A_n\}$, $\{\delta_n\}$ and $\{a_n\}$ such that the bound, $D(n, A_n, \delta_n, a_n, r)$ given by (4.1) approaches zero as n becomes infinite.

Using the same notation as in Theorem 1, let $0 < \delta < 1$ be such that $\pm\delta^{-\frac{1}{2}}$ are continuity points of $G(u)$. Using (2.9) define $g(n, a, \delta)$ by

$$g^a(n, m(\delta^{-\frac{1}{2}}, \delta), r) = g(n, a, \delta) + \{8r^{-1} \textstyle\int_{|u|>a} |u|^r \, dG(u)\}^{1/(1+r)}. \tag{4.2}$$

This leads to the main result of this section.

THEOREM 2. *Let (X_{nk}), $F_n(x)$ and $F(x)$ satisfy the conditions of Theorem 1. Assume further that the random variables (X_{nk}) are infinitesimal, $F_n(x)$ converges to $F(x)$ at continuity points of F, and that for some r, (without loss of generality assume $0 < r \leqq 1$)*

$$\textstyle\int_{-\infty}^{\infty} |u|^r \, dG(u) < \infty.$$

Then there exist sequences $\{a_n\}$ and $\{\delta_n\}$ such that, for large n

$$M_n = \sup_{-\infty<x<\infty} |F_n(x) - F(x)| \leqq D(n, \delta_n^{-\frac{1}{2}}, \delta_n, a_n, r), \quad \textit{and} \tag{4.3}$$

$$\lim_{n\to\infty} D(n, \delta_n^{-\frac{1}{2}}, \delta_n, a_n, r) = 0. \tag{4.4}$$

(The function D is given by (4.1) and (2.9)).

PROOF. Since (X_{nk}) are infinitesimal, (X_{nk}^a) as well as $(X_{nk}^a - \mu_{nk}(a))$ are also infinitesimal. By Theorems 3 and 6 of [8], if $\pm a$ are continuity points of G, we have $\lim_{n\to\infty} F_n^a(x) = F^a(x)$ and $\lim_{n\to\infty} \sigma_n^2(a) = \sigma^2(a)$. It follows from the proof of Theorem 5 of [6] that there exists a sequence $\{\delta_n(a)\}$ such that

$$\lim_{n\to\infty} g(n, a, \delta_n(a)) = 0. \tag{4.5}$$

By Lemma 5 we can find sequences $\{a_n\}$ and $\{\delta_n\}$ such that

$$\lim_{n\to\infty} g(n, a_n, \delta_n) = 0, \tag{4.6}$$

$a_n \leqq a_{n+1}$, and $\lim_{n\to\infty} a_n = \infty$. (Note that from (4.2) and (2.9) we see that (4.5) and (4.6) imply $\lim_{n\to\infty} \max_k \sigma_{nk}^2(a) = \lim_{n\to\infty} \max_k \sigma_{nk}^2(a_n) = 0$. This justifies the parenthetical remark in the statement of Theorem 1.) Clearly from the proof of Lemma 5, a_n can be chosen so that $\pm a_n$ are continuity points of $G(u)$ so that the conclusion of Lemma 4 holds. Now since $\int_{-\infty}^{\infty} |u|^r \, dG(u) < \infty$ it follows that $\lim_{n\to\infty} \int_{|u|>a_n} |u|^r \, dG(u) = 0$. Since

$$D(n, \delta_n^{-\frac{1}{2}}, \delta_n, a_n, r) = k(B)\{g(n, a_n, \delta_n) + [8r^{-1} \textstyle\int_{|u|>a_n} |u|^r \, dG(u)]^{1/1+r}\}$$
$$+ \textstyle\sum_{k=1}^{k_n} \{F_{nk}(-a) + 1 - F_{nk}(a)\}$$

we see that (4.4) holds. Finally from Theorem 1, as soon as n is so large that $a_n \geqq 1$ and $\max_k \sigma_{nk}^2(a_n) \leqq 1$, it follows that (4.3) holds.

5. Application to stable laws. The basic properties of stable distributions can be found in [4] or [5]. We recall that a distribution function $F(x)$ is stable if and only if there exists a sequence of independent identically distributed random variables $\{X_n\}$ and constants A_n and $B_n > 0$ such that

$$\lim_{n\to\infty} P\left\{\frac{X_1+\cdots+X_n}{B_n}-A_n \leqq x\right\} = F(x). \tag{5.1}$$

Since stable distributions are infinitely divisible, by letting $X_{nk} = X_k/B_n$ it can be seen that our previous results can be applied to limit theorems of the type (5.1).

As is well known to every stable distribution, there corresponds an exponent $\alpha(0 < \alpha \leqq 2)$. The case $\alpha = 2$ corresponds to the normal distribution and will not be discussed here. From the theorem on page 164 of [5] we know that for stable distributions with exponent α, $(0 < \alpha < 2)$, the functions M and N and the constant σ^2 in (2.3) are given

$$\begin{aligned} M(x) &= c_1|x|^{-\alpha}, \qquad x<0 \\ N(x) &= -c_2 x^{-\alpha}, \qquad x>0 \\ \sigma^2 &= 0 = G(0+)-G(0-) \end{aligned} \tag{5.2}$$

where c_1 and c_2 are nonnegative and $c_1+c_2 > 0$. From Section 36 of [5] it follows that all proper stable distributions (and hence all stable distributions with $0 < \alpha < 2$) have bounded derivatives of all orders of every point. Thus for (proper) stable distributions the assumption in Theorem 1 on the derivative of $F(x)$ is always satisfied.

The next lemma removes one of the hypotheses of Theorem 2.

LEMMA 6. *If $F(x)$ is a stable distribution function with corresponding function $G(u)$ given by the formula* (2.1), *then there exists a real number $r > 0$ such that*

$$\int_{-\infty}^{+\infty} |u|^r\, dG(u) < +\infty.$$

PROOF. From (5.2) and (2.3) we note that

$$\begin{aligned} dG(u) &= c_1\frac{|u|^{1-\alpha}}{1+u^2}du, \qquad \text{for} \quad u<0 \\ &= c_2\frac{u^{1-\alpha}}{1+u^2}du, \qquad \text{for} \quad u>0. \end{aligned} \tag{5.3}$$

Thus
$$\int_{-\infty}^{+\infty} |u|^r\, dG(u) = (c_1+c_2)\int_{0+}^{+\infty} \frac{x^{1-(\alpha-r)}}{1+x^2}\,dx$$

which is finite if $r < \alpha$.

The next two lemmas will be used to simplify the general estimate given in Theorem 1 to the stable case.

LEMMA 7. *Let $F(x)$ be a stable distribution function with representation given by* (2.2) *and* (5.2). *Then using the notation of Section* 2,

$$\mu(a) = \gamma + (c_1 - c_2)\left\{\int_a^{+\infty} \frac{du}{(1+u^2)u^\alpha} - \int_0^a \frac{u^{2-\alpha}\,du}{1+u^2}\right\}, \tag{5.4}$$

$$\sigma^2(a) = (c_1 + c_2)\frac{a^{2-\alpha}}{2-\alpha}, \tag{5.5}$$

and

$$\begin{aligned} K^a(v) &= 0, & v &< -a, \\ &= c_1\left(\frac{a^{2-\alpha}}{2-\alpha} - \frac{|v|^{2-\alpha}}{2-\alpha}\right), & -a &\leqq v < 0, \\ &= c_1\left(\frac{a^{2-\alpha}}{2-\alpha}\right) + c_2\left(\frac{v^{2-\alpha}}{2-\alpha}\right), & 0 &\leqq v < a \\ &= (c_1 + c_2)\left(\frac{a^{2-\alpha}}{2-\alpha}\right), & a &\leqq v. \end{aligned} \tag{5.6}$$

PROOF. From (5.3) and (2.6) we have

$$\gamma^a = \gamma + (c_1 - c_2)\int_a^{+\infty} \frac{du}{(1+u^2)u^\alpha}.$$

Hence, from (2.7)

$$\begin{aligned} \mu(a) &= \gamma^a + \textstyle\int_{-a}^{a} u\, dG(u) \\ &= \gamma^a + \int_{-a}^{0} u c_1 \frac{|u|^{1-\alpha}}{1+u^2}\,du + \int_0^a u c_2 \frac{u^{1-\alpha}}{1+u^2}\,du \\ &= \gamma^a - (c_1 - c_2)\int_0^a \frac{u^{2-\alpha}}{1+u^2}\,du \\ &= \gamma + (c_1 - c_2)\left\{\int_a^{+\infty} \frac{du}{(1+u^2)u^\alpha} - \int_0^a \frac{u^{2-\alpha}}{1+u^2}\,du\right\}. \end{aligned}$$

This proves (5.4). For $-a < v < 0$, we have

$$\begin{aligned} K^a(v) &= \textstyle\int_{-\infty}^{v} (1+u^2)\, dG^a(u) = \int_{-a}^{v} (1+u^2)\, dG(u) \\ &= \int_{-a}^{v} c_1 |u|^{1-\alpha}\,du = c_1\left(\frac{a^{2-\alpha}}{2-\alpha} - \frac{v^{2-\alpha}}{2-\alpha}\right). \end{aligned}$$

For $0 < v < a$, we have

$$\begin{aligned} K^a(v) &= \textstyle\int_{-\infty}^{v} (1+u^2)\, dG^a(u) \\ &= \textstyle\int_{-a}^{0} (1+u^2)\, dG(u) + [G(0+) - G(0-)] + \int_0^v (1+u^2)\, dG(u) \\ &= c_1\left(\frac{a^{2-\alpha}}{2-\alpha}\right) + c_2\left(\frac{v^{2-\alpha}}{2-\alpha}\right), \end{aligned}$$

using (5.3) and the fact that $G(0+)-G(0-) = \sigma^2 = 0$ from (5.2). From this (5.6) follows. Formula (5.5) follows from (5.6) and the fact that $\sigma^2(a) = K^a(+\infty)$.

LEMMA 8. *Let $F(x)$ be a stable distribution function as given in Lemma 7. If $0 < r < \alpha$, then*

$$\int_{|u|>a} |u|^r \, dG(u) \leqq \frac{c_1+c_2}{(\alpha-r)a^{\alpha-r}}.$$

PROOF. Using (5.3) we have

$$\begin{aligned}\int_{|u|>a} |u|^r \, dG(u) &= (c_1+c_2)\int_a^{+\infty} \frac{u^{1+r-\alpha}}{1+u^2}\, du \\ &\leqq (c_1+c_2)\int_a^{+\infty} \frac{u^{1+r-\alpha}}{u^2}\, du \\ &= \frac{c_1+c_2}{(\alpha-r)a^{\alpha-r}},\end{aligned}$$

which proves the lemma.

As was done at the beginning of this section, let $\{X_n\}$, $n = 1, 2, \cdots$ be a sequence of independent random variables with a common distribution function $\bar{F}(x)$. For each n, let

$$S_n = \frac{X_1 + \cdots + X_n}{B_n},$$

where B_n are suitably chosen positive constants. Let $F_n(x)$ denote the distribution function of S_n. To apply our general result, we let

$$X_{nk} = X_k/B_n.$$

The results expressed in (5.7)–(5.13) follow easily.

$$\begin{aligned} X^a_{nk} &= X_k/B_n \qquad -aB_n < X_k \leqq aB_n, \\ &= 0 \qquad \text{otherwise.}\end{aligned} \tag{5.7}$$

$$F_{nk}(x) = \bar{F}(xB_n). \tag{5.8}$$

$$\begin{aligned} F^a_{nk}(x) &= 0 && \text{for} \quad x \leqq -a, \\ &= \bar{F}(xB_n) - \bar{F}(-aB_n) && \text{for} \quad -a < x < 0, \\ &= \bar{F}(xB_n) + 1 - \bar{F}(aB_n) && \text{for} \quad 0 \leqq x \leqq a, \\ &= 1 && \text{for} \quad a \leqq x.\end{aligned} \tag{5.9}$$

$$\mu_{nk}(a) = B_n^{-1} \textstyle\int_{-aB_n}^{aB_n} x \, d\bar{F}(x) \quad \text{and} \quad \mu_n(a) = Bn_n^{-1} \int_{-aB_n}^{aB_n} x \, d\bar{F}(x). \tag{5.10}$$

$$\sigma^2_{nk}(a) = B_n^{-2} \{ \textstyle\int_{-aB_n}^{aB_n} x^2 \, d\bar{F}(x) - (\int_{-aB_n}^{aB_n} x \, d\bar{F}(x))^2 \} \quad \text{and} \tag{5.11}$$

$$\sigma_n{}^n(a) = Bn_n^{-2} \{ \textstyle\int_{-aB_n}^{aB_n} x^2 \, d\bar{F}(x) - (\int_{-aB_n}^{aB_n} x \, d\bar{F}(x))^2 \}.$$

$$(5.12)\qquad \begin{aligned} K_n{}^a(v) &= n\textstyle\int_{-\infty}^{v+\mu_{nk}(a)}(u-\mu_{nk}(a))^2\,dF^a_{nk}(u) \\ &= 0, \qquad v+\mu_{nk}(a) \leqq -a, \\ &= n\textstyle\int_{-a}^{v+\mu_{nk}(a)}(u-\mu_{nk}(a))^2\,d\bar{F}(uB_n), \qquad -a < v+\mu_{nk}(a) < 0, \\ &= n\textstyle\int_{-a}^{v+\mu_{nk}(a)}(u-\mu_{nk}(a))^2\,d\bar{F}(uB_n) \\ &\quad +[\mu_{nk}(a)]^2[1-\bar{F}(aB_n)+\bar{F}(-aB_n)], \qquad 0 \leqq v+\mu_{nk}(a) < a, \\ &= n\textstyle\int_{-a}^{a}(u-\mu_{nk}(a))^2\,d\bar{F}(uB_n) \\ &\quad +[\mu_{nk}(a)]^2[1-\bar{F}(aB_n)+\bar{F}(-aB_n)], \qquad a < v+\mu_{nk}(a). \end{aligned}$$

Finally we have

$$(5.13)\qquad \textstyle\sum_{k=1}^{k_n}\{F_{nk}(-a)+1-F_{nk}(a)\} = n\{\bar{F}(-aB_n)+1-\bar{F}(aB_n)\}.$$

The following theorem is an immediate consequence of Theorem 1, Lemmas 7 and 8, and (5.7)–(5.13).

THEOREM 3. *Let* $\{X_n\}$, $n = 1, 2, \cdots$ *be a sequence of independent random variables with a common distribution function* $\bar{F}(x)$. *For each* n, *let*

$$S_n = \frac{X_1+\cdots+X_n}{B_n},$$

where B_n *are positive constants. Let* $F_n(x)$ *denote the distribution function of* S_n. *Let* $F(x)$ *be a stable distribution function with exponent* α *given by* (2.2) *and* (5.2). *Assume that*

$$B_n{}^{-2}\{\textstyle\int_{-aB_n}^{aB_n}x^2\,dF(x)-(\int_{-aB_n}^{aB_n}x\,dF(x))^2\} \leqq 1.$$

(We note that this is $\sigma^2_{nk}(a) \leqq 1$ so that by the proof of Theorem 2 this is a weak assumption.)

Then for, $0 < r \leqq 1, a > 1$, *we have*

$$\begin{aligned} M_n &= \sup_{-\infty<x<+\infty}|F_n(x)-F(x)| \\ &\leqq kg^a(n, m(A,\delta), r)+n\{\bar{F}(-aB_n)+1-\bar{F}(aB_n)\}, \end{aligned}$$

where k *is a constant depending only on the bound of the derivative of* F, *and*

$$\begin{aligned} &g^a(n, m(A,\delta), r) \\ &= \left\{\frac{n}{3B_n{}^4}\left[\int_{-aB_n}^{aB_n}x^2\,dF(x)-\left(\int_{-aB_n}^{aB_n}x\,dF(x)\right)^2\right]^2\right\}^{1/5} \\ &\quad +\left\{\frac{5}{6}\delta\left[\frac{n}{B_n{}^2}\left\{\int_{-aB_n}^{aB_n}x^2\,dF(x)-\left(\int_{-aB_n}^{aB_n}x\,dF(x)\right)^2\right\}+\frac{(c_1+c_2)a^{2-\alpha}}{2-\alpha}\right]\right\}^{1/4} \\ &\quad +\{\tfrac{1}{2}\textstyle\sum_{i=0}^{m}|K_n{}^a(x_i)-K^a(x_i)|\}^{1/3} \\ &\quad +\left\{\frac{4}{A}\left[\frac{n}{B_n{}^2}\left(\int_{-aB_n}^{aB_n}x^2\,dF(x)-\left(\int_{-aB_n}^{aB_n}x\,dF(x)\right)^2\right)-K_n{}^a(A)+\frac{(c_1+c_2)a^{2-\alpha}}{2-\alpha}\right.\right. \\ &\quad \left.-K^a(A)+K_n{}^a(-A)+K^a(-A)\right]+2\left|\frac{n}{B_n}\int_{-aB_n}^{aB_n}x\,dF(x)-\gamma-(c_1-c_2)\right. \\ &\quad \left.\left.\times\left(\int_a^{+\infty}\frac{du}{(1+u^2)u^\alpha}-\int_0^a\frac{u^{2-\alpha}du}{1+u^2}\right)\right|\right\}^{1/2}+\left\{\frac{8(c_1+c_2)}{r(\alpha-r)a^{\alpha-r}}\right\}^{1+r^{-1}}. \end{aligned}$$

The functions $K_n^a(v)$ and $K^a(v)$ are given by (5.12) and (5.6), and A, δ, $m(A, \delta)$ are given in Section 2.

6. An example. As an example we consider a sequence of independent random variables $\{X_n\}$, $n = 1, 2, \cdots$ with a common distribution function $\bar{F}(x)$. Let $\bar{F}(x)$ have density function

$$\bar{f}(x) = \pi^{-1}(1-\cos x),$$

and consider the normed sums

$$S_n = n^{-1}(X_1 + \cdots + X_n).$$

Again, let $F_n(x)$ denote the distribution function of S_n. The characteristic function of $\bar{F}(x)$ (c.f. [2] page 94 ff) is

$$\begin{aligned} \bar{\varphi}(t) &= 1-|t| \quad \text{or} \quad |t| \leqq 1, \\ &= 0 \qquad \text{for} \quad |t| > 1 \end{aligned}$$

and hence the characteristic function of $F_n(x)$ is

$$\begin{aligned} \varphi_n(t) &= (1-n^{-1}|t|)^n \quad \text{for} \quad |t| \leqq n, \\ &= 0 \qquad\qquad \text{for} \quad |t| > n. \end{aligned}$$

Clearly $\varphi_n(t)$ converges to $\varphi(t) = e^{-|t|}$ which is the characteristic function of the well-known Cauchy distribution function

$$F(x) = \pi^{-1}(\tfrac{1}{2}\pi + \arctan x).$$

We shall establish that the rate of convergence of $\sup_{-\infty < x < +\infty} |F_n(x) - F(x)|$ to zero is bounded by $C/n^{1/15}$ where C is a constant.

As is well known, $F(x)$ is stable with (c.f. [9] Section 4) the constants in (5.2) and (2.2) given by

$$\begin{aligned} &\alpha = 1, \\ (6.1) \qquad &c_1 = c_2 = \pi^{-1} \quad \text{and} \\ &\gamma = 0. \end{aligned}$$

To apply the result of Theorems 1 or 3, we put $X_{nk} = n^{-1}X_k$. We have

$$\begin{aligned} \mu_{nk}(a) &= 0 = \mu_n(a), \\ \sigma_{nk}^2(a) &= \frac{1}{n^2}\int_{-na}^{na} x^2 \frac{1}{\pi}\frac{1-\cos x}{x^2}\,dx \\ &= \frac{2}{\pi n^2}(na - \sin na), \quad \text{and} \\ \sigma_n^2(a) &= \frac{2}{\pi n}(na - \sin na). \end{aligned}$$

For $-a < v < a$ we have

$$K_n^a(v) = \frac{n}{n^2}\int_{-na}^{nv} u^2 \frac{1}{\pi}\frac{1-\cos u}{u^2}\,du$$

$$= \frac{1}{n\pi}[n(v+a)-(\sin nv+\sin na)].$$

Hence

$$K_n^a(v) = 0 \quad \text{for} \quad v < -a,$$

$$= \frac{1}{n\pi}[n(v+a)-(\sin nv-\sin na)] \quad \text{for} \quad -a \leqq v < a,$$

$$= \frac{2}{n\pi}[na-\sin na] \quad \text{for} \quad a \leqq v.$$

From (6.1) and (5.3) we have $dG(u) = \pi^{-1}(1+u^2)^{-1}du$. Applying Lemma 7 we have

$$\mu(a) = 0, \qquad \sigma^2(a) = 2a/\pi,$$ and

$$K^a(v) = 0 \quad \text{for} \quad v < -a,$$

$$= (a+v)/\pi \quad \text{for} \quad -a \leqq v < a,$$

$$= 2a/\pi \quad \text{for} \quad a \leqq v.$$

For simplicity we let $a = A = \delta^{-\frac{1}{2}}$. Then we have

$$\left\{\frac{1}{3}\sigma_n^2(a)\max\sigma_{nk}^2(a)\right\}^{1/5} \leqq \left\{\frac{16}{3\pi^2 n\delta}\right\}^{1/5}, \tag{6.2}$$

$$\left\{\frac{5}{6}\delta(\sigma_n^2(a)+\sigma^2(a))\right\}^{1/4} \leqq \left\{\frac{5}{\pi}\sqrt{\delta}\right\}^{1/4}, \tag{6.3}$$ and

$$\left\{\frac{1}{2}\sum_{i=0}^{m}|K_n^a(x_i)-K^a(x_i)|\right\}^{1/3} = \left\{\frac{1}{2}\sum_{i=0}^{m}\frac{|\sin nx_i-\sin na|}{n\pi}\right\}^{1/3} \tag{6.4}$$

$$\leqq \left\{\frac{1}{2}(m+1)\frac{2}{n\pi}\right\}^{1/3} \leqq \left\{\frac{4}{n\pi\delta\sqrt{\delta}}\right\}^{1/3},$$

from the definition of m. Furthermore,

$$\{4A^{-1}(K_n^a(+\infty)-K_n^a(A)+K^a(+\infty)-K^a(A)+K_n^a(-A)+K^a(-A)) + 2|\mu_n(a)-\mu(a)|\}^{1/2} = 0. \tag{6.5}$$

Applying Lemma 8, and (6.1)–(6.5) to $g^a(n, m(A, \delta), r)$ as given in (2.9) we have

$$g^a(n, m(A,\delta), r) \leqq \left\{\frac{16}{3\pi^2 n\delta}\right\}^{1/5} + \left\{\frac{5}{\pi}\sqrt{\delta}\right\}^{1/4} + \left\{\frac{4}{\pi n\,\delta\sqrt{\delta}}\right\}^{1/3} + \left\{\frac{16}{\pi r(1-r)}\right\}^{(1+r)^{-1}} \{\sqrt{\delta}\}^{(1-r)/(1+r)}.$$

Applying (5.13) we have

$$\sum_{k=1}^{k_n} \{F_{nk}(-a)+1-F_{nk}(a)\} = n\{\bar{F}(-an)+1-\bar{F}(an)\}$$
$$= n\frac{2}{\pi}\int_{an}^{+\infty} \frac{1-\cos x}{x^2}\,dx \leqq \frac{4}{\pi a} = \frac{4}{\pi}\sqrt{\delta}.$$

From Theorem 1 or 3 we have

$$\sup_{-\infty<x<+\infty} |F_n(x)-F(x)| \leqq k\left\{\frac{16}{3\pi^2 n\delta}\right\}^{1/5} + \left\{\frac{5}{\pi}\sqrt{\delta}\right\}^{1/4}$$
$$+\left\{\frac{4}{\pi n\delta\sqrt{\delta}}\right\}^{1/3} + \left\{\frac{16}{\pi r(1-r)}\right\}^{(r+1)^{-1}} \{\sqrt{\delta}\}^{(1-r)/(1+r)} + \frac{4}{\pi}\sqrt{\delta}.$$

Taking for example $\delta = 1/n^{8/15}$, $r = \frac{3}{5}$, we find that

$$\sup_{-\infty<x<+\infty} |F_n(x)-F(x)| \leqq C/n^{1/15},$$

where C is a constant.

REFERENCES

[1] BERRY, A. C. (1941). The accuracy of the Gaussian approximation to the sum of independent variates. *Trans. Amer. Math. Soc.* **49** 122–136.

[2] CRAMÉR, H. (1946). *Mathematical Methods of Statistics*. Princeton Univ. Press.

[3] ESSEEN, C. G. (1945). Fourier analysis of distribution functions. *Acta Math.* **77** 1–125.

[4] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2**. Wiley, New York.

[5] GNEDENKO, B. V. and KOLMOGOROV, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables* (translation by K. L. Chung) Addison-Wesley, Reading, Mass.

[6] SHAPIRO, J. M. (1955). Error estimates for certain probability limit theorems. *Ann. Math. Statist.* **26** 617–630.

[7] SHAPIRO, J. M. (1956). A condition for existence of moments of infinitely divisible distribution functions. *Canad. J. Math.* **8** 69–71.

[8] SHAPIRO, J. M. (1957). Sums of independent truncated random variables, *Ann. Math. Statist.* **28** 754–761.

[9] SHAPIRO, J. M. (1965). Domains of attraction for absolute values of random variables. *J. Soc. Indust. Appl. Math.* **13** 129–135.